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Weighted H^2 rational approximation and consistency*

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Summary. We investigate consistency properties of rational approximation of prescribed type in the weighted Hardy space $H^2_-(\mu)$ for the exterior of the unit disk, where μ is a positive symmetric measure on the unit circle $\mathbb{T}$. The question of consistency, which is especially significant for gradient algorithms that compute local minima, concerns the uniqueness of critical points in the approximation criterion for the case when the approximated function is itself rational. In addition to describing some basic properties of the approximation problem, we prove for measures μ having a rational function distribution (weight) with respect to arclength on $\mathbb{T}$, that consistency holds only under rather restricted conditions.

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Notations

$\mathbb{T}$, $\mathbb{D}$, $\widetilde{\mathbb{D}}$	unit circle, open unit disk, complement in $\mathbb{C}$ of the closed unit disk
$\mathcal{P}_n$	space of real polynomials of degree at most n (if $n < 0$, $\mathcal{P}_n = \{0\}$)
$\mathcal{M}_n^1$	monic real polynomials of degree n having all their roots in $\mathbb{D}$

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$\widetilde{\mathcal{M}}_n^1$	real polynomials of degree at most n with constant coefficient equal to 1 having all their roots outside the closure of $\mathbb{D}$
$L^2(\mu)$, L^2	real Hilbert spaces of square-summable functions with respect to a finite positive measure μ symmetric on $\mathbb{T}$ (i.e. invariant by complex conjugation), with respect to the Lebesgue measure $d\theta$ on $\mathbb{T}$
$\langle\cdot,\cdot\rangle_\mu$, $\langle\cdot,\cdot\rangle$	(real valued) scalar products in $L^2(\mu)$, in L^2
$\|.\|_{2,\mu}$, $\|.\|_2$	norms in $L^2(\mu)$, in L^2
$H^2(\mu)$, H^2	real Hardy spaces of exponent 2 of the unit disk, closures of the space of real polynomials in $L^2(\mu)$, in L^2
$H^2_-(\mu)$, H^2_-	real Hardy spaces defined as the closures of the linear span of $\{1/z^k\ ,\ k>0\}$ in $L^2(\mu)$, in L^2
P_+^μ, P_+, P_-^μ, P_-	orthogonal projections $L^2(\mu)\to H^2(\mu)$, $L^2\to H^2$, $L^2(\mu)\to H^2_-(\mu)$, and $L^2\to H^2_-$
L^∞, H^∞	real Banach space of essentially bounded functions on $\mathbb{T}$, real subspace of H^2 of essentially bounded functions
$\mathcal{R}^-_{m,n}$	subset of H^2_- consisting of rational functions $z^{n-m-1}p/q$ with $p\in\mathcal{P}_m$ and $q\in\mathcal{M}_n^1$

1 Introduction

We investigate consistency properties of rational approximation in the complement in $\mathbb{C}$ of the closed unit disk, with the norm induced by that of a weighted real Hardy space $H^2_-(\mu)$, where μ is some finite positive measure symmetric on $\mathbb{T}$. Let us first state and comment on the rational approximation problem.

$\mathrm{Pb}^\mu(\widetilde{\mathbb{D}},m,n)$: *Given $f\in H^2_-(\mu)$ and nonnegative integers m, n, find rational functions $z^{n-m-1}p/q\in\mathcal{R}^-_{m,n}$ which minimize*

$$\|f-z^{n-m-1}p/q\|^2_{2,\mu}=\frac{1}{2\pi}\int_0^{2\pi}|(f-z^{n-m-1}p/q)(\mathrm{e}^{\mathrm{i}\theta})|^2d\mu(\theta). \tag{1.1}$$

The measure μ on $\mathbb{T}$ involved in the definition of the norm above is assumed to carry some kind of regularity: we consider the case of an absolutely continuous measure μ which belongs to the Szegő class, that is

$$d\mu=w(\theta)d\theta,\quad 0\le w\in L^1,\quad \log w\in L^1. \tag{1.2}$$

It then follows from properties of the corresponding weighted Hardy space $H^2_-(\mu)$ that the set $\mathcal{R}^-_{m,n}$ of approximating functions is contained in $H^2_-(\mu)$, see Sect. 2 below.

Problem $\mathrm{Pb}^\mu(\widetilde{\mathbb{D}},m,n)$ is standard in approximation theory. Our motivation for investigating rational approximation partly stems from system and control theory. Indeed, problem $\mathrm{Pb}^{\mu}(\widetilde{\mathbb{D}}, m, n)$ is a well-posed and efficient formulation for weighted model reduction issues in frequency domain identification of linear scalar systems that are stable, causal, and time invariant [15,23,25]. For multivariable systems, model reduction amounts to solving an H^2 matrix rational best approximation problem of bounded MacMillan degree [5].

The primary question of existence of a minimizer in problem $\mathrm{Pb}^{\mu}(\widetilde{\mathbb{D}}, m, n)$ can be answered positively as in the unweighted diagonal case ($m = n - 1$, $w = 1$), see [1].

Using the differential framework induced by the integral norm on $H^2_-(\mu)$, we can define the rational functions that are *critical points* of the criterion (1.1). These are the $z^{n-m-1}p/q \in \mathcal{R}^-_{m,n}$ at which the derivative of the norm of the error $\|f - z^{n-m-1}p/q\|_{2,\mu}$ with respect to the coefficients of p/q vanishes. Among them, lie local minima as well as global minima, but possibly also saddle points or local maximums. The analysis of these critical points forms the basis of our study of the above rational approximation problem and we spend some time characterizing them.

The unweighted scalar case with degree (m, n) satisfying $m \geq n - 1$ has already been studied in a series of papers. Let us briefly review some of the results that were obtained. Existence and generic uniqueness of a best approximant, asymptotic properties, as well as an index theorem that gives a global constraint on the set of critical points, have been established in [1, 4,6]; a gradient algorithm converging to a local minimum is also described in [3]. From the index theorem, uniqueness of a critical point (hence of a local and global minimum) is derived for some classes of functions, like Markov functions or exponentials; see [8] and the bibliography therein. Such uniqueness properties are of interest from a numerical viewpoint, since they ensure the convergence of the above algorithm to the best approximant.

A first step in the study of the weighted diagonal approximation problem $\mathrm{Pb}^{\mu}(\widetilde{\mathbb{D}}, n-1, n)$ in $H^2_-(\mu)$ appears in [19], where a resolution algorithm is proposed for weights w that are the square-modulus of reciprocals of polynomials. In this connection, weighted Hardy spaces $H^2(\mu)$, for measures μ belonging to the Szegő class (1.2), have also been considered in [9] to generalize results about uniform meromorphic approximation in the unit disk (commonly as Adamjan–Arov–Krein theory) rather than H^2 rational approximation.

We come now to the consistency issue. This is a quite standard notion in numerical analysis of differential equations. Let us first explain what we mean by consistency for problem $\mathrm{Pb}^{\mu}(\widetilde{\mathbb{D}}, m, n)$. When the function f itself is an irreducible fraction in $\mathcal{R}^-_{m,n}$, it is obvious that f minimizes (1.1), as its own approximant. Still, in this case, the question arises whether f is the only critical point among the set of approximants. By *consistency of* *the approximation problem*, we mean that the answer to this question is in the positive; see Definition 4.1 for a precise statement. For example, the unweighted diagonal case ($m = n - 1$) is known to be consistent [4] as well as its matrix valued generalization [5].

This consistency property has strong consequences when interpreting the local minima found by an algorithm, based on a gradient method. Indeed, if consistency fails, a local minimum, even of the right degree, that furnishes a small error when approximating a rational function, can happen to be far from the function to be identified. This pathological behaviour then leads to computational difficulties when trying to recover the right degree of the function, let alone the function itself. In a system theoretic setting, consistency-like properties have already been investigated for least–squares criteria. Sufficient conditions in the overparametrizing case (degree of the approximant larger than the degree of the system) for uniqueness of critical points and a counterexample to consistency in a matching order case (of degrees $m = 0$, $n = 2$, and a weight w of the form $|z^2 - \alpha^2|^4$, $\alpha \in (-1, 1)$) have been given in [26]. Further counterexamples in the unweighted case (with $m = 0$ and $n = 3$) and a discussion about the uniqueness of critical points in more general situations also appear in [11,21]. The consistency issue for different kinds of approximants, namely multivariate Padé approximants, has also been studied in [10].

In this paper, we consider rational weights w and demonstrate that consistency holds only in a few cases that we specify. For example, we show that the unweighted problem is consistent if and only if $n - 2 \leq m$. In most other cases, we construct counterexamples which show that consistency fails. In the special occurence where the degree of the numerator of the approximants equals zero ($m = 0$), we prove that situations with two distinct minima can occur (*strong nonconsistency*).

In Sect. 2, we describe the weighted Hardy spaces that we consider, along with their main properties. We also display some of the characteristics of our weighted rational approximation problem. In Sect. 3, we study the differential properties of $\mathrm{Pb}^{\mu}(\widetilde{\mathbb{D}}, m, n)$ and establish the critical points equations. In Sect. 4, we derive a sufficient condition for consistency to hold and exhibit cases where it is satisfied. In Sect. 5, we construct examples of nonconsistency and, using a topological argument, we refine these examples to get, under the additional condition that $m = 0$, cases of strong nonconsistency.

2 Weighted H^2 rational approximation

General properties of Hardy spaces can be found in [12,16,18,24]. Some results on weighted Hardy spaces are given in [16,22]. We summarize in the following proposition a few of the well–known facts about Hardy spaces associated with measures on $\mathbb{T}$ satisfying the Szegő condition (1.2).

Proposition 2.1 ([13,16]) *If μ satisfies* (1.2), *then:*

(i) The function $1/z$ does not belong to $H^2(\mu)$. In particular, $H^2(\mu) \neq L^2(\mu)$.
(ii) The Lebesgue measure is absolutely continuous with respect to μ.
(iii) The weight $w \in L^1$ can be written as

$$w = |h|^2, \tag{2.1}$$

for an outer function h in H^2, with $1/h$ in $H^2(\mu)$.

Note that h can be chosen real since the weight w satisfies $w(\theta) = w(-\theta)$, $\theta \in \mathbb{T}$. From Proposition 2.1, we deduce that the maps

$$f \mapsto hf, \quad f \mapsto \bar{h}f, \tag{2.2}$$

are isometric isomorphisms from $L^2(\mu)$ onto L^2. Moreover, the restriction of the first map to $H^2(\mu)$ is an isometric isomorphism onto H^2, while the restriction of the second map to $H^2_-(\mu)$ is an isometric isomorphism onto H^2_-. Since $1/h$ (resp. $1/\bar{h}$) defines an analytic function in $\mathbb{D}$ (resp. $\widetilde{\mathbb{D}}$), for any function f in $H^2(\mu)$ (resp. $H^2_-(\mu)$), there is a function $g = (hf)(1/h)$ (resp. $g = (\bar{h}f)(1/\bar{h})$) analytic in $\mathbb{D}$ (resp. $\widetilde{\mathbb{D}}$), such that the non–tangential limits of g agree with f almost everywhere with respect to the Lebesgue measure. Furthermore,

$$\lim_{r \to 1} \int_0^{2\pi} |g(\mathrm{e}^{\mathrm{i}\theta}) - g(r\mathrm{e}^{\mathrm{i}\theta})|^2 d\mu(\theta) = 0.$$

Hence, similar to the usual Hardy space H^2, the weighted space $H^2(\mu)$ can be considered as a Hilbert space of analytic functions in $\mathbb{D}$.

Finally, it is easily checked from the previous isomorphisms that the following two orthogonal decompositions of $L^2(\mu)$ hold:

$$L^2(\mu) = H^2(\mu) \oplus (\bar{h}/h)H^2_-(\mu) = (h/\bar{h})H^2(\mu) \oplus H^2_-(\mu), \tag{2.3}$$

and that, for any f in $L_2(\mu)$,

$$P^\mu_+(f) = h^{-1}P_+(hf), \quad P^\mu_-(f) = (\bar{h})^{-1}P_-(\bar{h}f). \tag{2.4}$$

Let us now describe some properties of the approximation problem.

First, observe that $\mathrm{Pb}^\mu(\widetilde{\mathbb{D}}, m, 0)$ simply amounts to the usual polynomial approximation of the function $z^{m+1}f(z)$ in $L^2(\mu)$, the unique solution of which is given by the $m+1$ first terms of its expansion in the Szegő basis associated with the measure μ, see [27, Chapter XI]. Note also that, in this case, the answer to the consistency issue is immediate since uniqueness of a critical point holds. In order to consider truly rational approximation, *we assume hereafter that the degree* n *of the denominators of the approximants satisfies* $n \geq 1$.

Problem $\mathrm{Pb}^{\mu}(\widetilde{\mathbb{D}}, m, n)$ can be transformed into an equivalent problem about functions that are analytic inside the unit disk. Indeed, applying the involutive isometry of $L^2(\mu)$ defined by:

$$g^{\sharp}(z) = z^{-1}\, g(z^{-1}),$$

which interchanges $H^2(\mu)$ and $H^2_-(\mu)$, we get

$\mathrm{Pb}^{\mu}(\mathbb{D}, m, n)$: *Given* $g \in H^2(\mu)$ *and nonnegative integers* m, n, *minimize*

$$\|g - \widetilde{p}/\widetilde{q}\|^2_{2,\mu},$$

as $\widetilde{p}/\widetilde{q}$ *ranges over the subset of* $H^2(\mu)$ *consisting of rational functions with* $\widetilde{p} \in \mathcal{P}_m$ *and* $\widetilde{q} \in \widetilde{\mathcal{M}}^1_n$.

Next, the problem of minimizing the distance

$$\|\varphi - p/q\|^2_{2,\mu}$$

between any function φ in the ambient space $L^2(\mu)$ and fractions p/q with $p \in \mathcal{P}_m$ and $q \in \mathcal{M}^1_n$ reduces to our problem $\mathrm{Pb}^{\mu}(\widetilde{\mathbb{D}}, m, n)$. Indeed,

$$\|\varphi - p/q\|^2_{2,\mu} = \|z^{n-m-1}\,\varphi - z^{n-m-1} p/q\|^2_{2,\mu},$$

and, using the second decomposition in (2.3), we have

$$z^{n-m-1}\,\varphi = \varphi_1 + \varphi_2, \quad \varphi_1 \in (h/\overline{h})H^2(\mu), \quad \varphi_2 \in H^2_-(\mu).$$

By orthogonality, we deduce, since $z^{n-m-1}p/q \in H^2_-(\mu)$, that

$$\|\varphi - p/q\|^2_{2,\mu} = \|\varphi_1\|^2_{2,\mu} + \|\varphi_2 - z^{n-m-1}p/q\|^2_{2,\mu}.$$

Hence, we are led to minimize $\|\varphi_2 - z^{n-m-1}p/q\|^2_{2,\mu}$, which is $\mathrm{Pb}^{\mu}(\widetilde{\mathbb{D}}, m, n)$ for the function φ_2.

In order to establish the next property, we need an additional assumption on the measure $d\mu$, namely that the outer function h such that (2.1) holds also satisfies

$$h \in H^{\infty}, \quad 1/h \in H^{\infty}. \tag{2.5}$$

This implies, in view of the isomorphisms (2.2), that

$$L^2(\mu) = L^2, \quad H^2(\mu) = H^2, \quad H^2_-(\mu) = H^2_-, \tag{2.6}$$

and that the norms $\|.\|_{2,\mu}$ and $\|.\|_2$ are equivalent. Note that the assumption (2.5) is also necessary in order to have the identities (2.6). Indeed, it is well-known that any multiplier on L^2 has to belong to L^{∞} (cf. [29, Theorem 13.14]). Hence, $h \in L^{\infty} \cap H^2 = H^{\infty}$, and the same holds for its reciprocal $1/h$.

Proposition 2.2 (Normality property) *Assume that the measure* μ *belongs to the Szegő class and satisfies assumption* (2.5). *Then, if* $f \in H^2_-(\mu)$ *is not a rational function belonging to* $\mathcal{R}^-_{m-1,n-1}$, *any local minimum* $z^{n-m-1}p/q$ *of* $\mathrm{Pb}^\mu(\widetilde{\mathbb{D}}, m, n)$ *with respect to* f *is such that* $\deg q = n$ *and* p, q *are coprime.*

Proof. Assume that $z^{n-m-1}p/q \in \mathcal{R}^-_{m,n}$ is a reducible local minimum of $\mathrm{Pb}^\mu(\widetilde{\mathbb{D}}, m, n)$ so that $z^{n-m-1}p/q \in \mathcal{R}^-_{m-\nu,n-\nu}$ with $0 < \nu \le \inf(m,n)$. To obtain our result, we use a perturbation argument as follows. For any closed subset K of $(-1,1)$ with nonempty interior, there exists a neighbourhood U of zero in $\mathbb{R}$ such that

$$\forall a \in U, \quad \forall b \in K,$$
$$\left\| f - z^{n-m-1}\left(\frac{p}{q} + \frac{az^{m-\nu+1}}{(z-b)q}\right)\right\|^2_{2,\mu} \ge \left\| f - z^{n-m-1}p/q\right\|^2_{2,\mu}.$$

Note that $z^{n-m-1}(p/q + az^{m-\nu+1}/(z-b)q) \in \mathcal{R}^-_{m,n}$. Expanding the norms in terms of scalar products yields

$$\left\langle \frac{a}{(z-b)q}, \frac{a}{(z-b)q}\right\rangle_\mu - 2\left\langle f - z^{n-m-1}\frac{p}{q}, \frac{az^{n-\nu}}{(z-b)q}\right\rangle_\mu \ge 0. \tag{2.7}$$

As a tends to zero, the left–hand side is of order

$$-2a\left\langle f - z^{n-m-1}p/q, z^{n-\nu}/(z-b)q\right\rangle_\mu.$$

Since (2.7) is satisfied regardless of the sign of a, we must have

$$\forall b \in K, \quad \left\langle f - z^{n-m-1}p/q, z^{n-\nu}/(z-b)q\right\rangle_\mu = 0. \tag{2.8}$$

On the other hand, the family $\{z^{n-\nu}/(z-b)q\}_{b\in K}$ spans a dense subspace $\mathcal{F}$ of H^2_-. Indeed, assume $g \in H^2_-$ is orthogonal to this family. Then, by definition of the scalar product in H^2_-, we have

$$\int \frac{z^{n-\nu}g^\sharp(z)}{(z-b)q(z)}dz = 0,$$

or equivalently, by Hermite formula,

$$b^{n-\nu}g^\sharp(b) - \mathcal{L}(b) = 0,$$

where $\mathcal{L} \in \mathcal{P}_{n-\nu-1}$ denotes the polynomial interpolating $z^{n-\nu}g^\sharp(z)$ at the roots of q. As this equality between two analytic functions in $\mathbb{D}$ holds for b in a subset K that admits accumulation points in $\mathbb{D}$, we deduce that $z^{n-\nu}g^\sharp(z) = \mathcal{L}(z)$, which, in turn, implies $g^\sharp = 0$, since $\deg\mathcal{L} = n-\nu-1$.

Consequently, the orthogonal complement of $\mathcal{F}$ in H^2_- is zero; whence $\overline{\mathcal{F}} = H^2_-$, as was to be proved. Next, by the assumption (2.5) on μ, H^2_- and $H^2_-(\mu)$ share the same topology, so that the subspace $\mathcal{F}$ is also dense in the weighted $H^2_-(\mu)$. Now, (2.8) means that $f - z^{n-m-1}p/q$ belongs to the orthogonal complement of $\mathcal{F}$ in $H^2_-(\mu)$; hence it is equal to zero, by the preceding remark. This yields a contradiction with the assumption made on f and finishes the proof. □

As mentioned in the introduction, we shall now restrict our study to the case of rational weights. It seems that such weights on $\mathbb{T}$ have not yet received much attention in the literature. The only reference that the authors know of is the article [17] where orthogonal polynomials with respect to rational weights are investigated. Let us now give the precise definition of the measures $d\mu$ on $\mathbb{T}$ associated with rational weights. These measures are the finite positive absolutely continuous ones, whose densities are equal to the square modulus of a rational function. Namely,

$$d\mu(\mathrm{e}^{\mathrm{i}\theta}) = |r(\mathrm{e}^{\mathrm{i}\theta})|^2\, d\theta\,,\ r = r_0/r_1\,, \tag{2.9}$$

for some polynomials r_0 and r_1. Throughout, we suppose that no roots of r_0 nor r_1 lie on $\mathbb{T}$, and that r_0 and r_1 are coprime. We also assume these two polynomials to be monic since this only requires multiplying the measure $d\mu$ by a positive constant. If

$$\deg r_0 = d_0\,, \quad \deg r_1 = d_1,$$

it will be convenient to say that the weight is of type (d_0, d_1). Moreover, note from the definition (2.9) of $d\mu$ that all the roots of r_0 and r_1 can be chosen in $\mathbb{D} \setminus \{0\}$. In this way, we get in particular that $r_0 \in \mathcal{M}^1_{d_0}$ and $r_1 \in \mathcal{M}^1_{d_1}$. Also, the derivative $d\mu/d\theta$ on $\mathbb{T}$ equals the square modulus of $\widetilde{r}_0/\widetilde{r}_1$ which, with our assumptions, belongs to H^∞, and the same is true for its reciprocal. Thus, condition (2.5) is satisfied, so that the identities (2.6) hold in the case of rational weights.

For convenience, when $d\mu$ is a measure as in (2.9), we rename the weighted rational approximation problem $\mathrm{Pb}^\mu(\widetilde{\mathbb{D}}, m, n)$ as:

$\mathrm{Pb}^r(m, n, d_0, d_1)$: *Given $f \in H^2_-$ and two integers $m, n \geq 0$, minimize*

$$\begin{aligned}\|f - z^{n-m-1}p/q\|^2_{2,\mu} \\ = \frac{1}{2\pi}\int_0^{2\pi} |(f - z^{n-m-1}p/q)(\mathrm{e}^{\mathrm{i}\theta})|^2|r(\mathrm{e}^{\mathrm{i}\theta})|^2 d\theta,\end{aligned} \tag{2.10}$$

as $z^{n-m-1}p/q$ ranges over $\mathcal{R}^-_{m,n}$.

One should note about $\mathrm{Pb}^r(m, n, 0, d_1)$ (that is, when the weight r only consists of the reciprocal of a polynomial) that situations where the degrees satisfy $m > n + d_1 - 1$ can always be reduced to the so-called canonical diagonal case where $m = n+d_1-1$. This merely means that, when $d_0 = 0$, some linear part of the approximants involved in the numerators does not play any role in our problem, as soon as it is of degree larger than some given constant. The proof is as follows. By definition of the weight in $H^2_-(\mu)$, we have:

$$\|f - z^{n-m-1}p/q\|_{2,\mu} = \|\frac{z^{d_1}f}{r_1} - \frac{z^{d_1+n-m-1}p}{r_1 q}\|_2.$$

In the unweighted space H^2, we can apply [7, Lemma 2.2, Eq. (2.7)], which implies that the last norm equals

$$\|P_-\left(z^{m-n+1}f/r_1\right) - \pi/r_1 q\|_2\,,$$

for some $\pi \in \mathcal{P}_{n+d_1-1}$. Hence,

$$\|f - z^{n-m-1}p/q\|_{2,\mu} = \|\frac{r_1}{z^{d_1}}\,P_-\left(\frac{z^{d_1}}{r_1}z^{m-n+1-d_1}f\right) - \frac{z^{-d_1}\,\pi}{q}\|_{2,\mu}.$$

Finally, observing that in the present situation the outer function h defined in (2.1) equals $1/\widetilde{r}_1$, we get by applying the second equality in (2.4) that

$$\|f - z^{n-m-1}p/q\|_{2,\mu} = \|P^\mu_-\left(z^{m-n+1-d_1}\,f\right) - z^{-d_1}\pi/q\|_{2,\mu}\,,$$

which is just the quantity to minimize when considering the approximation problem $\mathrm{Pb}^r(n+d_1-1, n, 0, d_1)$ for the function $P^\mu_-\left(z^{m-n+1-d_1}\,f\right)$.

3 Critical points

The aim of this section is to study our approximation problem with respect to differentiation.

First, the following definition has to be given. The reverse $\widetilde{p}$ of the polynomial $p \in \mathcal{P}_n$ is the polynomial such that

$$\widetilde{p}(z) = z^n\,p(1/z) \in \mathcal{P}_n.$$

If $n' > n$ and $p \in \mathcal{P}_n$ is considered as an element of $\mathcal{P}_{n'}$ (with vanishing leading coefficients), then the two definitions of $\widetilde{p}$ may be inconsistent. For this reason, we shall always specify which $\mathcal{P}_n$ is involved in the computation.

We identify the polynomial of $\mathcal{P}_n$,

$$p(z) = p_n z^n + \cdots + p_0,$$

with the vector $(p_n, \ldots, p_0)$. In this way, $\mathcal{P}_n$ is endowed with the Euclidean topology of $\mathbb{R}^{n+1}$. Similarly, identifying the polynomial of $\mathcal{M}^1_n$,

$$q(z) = z^n + q_{n-1}z^{n-1} + \cdots + q_0,$$

with the vector $(q_{n-1}, \ldots, q_0)$, $\mathcal{M}_n^1$ becomes an open subset of $\mathbb{R}^n$. Differentiating under the integral sign in (2.10), we see that the map $\mathcal{P}_m \times \mathcal{M}_n^1 \to \mathbb{R}$ given by

$$(p, q) \to \|f - z^{n-m-1} p/q\|_{2,\mu}^2 \tag{3.1}$$

is *smooth*, i.e. all partial derivatives of every order exist and are continuous. Next, in order to obtain a criterion depending on the denominator q only, we characterize the optimal numerator p_* from f and q. Considering the partial derivative of (3.1) with respect to the numerator p, we obtain

$$\left\langle f - \frac{p_*}{z^{m-n+1} q}, \frac{\mathcal{P}_m}{z^{m-n+1} q} \right\rangle_\mu = 0, \tag{3.2}$$

which means that the unique optimal p_* is the numerator of the orthogonal projection of f onto the subspace $\mathcal{P}_m / z^{m-n+1} q \subset H^2_-(\mu)$. Denoting this numerator by $L_{m,n}^{f,\mu}(q) \in \mathcal{P}_m$, we define a new map $\psi_{m,n}^{f,\mu} : \mathcal{M}_n^1 \to \mathbb{R}$ as a function of q only, by

$$\begin{aligned} \psi_{m,n}^{f,\mu}(q) &= \|f - L_{m,n}^{f,\mu}(q)/z^{m-n+1} q\|_{2,\mu}^2 \\ &= \|f\|_{2,\mu}^2 - \|L_{m,n}^{f,\mu}(q)/z^{m-n+1} q\|_{2,\mu}^2. \end{aligned} \tag{3.3}$$

In particular, the map $\psi_{m,n}^{f,\mu}$ is bounded by $\|f\|_{2,\mu}^2$ on $\mathcal{M}_n^1$.

Theorem 3.1 *Let $d\mu$ be a measure on $\mathbb{T}$ such that* (2.9) *holds. Then, the map $\psi_{m,n}^{f,\mu} : \mathcal{M}_n^1 \to \mathbb{R}$ is smooth.*

Proof. It is sufficient to prove the smoothness of the map $q \mapsto L_{m,n}^{f,\mu}(q)$. Let $\{\Phi_j^{\mu,q}\}_{j \geq 0}$ denote the system of orthonormal polynomials on $\mathbb{T}$ for the measure $d\mu/|q|^2$. The orthogonal polynomial $\Phi_j^{\mu,q}$ has precisely degree j and its roots lie in $\mathbb{D}$ [27, Thm. 11.4.1].

Choosing $\{\Phi_j^{\mu,q}\}_{0 \leq j \leq m}$ as a basis for $\mathcal{P}_m$ and applying the Christoffel–Darboux formula for the Szegő kernel (see [27, Chapter XI]), we first get, as in [2] or [19, Proposition 2], that the polynomial $L_{m,n}^{f,\mu}(q) \in \mathcal{P}_m$ is given by

$$L_{m,n}^{f,\mu}(q)(z) = \left\langle f, \frac{\widetilde{\Phi}_{m+1}^{\mu,q}(\xi) \widetilde{\Phi}_{m+1}^{\mu,q}(z) - \Phi_{m+1}^{\mu,q}(\xi) \Phi_{m+1}^{\mu,q}(z)}{\xi^{m-n+1}(1 - \xi z) q(\xi)} \right\rangle_\mu .$$

Next, recall that μ is defined by (2.9) and introduce the measure ϱ defined on $\mathbb{T}$ by $d\varrho(\mathrm{e}^{\mathrm{i}\theta}) = d\theta / |r_1(\mathrm{e}^{\mathrm{i}\theta})|^2$; whence $d\mu(\mathrm{e}^{\mathrm{i}\theta}) = |r_0(\mathrm{e}^{\mathrm{i}\theta})|^2 \, d\varrho(\mathrm{e}^{\mathrm{i}\theta})$. Write $\{\Phi_j^{\varrho,q}\}_{j \geq 0}$ for the system of orthonormal polynomials on $\mathbb{T}$ associated with the measure $d\varrho/|q|^2$. Because $r_0 \in \mathcal{M}_{d_0}^1$ and $|r_0(z)|^2 = z^{-d_0} \, r_0(z) \, \widetilde{r}_0(z)$ for $z \in \mathbb{T}$, we get from [17, Theorem 1] that the orthonormal polynomials associated with the distributions $d\varrho$ and $d\mu = |r_0|^2 \, d\varrho$ are linked by a determinantal relation of the form

$$r_0(z)\, \widetilde{r}_0(z)\, \Phi_k^{\mu,q}(z) = \alpha_k(z)\, \widetilde{\Phi}_{k+d_0}^{\varrho,q}(z) + \beta_k(z)\, \Phi_{k+d_0}^{\varrho,q}(z), \quad k \geq 0,$$

where $\alpha_k \in \mathcal{P}_{d_0-1}$ and $\beta_k \in \mathcal{P}_{d_0}$. Moreover, the coefficients of the polynomials α_k and β_k are given by determinants whose entries are the values of $z^j \widetilde{\Phi}_{k+d_0}^{\varrho,q}(z)$ and $z^j \Phi_{k+d_0}^{\varrho,q}(z)$ at the roots of r_0 and $\widetilde{r}_0$, for $j = 0, \ldots, d_0$.

Finally, the map $q \mapsto \Phi_j^{\varrho,q}$ is smooth on $\mathcal{M}_n^1$. Indeed, the classical induction formulas (see [27]):

$$\begin{cases} \widetilde{\Phi}_j^{\varrho,q}(0)\, \widetilde{\Phi}_j^{\varrho,q}(z) = \widetilde{\Phi}_{j+1}^{\varrho,q}(0)\, \widetilde{\Phi}_{j+1}^{\varrho,q}(z) \; - \; \Phi_{j+1}^{\varrho,q}(0) \Phi_{j+1}^{\varrho,q}(z)\, , \; j \geq 0\, , \\ \widetilde{\Phi}_j^{\varrho,q}(0)^2 = \widetilde{\Phi}_{j+1}^{\varrho,q}(0)^2 \; - \; \Phi_{j+1}^{\varrho,q}(0)^2 = \sum_{k=0}^{j} \Phi_k^{\varrho,q}(0)^2\, , \end{cases}$$

can be initialized backwards for such measures $d\varrho$ since we have the relations

$$\Phi_j^{\varrho,q}(z) = \; z^{j-n-d_1}\, q(z)\, r_1(z), \quad j \geq n + d_1\, .$$

The last equation directly yields the smoothness of $q \mapsto \Phi_j^{\varrho,q}$ for $j \geq n+d_1$. The induction formula then allows one to deduce the same result for $0 \leq j < n+d_1$ as soon as $\widetilde{\Phi}_j^{\varrho,q}(0)^2 = \widetilde{\Phi}_{j+1}^{\varrho,q}(0)^2 - \Phi_{j+1}^{\varrho,q}(0)^2 \neq 0$. This is always true for $q \in \mathcal{M}_n^1$, for the polynomial $\Phi_{j+1}^{\varrho,q}(z)/\widetilde{\Phi}_{j+1}^{\varrho,q}(0)$ belongs to $\mathcal{M}_{j+1}^1$; hence, $\Phi_{j+1}^{\varrho,q}(0)/\widetilde{\Phi}_{j+1}^{\varrho,q}(0)$ which is the product of its zeros, necessarily satisfies $|\Phi_{j+1}^{\varrho,q}(0)/\widetilde{\Phi}_{j+1}^{\varrho,q}(0)| < 1$.

This establishes the claim, since each step of the composed map $q \mapsto \Phi_{m+1+d_0}^{\varrho,q} \mapsto \Phi_{m+1}^{\mu,q} \mapsto L_{m,n}^{f,\mu}(q)$ is smooth for $q \in \mathcal{M}_n^1$. □

By definition, a *critical point* of $\psi_{m,n}^{f,\mu}$ will be any $q \in \mathcal{M}_n^1$ such that the derivative of $\psi_{m,n}^{f,\mu}$ vanishes at this point. We first use the integral form of (3.2) to obtain in Proposition 3.2 a characterization of the optimal numerator p in terms of division relations. Then, in Proposition 3.4, we characterize in the same manner critical points of $\mathrm{Pb}^r(m, n, d_0, d_1)$.

In the remainder of this paper, we assume the measure μ to be of the type (2.9) and we set throughout

$$l = n - m + d_1 - d_0 - 1\, . \tag{3.4}$$

Proposition 3.2 *Assume* $q \in \mathcal{M}_n^1$ *is fixed in* $\mathrm{Pb}^r(m, n, d_0, d_1)$ *and set*

$$E = f^\sharp\, \widetilde{q} - \widetilde{p}, \quad p \in \mathcal{P}_m.$$

Then, the following two assertions are equivalent:

(i) The polynomial $p \in \mathcal{P}_m$ *minimizes the weighted norm* (2.10).

(ii) If $l > 0$*, there exist a polynomial* $A \in \mathcal{P}_{l+d_0-1}$ *and a function* $B \in H^2$ *such that*

$$z^l \, r_0 \, \widetilde{r}_0 \, E - \widetilde{r}_1 \, \widetilde{q} \, A = r_1 \, q \, B \, . \tag{3.5}$$

If $l \leq 0$*, there exist a polynomial* $A \in \mathcal{P}_{d_0-1}$ *and a function* $B \in H^2$ *such that*

$$r_0 \, \widetilde{r}_0 \, E - \widetilde{r}_1 \, \widetilde{q} \, A = z^{-l} \, r_1 \, q \, B \, . \tag{3.6}$$

Remark. Assertion (ii) can be rephrased as follows. If $l > 0$, there exists a polynomial $A \in \mathcal{P}_{l+d_0-1}$ such that $r_1 q$ divides $z^l \, r_0 \, \widetilde{r}_0 \, E - \widetilde{r}_1 \, \widetilde{q} \, A$ in H^2. If $l \leq 0$, there exists a polynomial $A \in \mathcal{P}_{d_0-1}$ such that $z^{-l} r_1 q$ divides $r_0 \, \widetilde{r}_0 \, E - \widetilde{r}_1 \, \widetilde{q} \, A$ in H^2.

Proof. By differentiating the quantity (2.10) with respect to the coefficients of p, we get that p/q leads to a minimum if and only if

$$\begin{aligned} &\left\langle z^{n-m-1+i}/q, f - z^{n-m-1} p/q \right\rangle_\mu \\ &\quad = \left\langle z^{n-m-1+i} r/q, \left(f - z^{n-m-1} p/q\right) r \right\rangle = 0 \, , \; i = 0, \ldots, m \, . \end{aligned}$$

Using the integral representation of the H^2 scalar product and taking linear combinations of the previous equations, we get

$$\frac{1}{2i\pi} \int_{\mathbb{T}} \left(f^\sharp(z) - \frac{\widetilde{p}(z)}{\widetilde{q}(z)} \right) \, z^{n-m-1} \, \frac{\pi(z)}{q(z)} \, |r(z)|^2 \, dz = 0 \, , \; \forall \pi \in \mathcal{P}_m \, ,$$

or, equivalently,

$$\frac{1}{2i\pi} \int_{\mathbb{T}} z^l \, \frac{E \, r_0 \, \widetilde{r}_0}{\widetilde{q} \, \widetilde{r}_1}(z) \, \frac{\pi}{q \, r_1}(z) \, dz = 0 \, , \; \forall \pi \in \mathcal{P}_m \, . \tag{3.7}$$

Assume first that $l > 0$. Write

$$q \, r_1 = \pi_1 \, \pi_2 \, , \quad \pi_1 \in \mathcal{P}_{n-m+d_1-1} \, , \quad \pi_2 \in \mathcal{P}_{m+1} \, .$$

Moreover, let F be the H^2 function defined by:

$$F(u) = \frac{\pi_1(u)}{2i\pi} \int_{\mathbb{T}} z^l \, \frac{E \, r_0 \, \widetilde{r}_0}{\widetilde{q} \, \widetilde{r}_1}(z) \, \frac{dz}{\pi_1(z) \, (z-u)} \, , \; \forall u \in \mathbb{D} \, . \tag{3.8}$$

On the one hand, if α denotes a root of $q \, r_1$ with multiplicity ℓ, writing $F^{(s)}$ for the s^{th} derivative of F, it holds that

$$F^{(s)}(\alpha) = 0 \quad \text{for} \quad 0 \leq s \leq \ell - 1 \, , \tag{3.9}$$

or, equivalently, that $q \, r_1$ divides F in H^2. Indeed,

$$\begin{aligned} F^{(s)}(u) = \sum_{j=0}^{s} \frac{s!}{j!} \, \frac{\pi_1^{(j)}(u)}{2i\pi} \int_{\mathbb{T}} z^l \\ \times \frac{E \, r_0 \, \widetilde{r}_0}{\widetilde{q} \, \widetilde{r}_1}(z) \, \frac{dz}{\pi_1(z) \, (z-u)^{s-j+1}} \, , \; \forall u \in \mathbb{D} \, . \end{aligned} \tag{3.10}$$

Let $\ell = \ell_1 + \ell_2$ where ℓ_i is the multiplicity of α as a root of π_i, $i = 1, 2$. When $0 \leq j \leq \ell_1 - 1$, then $\pi_1^{(j)}(\alpha) = 0$. When $\ell_1 \leq j \leq \ell - 1$, then $1 \leq s - j + 1 \leq \ell_2$ for $0 \leq j \leq s$; thus, $\sigma_1(z) = \pi_1(z)\,(z-\alpha)^{s-j+1}$ is a factor of $q\,r_1$ of degree larger than or equal to $n - m + d_1$ and $q\,r_1 = \sigma_1\,\sigma_2$ for some $\sigma_2 \in \mathcal{P}_m$. Applying (3.7) with $\pi = \sigma_2$, we get that the integrand in (3.10) vanishes at α, thereby establishing (3.9).

On the other hand, the residue formula applied to expression (3.8) implies that there exists some polynomial $A \in \mathcal{P}_{n-m+d_1-2}$ such that

$$F(u) = u^l\, \frac{E\,r_0\,\widetilde{r}_0}{\widetilde{q}\,\widetilde{r}_1}(u) - A(u)\,,\ \forall u \in \mathbb{D}\,.$$

This, together with (3.9) and the observation that $n-m+d_1-2 = l+d_0-1$, establishes (3.5).

Conversely, assume that (3.5) holds. The integral in (3.7) then becomes

$$\frac{1}{2i\pi}\int_{\mathbb{T}} \frac{A\,\pi}{q\,r_1}(z)\,dz + \frac{1}{2i\pi}\int_{\mathbb{T}} \frac{B\,\pi}{\widetilde{q}\,\widetilde{r}_1}(z)\,dz\,.$$

However, we get from Cauchy's theorem that the two integrals above vanish for $\pi \in \mathcal{P}_m$. Indeed, in the first one, $A\,\pi$ is of degree at most $n + d_1 - 2$ while $q\,r_1$ is exactly of degree $n + d_1$ and has no zero outside the closure of $\mathbb{D}$. In the second one, the integrand belongs to H^2. This completes the proof of the equivalence of (i) and (ii) when $l > 0$.

Second, assume $l \leq 0$ and write this time

$$z^{-l}\,q\,r_1 = \pi_1\,\pi_2\,, \quad \pi_1 \in \mathcal{P}_{d_0}\,, \quad \pi_2 \in \mathcal{P}_{m+1}\,.$$

Define $F \in H^2$ as

$$F(u) = \frac{\pi_1(u)}{2i\pi}\int_{\mathbb{T}} \frac{E\,r_0\,\widetilde{r}_0}{\widetilde{q}\,\widetilde{r}_1}(z)\,\frac{dz}{\pi_1(z)\,(z-u)}\,,\ \forall u \in \mathbb{D}\,. \tag{3.11}$$

Again, we get that (3.9) holds for every root α of $z^{-l}\,q\,r_1$ with multiplicity ℓ and as before:

$$F(u) = \frac{E\,r_0\,\widetilde{r}_0}{\widetilde{q}\,\widetilde{r}_1}(u) - A(u)\,,\ \forall u \in \mathbb{D}\,,$$

for some $A \in \mathcal{P}_{d_0-1}$. Note that whenever $d_0 = 0$, then $A = 0$. Finally, the assertion that (ii) implies (i) can be proved as in the case $l > 0$. □

From Proposition 3.2, we can gain some information on the behaviour of the map $\psi_{m,n}^{f,\mu}$ on the boundary $\partial\mathcal{M}_n^1$ of the domain $\mathcal{M}_n^1$. This is the content of the next corollary. It will be useful in the proof of Theorem 5.3. Note that $\partial\mathcal{M}_n^1$ consists of monic polynomials whose roots are all of modulus at most 1 and at least one root is exactly of modulus 1.

Corollary 3.3 *If the function $f \in H^2_-$ is continuous on the closure of $\widetilde{\mathbb{D}}$, then the maps $L^{f,\mu}_{m,n}$ and $\psi^{f,\mu}_{m,n}$ admit continuous extensions to the boundary $\partial\mathcal{M}^1_n$ of the domain $\mathcal{M}^1_n$. Moreover, if the limit point $q \in \partial\mathcal{M}^1_n$ has more than m roots of modulus 1, these continuity properties still hold at q without the above assumption on f. In this case, we also have $L^{f,\mu}_{m,n}(q) = 0$ and the map $\psi^{f,\mu}_{m,n}$ assumes its maximal value $\|f\|^2_{2,\mu}$ at q.*

Proof. Let (q_i) be a sequence of polynomials in $\mathcal{M}^1_n$ which tends to a polynomial $q = q_1q_2 \in \partial\mathcal{M}^1_n$, with q_1 a monic polynomial of degree $n_1 > 0$ having all its roots of modulus 1 and $q_2 \in \mathcal{M}^1_{n-n_1}$. The sequence $\|L^{f,\mu}_{m,n}(q_i)/q_i\|^2_{2,\mu}$ remains bounded by $\|f\|^2_{2,\mu}$. Consider any subsequence (q_{i_ν}) such that $L^{f,\mu}_{m,n}(q_{i_\nu})/q_{i_\nu}$ converges to some rational function p_ν/q, $p_\nu \in \mathcal{P}_m$. This limit has a finite L^2 norm; whence q_1 divides p_ν. Hence, if $n_1 > m$, we have $p_\nu = 0$, so that the whole sequence $L^{f,\mu}_{m,n}(q_i)/q_i$ tends to 0, which finishes the proof of the corollary in this case. If $n_1 \le m$, we consider the limits in relations (3.5) and (3.6) when q_{i_ν} tends to q. Indeed, one can check that the polynomial A admits a limit by recalling that it is obtained from the residue formula applied to expression (3.8) and that the function f is continuous up to $\mathbb{T}$, and in particular at the roots of q_1. Consequently, B admits a limit as well and we get

$$z^l\, r_0\, \widetilde{r}_0\, (f^\sharp \widetilde{q} - \widetilde{p}_\nu) - \widetilde{r}_1\, \widetilde{q}\, A = r_1\, q\, B, \quad A \in \mathcal{P}_{l+d_0-1}, \quad B \in H^2,$$

if $l > 0$, and

$$r_0\, \widetilde{r}_0\, (f^\sharp \widetilde{q} - \widetilde{p}_\nu) - \widetilde{r}_1\, \widetilde{q}\, A = z^{-l}\, r_1\, q\, B, \quad A \in \mathcal{P}_{d_0-1}, \quad B \in H^2,$$

if $l \le 0$. We set $p^*_\nu = p_\nu/q_1 \in \mathcal{P}_{m-n_1}$. Since q_1 is also a factor of $\widetilde{p}_\nu$, dividing the two previous relations by q_1 leads to

$$z^l\, r_0\, \widetilde{r}_0\, (f^\sharp \widetilde{q}_2 - \widetilde{p^*_\nu}) - \widetilde{r}_1\, \widetilde{q}_2\, A = r_1\, q_2\, B, \quad A \in \mathcal{P}_{l+d_0-1}, \quad B \in H^2,$$

if $l > 0$, and

$$r_0\, \widetilde{r}_0\, (f^\sharp \widetilde{q}_2 - \widetilde{p^*_\nu}) - \widetilde{r}_1\, \widetilde{q}_2\, A = z^{-l}\, r_1\, q_2\, B, \quad A \in \mathcal{P}_{d_0-1}, \quad B \in H^2,$$

if $l \le 0$. Consequently, applying Proposition 3.2, we derive that

$$p^*_\nu = L^{f,\mu}_{m-n_1,n-n_1}(q_2),$$

which shows that the limit $p_\nu/q = p^*_\nu/q_2$ is independent of the particular subsequence (q_{i_ν}). Hence, the whole sequence $L^{f,\mu}_{m,n}(q_i)/q_i$ converges to $L^{f,\mu}_{m-n_1,n-n_1}(q_2)/q_2$ if $n_1 \le m$ and the maps $L^{f,\mu}_{m,n}$ and $\psi^{f,\mu}_{m,n}$ admit a continuous extension to the boundary of $\mathcal{M}^1_n$. □

We now proceed with characterizing the critical points of $\mathrm{Pb}^r(m, n, d_0, d_1)$.

Proposition 3.4 *Let $z^{n-m-1}\,p/q$ be a rational function in $\mathcal{R}^-_{m,n}$ such that*

$$p = \delta p_0\,,\ q = \delta q_0\,,$$

for polynomials p_0, q_0 such that p_0/q_0 is irreducible and $\delta \in \mathcal{M}^1_\nu$, $0 \le \nu \le n$. Set

$$E_0 = f^\sharp\,\widetilde{q}_0 - \widetilde{p}_0. \tag{3.12}$$

Then, the following two assertions are equivalent:

(i) The polynomial q is a critical point of the map $\psi^{f,\mu}_{m,n}$ defined by (3.3).

(ii) If $l > 0$, there exist a polynomial $A \in \mathcal{P}_{l+d_0-1}$ and a function $B \in H^2$ such that

$$z^l\,r_0\,\widetilde{r}_0\,E_0 - \widetilde{r}_1\,\widetilde{q}_0\,A = r_1\,q_0^2\,\delta\,B\,. \tag{3.13}$$

If $l \le 0$, there exist a polynomial $A \in \mathcal{P}_{d_0-1}$ and a function $B \in H^2$ such that

$$r_0\,\widetilde{r}_0\,E_0 - \widetilde{r}_1\,\widetilde{q}_0\,A = z^{-l}\,r_1\,q_0^2\,\delta\,B\,. \tag{3.14}$$

Remarks. In the above reduction of p/q, note that either $p_0 = 0$ or $\nu \le m$. Assertion (ii) can be rephrased as follows. If $l > 0$, there exists a polynomial $A \in \mathcal{P}_{l+d_0-1}$ such that $r_1 q_0^2 \delta$ divides $z^l r_0 \widetilde{r}_0 E_0 - \widetilde{r}_1 \widetilde{q}_0 A$ in H^2. If $l \le 0$, there exists a polynomial $A \in \mathcal{P}_{d_0-1}$ such that $z^{-l} r_1 q_0^2 \delta$ divides $r_0\,\widetilde{r}_0\,E_0 - \widetilde{r}_1\,\widetilde{q}_0\,A$ in H^2.

Proof. By differentiating the criterion with respect to the coefficients of p and q, we get that p/q is a critical point for (2.10) if and only if, for $i = 0, \ldots, m$ and $j = 0, \ldots, n$, we have

$$\begin{aligned}&\left\langle z^{n-m-1}\frac{z^i q + z^j p}{q^2}, f - \frac{z^{n-m-1}p}{q}\right\rangle_\mu \\ &= \left\langle z^{n-m-1}\frac{z^i q + z^j p}{q^2} r, \left(f - \frac{z^{n-m-1}p}{q}\right) r\right\rangle = 0\,.\end{aligned}$$

This is equivalent to

$$\frac{1}{2i\pi}\int_{\mathbb{T}} \left(f^\sharp(z) - \frac{\widetilde{p}(z)}{\widetilde{q}(z)}\right)\, z^{n-m-1}\,\frac{\pi(z)\,\delta(z)}{q^2(z)}\,|r(z)|^2\,dz = 0\,,\qquad \forall \pi \in \mathcal{P}_{m+n-\nu}\,,$$

since the spaces $\mathcal{P}_m q_0 + \mathcal{P}_n p_0$ and $\mathcal{P}_{m+n-\nu}$ coincide, by coprimeness of p_0 and q_0. Substituting r_0/r_1 for r and using (3.12), we obtain

$$\frac{1}{2i\pi}\int_{\mathbb{T}} z^l\,\frac{E_0\,r_0\,\widetilde{r}_0}{\widetilde{q}_0\,\widetilde{r}_1}(z)\,\frac{\pi}{q_0^2\,\delta\,r_1}(z)\,dz = 0\,,\ \forall \pi \in \mathcal{P}_{m+n-\nu}\,. \tag{3.15}$$

Now, the remainder of the proof follows as in the proof of Proposition 3.2.

If $l > 0$, we choose

$$F(u) = \frac{\pi_1(u)}{2i\pi} \int_{\mathbb{T}} z^l \frac{E_0\, r_0\, \widetilde{r}_0}{\widetilde{q}_0\, \widetilde{r}_1}(z)\, \frac{dz}{\pi_1(z)\,(z-u)}\,,\ \forall u \in \mathbb{D}\,,$$

where

$$q_0^2\, \delta\, r_1 = \pi_1\, \pi_2, \quad \pi_1 \in \mathcal{P}_{n-m+d_1-1}, \quad \pi_2 \in \mathcal{P}_{m+n-\nu+1}.$$

If $l \leq 0$, we choose

$$F(u) = \frac{\pi_1(u)}{2i\pi} \int_{\mathbb{T}} \frac{E_0\, r_0\, \widetilde{r}_0}{\widetilde{q}_0\, \widetilde{r}_1}(z) \frac{dz}{\pi_1(z)\,(z-u)}\,,\ \forall u \in \mathbb{D}\,,$$

where

$$z^{-l} q_0^2\, \delta\, r_1 = \pi_1\, \pi_2, \quad \pi_1 \in \mathcal{P}_{d_0}, \quad \pi_2 \in \mathcal{P}_{m+n-\nu+1}.$$

Here, we get that $q_0^2\, \delta\, r_1$ divides F in the first case, while $z^{-l} q_0^2\, \delta\, r_1$ divides F in the second case. Using the residue formula to express F, one obtains (3.13) and (3.14). Again, the converse follows from Cauchy's theorem by plugging (3.13) and (3.14) into (3.15). □

4 Consistency properties of $\mathrm{Pb}^r(m, n, d_0, d_1)$

First, we state precisely what we mean by consistency. For fixed values of the degrees m, n, d_0, d_1, introduce $\mathcal{P}\mathrm{b}(m, n, d_0, d_1)$ as the family of approximation problems $\{\mathrm{Pb}^r(m, n, d_0, d_1)\}$, when r describes the set of rational weights of type (d_0, d_1) defined in Sect. 2. Observe that the unweighted family $\mathcal{P}\mathrm{b}(m, n, 0, 0)$ contains only problem $\mathrm{Pb}^1(m, n, 0, 0)$.

Definition 4.1 *The family $\mathcal{P}\mathrm{b}(m, n, d_0, d_1)$ is consistent if, for any weight r of type (d_0, d_1) and any function f which is a rational function:*

(4.1) $$f(z) = z^{n-m-1} \frac{P}{Q}(z) \in \mathcal{R}^-_{m,n},$$

with P and Q coprime, the criterion $\psi^{f,\mu}_{m,n}$, with $d\mu(\mathrm{e}^{\mathrm{i}\theta}) = |r(\mathrm{e}^{\mathrm{i}\theta})|^2 d\theta$, admits f as its unique critical point. On the contrary, if there exist a weight r and a function f of the form (4.1) *such that $\psi^{f,\mu}_{m,n}$ admits several critical points, then $\mathcal{P}\mathrm{b}(m, n, d_0, d_1)$ is nonconsistent. If at least two distinct local minima exist, then $\mathcal{P}\mathrm{b}(m, n, d_0, d_1)$ is strongly nonconsistent.*

In this section, we prove the following theorem.

Theorem 4.2 *The family* $\mathcal{P}\mathrm{b}(m,n,d_0,d_1)$, $n \geq 1$, *is consistent whenever* $d_0 \in \{0,1\}$ *and* $n-2 \leq m+d_1$. *In particular, the family* $\mathrm{Pb}^1(m,n,0,0)$ *of unweighted approximation problems of degree* (m,n) *is consistent whenever* $n-2 \leq m$.

Before displaying the proof, we need to restate Proposition 3.4 when the function f is of the form (4.1).

Proposition 4.3 *Let* f *be given by* (4.1) *and let* $z^{n-m-1}\,p/q$ *be a rational function in* $\mathcal{R}^-_{m,n}$ *as in Proposition* 3.4. *Set*

$$S_0 = \widetilde{P}\,\widetilde{q}_0 - \widetilde{Q}\,\widetilde{p}_0. \tag{4.2}$$

Then the following two assertions are equivalent:

(i) The polynomial q *is a critical point of the map* $\psi^{f,\mu}_{m,n}$ *defined by* (3.3).
(ii) If $l > 0$, *there exist polynomials* A *and* B *in* $\mathcal{P}_{l+d_0-1}$ *such that*

$$z^l\,r_0\,\widetilde{r}_0\,S_0 - \widetilde{r}_1\,\widetilde{q}_0\,\widetilde{Q}\,A = r_1\,q_0^2\,\delta\,B\,. \tag{4.3}$$

If $l \leq 0$, *there exist polynomials* A *and* B *in* $\mathcal{P}_{d_0-1}$ *such that*

$$r_0\,\widetilde{r}_0\,S_0 - \widetilde{r}_1\,\widetilde{q}_0\,\widetilde{Q}\,A = z^{-l}\,r_1\,q_0^2\,\delta\,B\,. \tag{4.4}$$

In particular, if $l \leq 0$ *and* $d_0 = 0$, *equation* (4.4) *reduces to* $S_0 \equiv 0$.

Proof. Multiplying (3.13) by $\widetilde{Q}$ and renaming $\widetilde{Q}B$ as B, we get (4.3) with B in H^2. Moreover, B is obtained by dividing the polynomial on the left-hand side of (4.3) by a polynomial with all roots in $\mathbb{D}$. Since B is in H^2, these roots necessarily cancel with some of the numerator, implying that B itself is a polynomial. Further,

$$\deg z^l r_0 \widetilde{r}_0 S_0 \leq 2n + d_0 + d_1 - \nu - 1 \quad \text{and}$$
$$\deg \widetilde{r}_1 \widetilde{q}_0 \widetilde{Q} A \leq 3n - m + 2d_1 - \nu - 2.$$

The second upper bound is larger than the first one by $l > 0$, so that B can be chosen of degree at most

$$3n - m + 2d_1 - \nu - 2 - \deg r_1 q_0^2 \delta = n - m + d_1 - 2 = l + d_0 - 1.$$

This establishes (ii) when $l > 0$. The proof in the case $l \leq 0$ is similar. The converse is immediate. □

Next, evaluate (4.3) at the roots of r_0 and $\widetilde{r}_0$ (if $d_0 > 0$) on the one hand, and identify the coefficients of z^k in the polynomials in the right and left-hand sides of (4.3), for k from 0 to $l-1$ and from $2n+d_0+d_1-\nu$ to $3n-m+2d_1-\nu-2$ on the other hand. This gives rise to $2(l+d_0)$ equations that completely determine the two polynomials A and B in $\mathcal{P}_{l+d_0-1}$. Similarly, if

$d_0 > 0$, evaluating (4.4) at the roots of r_0 and $\widetilde{r}_0$, gives rise to $2d_0$ equations that completely determine the two polynomials A and B in $\mathcal{P}_{d_0-1}$. These sets of equations comprise two linear systems with respect to the coefficients of A and B. They are described in the next proposition, after some additional notations that we now introduce.

Set

$$r_0(z) = \prod_{i=1}^{d_0}(z-\alpha_i), \quad \alpha_i \in \mathbb{D}\setminus\{0\}. \tag{4.5}$$

Also, put

$$N = 2n + d_1 - \nu\,,$$

and

$$\begin{aligned}&a(z) = r_1 q_0 Q(z) \in \mathcal{P}_N, \quad b(z) = r_1 q_0^2 \delta(z) \in \mathcal{P}_N, \quad \text{for } l>0,\\ &a(z) = z^{-l} r_1 q_0 Q(z) \in \mathcal{P}_{N-l}, \quad b(z) = z^{-l} r_1 q_0^2 \delta(z) \in \mathcal{P}_{N-l},\\ &\text{for } l \le 0.\end{aligned}$$

Let a_i and b_i be the coefficients of z^i in the polynomials a and b. Set $d = \deg A = \deg B$, and let A_i, B_i be the coefficients of z^i in the polynomials A and B. Define the vector U of dimension $2(d+1)$:

$$U = [A_0, \dots, A_d, B_0, \dots, B_d]^{\mathrm{t}}, \tag{4.6}$$

and, for some nonnegative integer k, the matrix $\mathcal{M}(k)$ of dimensions $2d_0 \times 2k$:

$$\mathcal{M}(k) = \begin{bmatrix} \widetilde{a}(\alpha_1) & \dots\ \alpha_1^{k-1}\widetilde{a}(\alpha_1) & b(\alpha_1) & \dots\ \alpha_1^{k-1}b(\alpha_1) \\ \vdots & \vdots & \vdots & \vdots \\ \widetilde{a}(\alpha_{d_0}) & \dots\ \alpha_{d_0}^{k-1}\widetilde{a}(\alpha_{d_0}) & b(\alpha_{d_0}) & \dots\ \alpha_{d_0}^{k-1}b(\alpha_{d_0}) \\ \alpha_1^{k-1}a(\alpha_1)\ \dots & a(\alpha_1) & \alpha_1^{k-1}\widetilde{b}(\alpha_1)\ \dots & \widetilde{b}(\alpha_1) \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{d_0}^{k-1}a(\alpha_{d_0})\ \dots & a(\alpha_{d_0}) & \alpha_{d_0}^{k-1}\widetilde{b}(\alpha_{d_0})\ \dots & \widetilde{b}(\alpha_{d_0}) \end{bmatrix}.$$

Observe that the equation $\mathcal{M}(d+1)U = 0$ expresses the vanishing of the polynomials $\widetilde{a}A + bB$ and $a\widetilde{A} + \widetilde{b}\widetilde{B}$ at $\alpha_1, \dots, \alpha_{d_0}$. For zeros of r_0 of multiplicity h, $h > 1$, we replace the corresponding rows in the matrix by those corresponding to the vanishing of the derivatives of order $0, 1, \dots, h-1$ of the polynomials $\widetilde{a}A + bB$ and $a\widetilde{A} + \widetilde{b}\widetilde{B}$, evaluated at that zero.

Furthermore, when $l > 0$, introduce the matrix $\mathcal{N}$ of size $2l \times 2(l+d_0)$:

$$
\mathcal{N} = \left[\begin{array}{cccccccccccc}
1 & & & & & & b_0 & & & & & \\
a_{N-1} & \ddots & & 0 & & & b_1 & \ddots & & 0 & & \\
\vdots & \ddots & \ddots & & & O_{l,d_0} & \vdots & \ddots & \ddots & & & O_{l,d_0} \\
\vdots & \ddots & \ddots & \ddots & & & \vdots & \ddots & \ddots & \ddots & & \\
a_{N-l+1} & \cdots & \cdots & a_{N-1} & 1 & & b_{l-1} & \cdots & \cdots & b_1 & b_0 & \\
& a_0 & a_1 & \cdots & \cdots & a_{l-1} & & 1 & b_{N-1} & \cdots & \cdots & b_{N-l+1} \\
& & \ddots & \ddots & \ddots & \vdots & & & \ddots & \ddots & \ddots & \vdots \\
O_{l,d_0} & & & \ddots & \ddots & \vdots & O_{l,d_0} & & & \ddots & \ddots & \vdots \\
& & 0 & & \ddots & a_1 & & & 0 & & \ddots & b_{N-1} \\
& & & & & a_0 & & & & & & 1
\end{array}\right],
$$

where O_{l,d_0} stands for the zero matrix of size $l \times d_0$.

Remark. The above matrix bears some resemblance with the matrices whose determinants are involved in the Schur-Cohn algorithm concerning the number of zeros of a given polynomial π in the unit disk [20, Chapter X, Sect. 43]. When $d_0 = 0$, they are identical except that, here, two sequences of coefficients must be considered, namely those of the polynomials a and b, whereas in the Schur-Cohn algorithm, only one sequence of coefficients appears, namely that of the polynomial π.

The following proposition, which is easily checked, describes the above mentioned sets of linear equations using the notations just introduced.

Proposition 4.4 *Let f be given by* (4.1) *and let $z^{n-m-1}\, p/q$ be a critical point of* $\mathrm{Pb}^r(m, n, d_0, d_1)$. *Moreover, let U be the vector* (4.6)*, consisting of the coefficients of the polynomials A and B satisfying* (4.3) *or* (4.4)*. Then*

$$
\begin{bmatrix} \mathcal{M}(d_0 + l) \\ \mathcal{N} \end{bmatrix} U = 0, \qquad \text{if} \quad l > 0, \tag{4.7}
$$

and

$$
\mathcal{M}(d_0)\, U = 0 \qquad \text{if} \quad l \leq 0. \tag{4.8}
$$

In particular, when $l > 0$ and $d_0 = 0$, equation (4.7) *reduces to $\mathcal{N}\, U = 0$, whereas, when $l \leq 0$ and $d_0 = 0$, equation* (4.8) *is empty.*

We fix the values of the various degrees m, n, d_0, d_1 involved in $\mathrm{Pb}^r(m, n, d_0, d_1)$ in order to discuss consistency properties. From Definition 4.1, consistency holds if for any f given by (4.1) and any critical point $z^{n-m-1}\, p/q$, one has $p = p_0 = P$ and $q = q_0 = Q$, or equivalently $S_0 = 0$, for S_0 defined in (4.2). However, depending on the value of l, it follows from (4.3) or (4.4) that if the polynomials A and B do vanish identically, then so does

S_0. Further, $A = B = 0$ can be stated as $U = 0$. Hence, from Proposition 4.4, we get the following sufficient condition for consistency to hold:

if, for some prescribed l *and* d_0, *equation* (4.7) *or* (4.8) *admits* $U = 0$ *as its unique solution, then* $\mathrm{Pb}^r(m, n, d_0, d_1)$ *is consistent for all integers* m, n, d_1 *such that* $n - m + d_1 = l + d_0 + 1$.

Using this property, we now proceed with the proof of Theorem 4.2.

Proof of Theorem 4.2. First, for $d_0 = 0$ and $l \leq 0$, the assertion follows immediately from the last observation of Proposition 4.3.

Second, consider $d_0 = 0$ and $l = 1$. In view of Proposition 4.4, we look for vectors U of size 2 such that:

$$\mathcal{N} U = 0\,,$$

for the 2×2 matrix $\mathcal{N}$ given by

$$\mathcal{N} = \begin{bmatrix} 1 & b_0 \\ a_0 & 1 \end{bmatrix}.$$

The determinant of $\mathcal{N}$ is equal to $1 - a_0\, b_0$ which cannot vanish, because the polynomials a and b have all their roots in $\mathbb{D}$. Hence, in this case, (4.7) admits the unique solution $U = 0$.

Finally, take $d_0 = 1$ and $l \leq 0$. Again from Proposition 4.4, we look for vectors U of size 2 that are solutions to (4.8), where the 2×2 matrix $\mathcal{M}(1)$ is given by

$$\mathcal{M}(1) = \begin{bmatrix} \widetilde{a}(\alpha_1) & b(\alpha_1) \\ a(\alpha_1) & \widetilde{b}(\alpha_1) \end{bmatrix}.$$

The determinant of $\mathcal{M}(1)$ is equal to $(\widetilde{a}\,\widetilde{b} - a\,b)(\alpha_1)$ which cannot vanish, since both Blaschke products $a/\widetilde{a}$ and $b/\widetilde{b}$ are of modulus less than 1 in $\mathbb{D}$. □

5 Counterexamples to consistency

We now turn to cases where consistency fails. The nonconsistency will be proved by exhibiting some critical points $z^{n-m-1}p/q$ different from the rational f. We will look at particular critical points, namely those with vanishing numerator[1] $p = 0$. In view of (3.3), these points always correspond to maximums of the criterion $\psi^{f,\mu}_{m,n}$.

Theorem 5.1 *The family* $\mathcal{P}\mathrm{b}(m, n, d_0, d_1)$ *is nonconsistent whenever*

[1] Such points have been termed degenerate in [26], but we avoid this terminology, since it may be confused with the usual one from differential geometry, where it means critical points whose matrix of second order derivatives (the Hessian matrix) is degenerate.

(i) $l \geq 2$ *and* $d_1 \leq m+1$*, or*

(ii) $l = 1$ *and* $1 \leq d_0 \leq n-2$*, or*

(iii) $l \leq 0$ *and* $2 \leq d_0 \leq n-1$.

More precisely, for any polynomial $r_0 \in \mathcal{M}^1_{d_0}$ *(having at least one (resp. two distinct) real roots when* $l = 1$ *(resp.* $l \leq 0$*)), there exist polynomials* $r_1 \in \mathcal{M}^1_{d_1}$*,* $P \in \mathcal{P}_m$ *and* $Q, q \in \mathcal{M}^1_n$ *such that the reducible rational function* $z^{n-m-1}p/q$*, with* $p = 0$*, is a critical point of* $\psi^{f,\mu}_{m,n}$*, with* $d\mu = |r_0/r_1(\mathrm{e}^{\mathrm{i}\theta})|^2 d\theta$*, when approximating the rational function* $z^{n-m-1}P/Q$*.*

Remark. In case (i), we get from the definition (3.4) of l and the inequality $d_1 \leq m+1$ that the degree d_0 of the weight's numerator satisfies $d_0 \leq n-2-(m-d_1+1)$.

Corollary 5.2 *The unweighted problem* $\mathrm{Pb}^1(m,n,0,0)$ *is consistent if and only if* $n-2 \leq m$.

Proof. This is a direct consequence of Theorem 4.2 and assertion (i) of Theorem 5.1. □

Remark. Here, it may be interesting to note that the same condition on the degrees also appears in a totally different problem, namely that of comparing the errors $E^r(f)$ and $E^c(f)$, f a continuous real function on the interval $[-1,1]$, in real and complex rational Chebyshev approximation of fixed degree (m,n). It was proved in [14,28] that the infimum of the ratios $E^c(f)/E^r(f)$, as f describes the set of continuous real–valued functions on $[-1,1]$ distinct from the rational functions of type (m,n) with coefficients in $\mathbb{R}$, equals zero if $n-2 > m$. It was also conjectured that the above infimum is positive when $n-2 \leq m$.

Figure 1 illustrates the results of Theorems 4.2 and 5.1. It shows the different known cases of consistency (represented by "$*$") or nonconsistency (represented by "□") according to the various degrees m, n, d_0, d_1. The horizontal dashed lines correspond to the values $n-1$, $n-2$, or $n-2-(m-d_1+1)$ of d_0 when $l \leq 0$, $l = 1$, or $l \geq 2$, respectively. It is very likely that larger values of d_0 also correspond to cases of nonconsistency. Also, note that the definition of l and the fact that d_1 is a nonnegative integer implies $d_0 \geq (n-m-1)-l$. Hence, in Fig. 1, only points located above the line L with equation $d_0 = (n-m-1)-l$ are to be taken into consideration.

In the special case when the degree of the numerator equals zero, we strengthen our results to strong nonconsistency properties. This will be achieved by following an observation of Söderström (cf. [26]).

Theorem 5.3 *The family* $\mathcal{P}\mathrm{b}(m,n,d_0,d_1)$ *is strongly nonconsistent whenever* $m = 0$ *and one of the three conditions of Theorem* 5.1 *holds.*

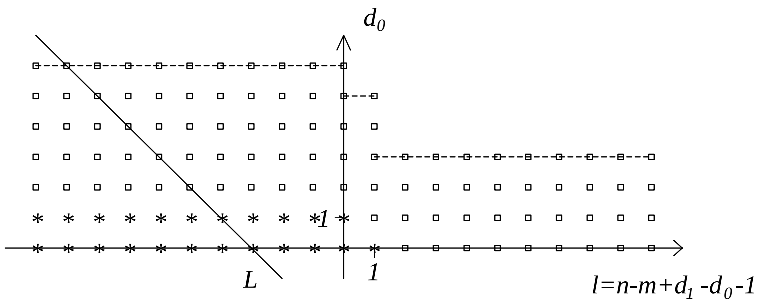

Fig. 1. Consistency ($*$) or nonconsistency ($\square$) according to the values of l and d_0

Proof of Theorem 5.1. As mentioned at the beginning of this section, we consider critical points $z^{n-m-1}p/q$ with vanishing numerator $p = 0$. With the notations of Proposition 3.4, such points satisfy

$$\delta = q, \quad q_0 = 1, \quad p_0 = 0.$$

Plugging these identities in the critical point equations (4.3) and (4.4) leads to

$$z^l r_0 \widetilde{r}_0 \widetilde{P} - \widetilde{r}_1 \widetilde{Q} A = r_1 q B, \qquad \text{if} \quad l > 0, \tag{5.1}$$

and

$$r_0 \widetilde{r}_0 \widetilde{P} - \widetilde{r}_1 \widetilde{Q} A = z^{-l} r_1 q B, \qquad \text{if} \quad l \leq 0, \tag{5.2}$$

where A and B have degrees at most $l + d_0 - 1$ in (5.1) and $d_0 - 1$ in (5.2). In order to construct explicitly critical points, we need to reduce the size of the matrices in (4.7) and (4.8) or, equivalently, to reduce the degrees of the involved polynomials A and B. To achieve this, the main idea will be to choose both denominators Q and q as multiples of the weight's numerator r_0 (or of a large part of it).

Case $l \geq 2$. We show that, for any numerator $r_0 \in \mathcal{M}^1_{d_0}$ of a measure $d\mu$ as in (2.9), equation (5.1) is satisfied by some polynomials P, Q, and q such that P/Q is an irreducible fraction, hence distinct from the critical point $z^{n-m-1}p/q \equiv 0$. We choose both Q and q as multiples of $z^{l-2}r_0$, which is always possible since the assumption $d_1 \leq m + 1$ implies

$$\deg z^{l-2} r_0 = n - m + d_1 - 3 < n = \deg Q = \deg q,$$

and write

$$Q(z) = z^{l-2} r_0(z) Q^*(z), \qquad q(z) = z^{l-2} r_0(z) q^*(z), \tag{5.3}$$

where both Q^* and q^* belong to $\mathcal{M}^1_{m-d_1+3}$. We then have

$$\deg z^l r_0 \widetilde{r}_0 \widetilde{P} \leq n + d_0 + d_1 - 1, \qquad \deg \widetilde{r}_1 \widetilde{Q} A \leq n + d_0 + d_1 + 1,$$

which, in view of (5.1), implies that $\deg r_1 q B \leq n + d_0 + d_1 + 1$; that is, $\deg B = d_0 + 1$. Moreover, as $z^{l-2} r_0$ divides q and is prime with $\widetilde{r}_1 \widetilde{Q}$, we deduce from (5.1) that it also divides A. Similarly, since $\widetilde{r}_0$ divides $\widetilde{Q}$ and is prime with $r_1 q$, it divides B. Performing these simplifications in (5.1) leads to

$$z^2 \widetilde{P} - \widetilde{r}_1 \widetilde{Q}^* A^* = r_1 q^* B^*, \tag{5.4}$$

with $\deg A^*$, $\deg B^* \leq 1$. Put

$$A^*(z) = A_0 + A_1 z, \qquad B^*(z) = B_0 + B_1 z.$$

We also set

$$m' = m - d_1 + 3 \geq 2$$

and

$$Q^*(z) = z^{m'} + Q_{m'-1} z^{m'-1} + \cdots + Q_0. \tag{5.5}$$

Moreover, we fix q^* and r_1 as[2]

$$q^*(z) = z^{m'} + q_0^*, \qquad r_1(z) = z^{d_1} + c, \tag{5.6}$$

where q_0^* and c are some real numbers in $(-1, 1)$, $c \neq 0$. Equation (5.4) entails that the two first and last coefficients of the polynomial $\widetilde{r}_1 \widetilde{Q}^* A^* + r_1 q^* B^*$ of degree $m + 4$ vanish. This can be written in matrix form as

$$\begin{bmatrix} 1 & 0 & cq_0^* & 0 \\ Q_{m'-1} & 1 & 0 & cq_0^* \\ cQ_0 & cQ_1 & 1 & 0 \\ 0 & cQ_0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ B_0 \\ B_1 \end{bmatrix} = 0, \tag{5.7}$$

whose determinant equals

$$(1 - c^2 q_0^* Q_0)^2 + c^2 q_0^* Q_1 Q_{m'-1}. \tag{5.8}$$

The values of this expression at $q_0^* = 1$ and $q_0^* = -1$ are

$$1 + c^4 Q_0^2 + c^2 (Q_1 Q_{m'-1} - 2Q_0) \quad \text{and} \quad 1 + c^4 Q_0^2 - c^2 (Q_1 Q_{m'-1} - 2Q_0),$$

respectively, which are of opposite signs if and only if

$$1 + c^4 Q_0^2 < |c^2 (Q_1 Q_{m'-1} - 2Q_0)|. \tag{5.9}$$

[2] Although more general denominators r_1 can be chosen for the measure $d\mu$, we stick here to this simple form, in order not to complicate the exposition and the computations.

There exist polynomials $Q^* \in \mathcal{M}^1_{m'}$ such that the previous inequality is met, for some $c \in (-1, 1)$. For example, choosing the roots of Q^* sufficiently close to 1, as in

$$\begin{aligned} Q^*(z) &= \left(z + \left(1 - \frac{1}{m'}\right)\right)^{m'} \\ &= z^{m'} + m'\left(1 - \frac{1}{m'}\right) z^{m'-1} + \cdots \\ &\quad + m'\left(1 - \frac{1}{m'}\right)^{m'-1} z + \left(1 - \frac{1}{m'}\right)^{m'}, \end{aligned} \tag{5.10}$$

satisfies the property, providing that the modulus of c is not too small. Indeed, the function $x \to \left(1 - \frac{1}{x}\right)^x$ is increasing for positive values of x; whence the following inequalities hold for $m' \geq 3$,

$$\left(\frac{2}{3}\right)^3 \leq \left(1 - \frac{1}{m'}\right)^{m'} \leq 1.$$

We have $1 + c^4 Q_0^2 < 2$, while

$$\begin{aligned} Q_1 Q_{m'-1} - 2Q_0 &= m'^2 \left(1 - \frac{1}{m'}\right)^{m'} - 2\left(1 - \frac{1}{m'}\right)^{m'} \\ &\geq 7\left(\frac{2}{3}\right)^3 = 2 + \frac{2}{27}, \end{aligned}$$

which implies (5.9), if c has a modulus sufficiently close to 1. Choosing Q^* as in (5.10), we deduce that there exists some $q_0^* \in (-1, 1)$, hence some q^* as in (5.6), such that the determinant (5.8) vanishes. Thus, there exists an associated nonzero solution to the system (5.7). Equivalently, we obtain two polynomials A^* and B^* of degree 1 with $A^* \neq 0$ or $B^* \neq 0$, such that

$$\widetilde{r}_1 \widetilde{Q}^* A^* + r_1 q^* B^* = z^2 P^*, \tag{5.11}$$

where P^* is some polynomial of degree m. It follows from the previous equation and the fact that A^* or B^* does not identically vanish that, actually, both A^* and B^* are distinct from zero. Moreover, P^* cannot vanish as well since, otherwise, q^* would divide $\widetilde{r}_1 \widetilde{Q}^* A^*$. But this is impossible as q^* has all its roots in $\mathbb{D}$ whereas $\widetilde{r}_1 \widetilde{Q}^*$ has all its roots outside $\mathbb{D}$ on one hand, and q^* has degree $m' = m - d_1 + 3 \geq 2$ larger than $\deg A^* \leq 1$ on the other hand. In view of (5.4), we set $P = \widetilde{P}^*$ and obtain that $0 = z^{n-m-1}(0/q)$ is a critical point of $\mathrm{Pb}^r(m, n, d_0, d_1)$ when approximating the nonzero rational $z^{n-m-1}P/Q$.

It remains to check that P/Q is irreducible. From the factorization of Q in (5.3) and equality (5.4), this is equivalent to the coprimeness of P and zr_0

on one hand, and to the coprimeness of $\widetilde{Q}^*$ and B^* on the other hand. First, if $P(0) = 0$, then the degree of $\widetilde{P}$ is less than m and equating coefficients of degree $m+2$ in (5.4) would give

$$Q_1 A_0 + Q_2 A_1 = 0.$$

But, from the system (5.7), we also deduce

$$(1 - c^2 q_0^* Q_0) A_0 - c^2 q_0^* Q_1 A_1 = 0.$$

The determinant of the two previous linear equations is $c^2 q_0^* Q_1^2 + Q_2(1 - c^2 q_0^* Q_0)$. One then checks that, for Q^* as in (5.10) and $c, q_0^* \in (-1,1)$, this determinant and the determinant (5.8) cannot vanish simultaneously. Hence, the two linear equations are independent, whence $A^* = 0$, which we know is false. Consequently, the following assertion holds

(i) *the polynomial P does not vanish at zero.*

Second, $\widetilde{Q}^*$ and B^* are coprime. Indeed, from the system (5.7), we get

$$B^*(z) = Q_1 + Q_0(1 - c^2 q_0^* Q_0) z,$$

up to some nonzero multiplicative constant. With the choice (5.10) of Q^*, the root of B^* equals

$$\frac{Q_1}{Q_0(c^2 q_0^* Q_0 - 1)} = \frac{2}{(m'-1)(\sqrt{1 - 4/m'^2} - 1)},$$

which is distinct from the root $-m'/(m'-1)$ of multiplicity m' of $\widetilde{Q}^*$. Hence, we have proved

(ii) *the polynomials P and Q^* are coprime.*

Finally, if P and r_0 are coprime, the fraction P/Q is irreducible from the previous discussion, and we are done. If not, there exists a partition $\{1, \ldots, d_0\} = I \cup J$, $J \neq \emptyset$, such that

$$P(\alpha_i) \neq 0, \quad i \in I, \tag{5.12}$$

$$P(\alpha_i) = 0, \quad i \in J, \tag{5.13}$$

where the α_i's denote the roots of r_0 as in (4.5). By (5.4), (5.13) is equivalent to

$$(\widetilde{r}_1 \widetilde{Q}^* A^* + r_1 q^* B^*)(1/\alpha_i) = 0, \qquad i \in J. \tag{5.14}$$

Since q^* and A^*, B^* are respectively obtained from equating the determinant (5.8) to zero and solving the system (5.7), we can consider the expression in the left-hand side of (5.14) as a function whose variables are the real numbers $c, Q_0, Q_1, \ldots, Q_{m'-1}$ or equivalently the polynomials r_1 and Q^* of the forms given in (5.6) and (5.5). We will denote by F_{α_i} this function parameterized by the root α_i. Also, observe that q^*, A^*, and B^* are determined in a way which makes F_{α_i} a real analytic function of its variables, defined on the subset of $\mathbb{R}^{m'+1}$ where (5.8) vanishes for some $q_0^* \in (-1, 1)$. Actually, the coefficients of A^* and B^* are determined up to some multiplicative constant, but this does not matter here, since (5.14) is homogeneous. Next, we show that the F_{α_i}, $i \in J$, are all distinct from the zero function by computing their values for $c = 1$, $Q_0 = 0$, $Q_1 = 2$, $Q_{m'-1} = 1$, and $Q_k = 0$, $1 < k < m' - 1$, when $m' > 2$, where they are equal to

$$(\alpha_i^{d_1} + 1)(2 - \alpha_i^{m'-3})/\alpha_i^{m+2},$$

and for $c = 2$, $Q_0 = 0$, when $m' = 2$, where they are equal to

$$3(2Q_1^2\alpha_i^{d_1-2} + 1)/(2\alpha_i^{m+1}).$$

Note that the normalization $A_0 = 1$ has been chosen and that, here, arbitrary values of the variables $c, Q_0, Q_1, \ldots, Q_{m'-1}$ such that the determinant (5.8) vanishes for some real number q_0^* can be considered. Since α_i is a nonzero complex number of modulus less than 1, the two previous quantities cannot vanish (when $m' = 2$, choose e.g. $Q_1 = 1/2$ if $d_1 \geq 2$ and $Q_1 = 1$ otherwise), which proves our contention. Denote by $\mathcal{M}$, the point in $\mathbb{R}^{m'+1}$ corresponding to the polynomial Q^* in (5.10) and the polynomial r_1 in (5.6), so that (5.14) rewrites

$$F_{\alpha_i}(\mathcal{M}) = 0, \qquad i \in J.$$

Since the product $\prod_{i \in J} F_{\alpha_i}$ is a nonzero real analytic function, it cannot be identically zero in a neighbourhood of $\mathcal{M}$, in particular in a neighbourhood $\mathcal{U}_0$ such that assertions (i), (ii) above and also (5.12) remain satisfied. Hence, there exists a point $\mathcal{M}' \in \mathcal{U}_0$ such that $\prod_{i \in J} F_{\alpha_i}(\mathcal{M}') \neq 0$ and it leads to an irreducible fraction P/Q which satisfies all of our requirements. This achieves the construction of an irreducible fraction $f = z^{n-m-1}P/Q$ such that $\mathrm{Pb}^r(m, n, d_0, d_1)$ admits a critical point distinct from f, when $l \geq 2$.

Case $l \leq 0$. Here, we assume that the polynomial r_0, which is of degree $d_0 \geq 2$ by assumption, has at least two distinct real roots α_1 and α_2 so that

$$r_0(z) = (z - \alpha_1)(z - \alpha_2)r_{0,d_0-2}, \quad r_{0,d_0-2} \in \mathcal{M}^1_{d_0-2}.$$

Choose then Q and q as multiple of $(z-\alpha_2)r_{0,d_0-2}$ and r_{0,d_0-2}, respectively (recall that $d_0 \leq n - 1$), and write

$$(5.15) \quad Q(z) = (z - \alpha_2)r_{0,d_0-2}Q^*(z), \qquad q(z) = r_{0,d_0-2}q^*(z),$$

where Q^* and q^* belong to $\mathcal{M}^1_{n-d_0+1}$ and $\mathcal{M}^1_{n-d_0+2}$. As r_{0,d_0-2} divides q and is prime with $\widetilde{r}_1\widetilde{Q}$, we deduce from (5.2) that it also divides A. Similarly, since $(1-\alpha_2 z)\widetilde{r}_{0,d_0-2}$ divides $\widetilde{Q}$ and is prime with $z^{-l}r_1 q$, it divides B. Hence, performing simplification in (5.2) leads to

$$(5.16)\qquad (z-\alpha_1)(z-\alpha_2)(1-\alpha_1 z)\widetilde{P}-\widetilde{r}_1\widetilde{Q}^* A^* = z^{-l}r_1 q^* B^*,$$

with $\deg A^* \leq 1$ and $\deg B^* = 0$. Equation (5.16) implies that the polynomial $\widetilde{r}_1\widetilde{Q}^* A^* + z^{-l}r_1 q^* B^*$ vanishes at α_1, α_2 and $1/\alpha_1$. With

$$A^*(z) = A_0 + A_1 z, \qquad B^*(z) = B_0,$$

we get three linear equations which can be written in matrix form as

$$(5.17)\quad \begin{bmatrix} \widetilde{r}_1\widetilde{Q}^*(\alpha_1) & \alpha_1\widetilde{r}_1\widetilde{Q}^*(\alpha_1) & \alpha_1^{-l}r_1 q^*(\alpha_1) \\ \widetilde{r}_1\widetilde{Q}^*(\alpha_2) & \alpha_2\widetilde{r}_1\widetilde{Q}^*(\alpha_2) & \alpha_2^{-l}r_1 q^*(\alpha_2) \\ \alpha_1^{-l+1}r_1 Q^*(\alpha_1) & \alpha_1^{-l}r_1 Q^*(\alpha_1) & \widetilde{r}_1\widetilde{q}^*(\alpha_1) \end{bmatrix} \begin{bmatrix} A_0 \\ B_0 \\ B_1 \end{bmatrix} = 0.$$

As in the previous case, we show that the determinant of this system vanishes. Suppose first that we are in the limit case where the polynomial $r_1(z) = z^{d_1}+1 = \widetilde{r}_1(z)$. Then, r_1 can be factorized in (5.17) and we ignore it in the following computations. We set $n' = n - d_0 + 1 \geq 2$ and choose Q^* and q^* of the special forms

$$(5.18)\qquad Q^*(z) = (z+Q_0)^{n'}, \qquad q^*(z) = z^{n'+1} + q_0^*,$$

where $Q_0\,,\ q_0^* \in (-1,1)$. Then, the 3×3 matrix in (5.17) can be rewritten as

$$(5.19)\quad \begin{bmatrix} (1+Q_0\alpha_1)^{n'} & \alpha_1(1+Q_0\alpha_1)^{n'} & \alpha_1^{-l}(\alpha_1^{n'+1}+q_0^*) \\ (1+Q_0\alpha_2)^{n'} & \alpha_2(1+Q_0\alpha_2)^{n'} & \alpha_2^{-l}(\alpha_2^{n'+1}+q_0^*) \\ \alpha_1^{-l+1}(\alpha_1+Q_0)^{n'} & \alpha_1^{-l}(\alpha_1+Q_0)^{n'} & 1+q_0^*\alpha_1^{n'+1} \end{bmatrix}.$$

We compute its determinant in the limit case where the moduli of Q_0 and q_0^* go to 1. For instance, we get for $Q_0 = 1$ and $q_0^* = 1$,

$$(5.20)\qquad \begin{aligned} &(1+\alpha_1)^{n'}[(1+\alpha_1)^{n'}\alpha_1^{-l}\alpha_2^{-l}(\alpha_1^2-1) \\ &\quad\times(1+\alpha_2^{n'+1}) + (1+\alpha_2)^{n'}C_1], \end{aligned}$$

and, for $Q_0 = 1$ and $q_0^* = -1$,

$$(5.21)\qquad \begin{aligned} &(1+\alpha_1)^{n'}[(1+\alpha_1)^{n'}\alpha_1^{-l}\alpha_2^{-l}(\alpha_1^2-1) \\ &\quad\times(\alpha_2^{n'+1}-1) + (1+\alpha_2)^{n'}C_2], \end{aligned}$$

where C_1 and C_2 are some algebraic quantities depending on α_1 and α_2. Let us denote respectively by $D_1(\alpha_1,\alpha_2)$ and $D_2(\alpha_1,\alpha_2)$ the second factor in (5.20) and (5.21).

If n' is even, we have

$$D_1(\alpha_1, -1) = 0, \quad D_2(\alpha_1, -1) = 2(-\alpha_1)^{-l}(1-\alpha_1^2)(1+\alpha_1)^{n'},$$

and

$$\frac{\partial D_1}{\partial \alpha_2}(\alpha_1, -1) = (-1)^{-l}(n'+1)\alpha_1^{-l}(\alpha_1^2-1)(1+\alpha_1)^{n'},$$

while if n' is odd, we have

$$D_1(\alpha_1, -1) = -2(-\alpha_1)^{-l}(1-\alpha_1^2)(1+\alpha_1)^{n'}, \quad D_2(\alpha_1, -1) = 0,$$

and

$$\frac{\partial D_2}{\partial \alpha_2}(\alpha_1, -1) = -(-1)^{-l}(n'+1)\alpha_1^{-l}(\alpha_1^2-1)(1+\alpha_1)^{n'}.$$

Note that, since $n' \geq 2$, the above derivatives do not depend on the actual values of C_1 and C_2. When n' is even, the sign of the derivative of D_1 at $\alpha_2 = -1$ is opposite to the sign of D_2 and, when n' is odd, the sign of the derivative of D_2 is opposite to the sign of D_1. It follows that, in all cases, (5.20) and (5.21) are of opposite signs when α_2 tends to -1 from above. By continuity, this is still true when Q_0 remains close to 1 and $Q_0 < 1$. Hence, for such a value of Q_0, we infer that there exists some q_0^* in $(-1, 1)$ at which the determinant of (5.19) vanishes. To summarize, we have shown the existence of α_1 and α_2 in $(-1, 1)$, $Q^* \in \mathcal{M}_{n'}^1$ and $q^* \in \mathcal{M}_{n'+1}^1$ of the forms (5.18) such that (5.19) has zero determinant. Hence, when considering these special values, we get a nonzero solution to the system (5.17). Equivalently, we obtain two polynomials A^* and B^* of degree at most 1 and 0 with $A^* \neq 0$ or $B^* \neq 0$, such that

$$\widetilde{Q}^* A^* + z^{-l} q^* B^* = (z-\alpha_1)(z-\alpha_2)(1-\alpha_1 z)P^*, \tag{5.22}$$

where P^* is some polynomial of degree m. Observe, from the previous equation that, actually, both A^* and B^* are distinct from zero. Also, P^* cannot identically vanish, since, otherwise, q^* would divide $\widetilde{Q}^* A^*$, hence A^*. But this is impossible as q^* has degree $n' \geq 2$ larger than $\deg A^* = 0$. In view of (5.16), we set $P = \widetilde{P}^*$ and obtain that $0 = z^{n-m-1}(0/q)$ is a critical point of $\mathrm{Pb}^r(m, n, d_0, d_1)$ when approximating the nonzero rational $z^{n-m-1}P/Q$.

Next, concerning the irreducibility of P/Q, from the factorization of Q in (5.15), we have to check that P does not vanish at the roots of r_0 (excepted α_1) on one hand, and that P and Q^* are coprime on the other hand. Using (5.22), this last assertion is equivalent to the coprimeness of B^* and $\widetilde{Q}^*$, which is obvious since B^* is a nonzero real number. If P and

$r_0(z)/(z-\alpha_1)$ are not coprime, we can again use a perturbation argument as in the previous case, by considering the expressions

$$(\widetilde{Q}^* A^* + z^{-l} q^* B^*)(1/\alpha_i), \quad i = 2, \ldots, d_0,$$

where the α_i's denote, as usual, the roots of r_0. Since q^*, A^*, and B^* are determined by the vanishing of the determinant of (5.19) and from the system (5.17), these expressions can still be seen as real analytic functions of the coefficients of the polynomial Q^*, parameterized by the root α_i. Choosing some special value for Q^*, one can check that these functions are all distinct from zero, hence that a slight change of Q is possible, if necessary, in order to get an irreducible fraction P/Q.

In order to consider nontrivial weights r_1, observe finally that, by continuity, all the previous discussion remains correct when perturbing the weight $r_1(z) = z^{d_1} + 1$ to $r_1(z) = z^{d_1} + c$, $c < 1$ in a neighbourhood of 1.

Case $l = 1$. As will become clear, this case may be seen as a combination of the two previous cases. Here, we assume that the polynomial r_0, which is of degree $d_0 \geq 1$ by assumption, has at least one real root α_1 so that r_0 factorizes as

$$r_0(z) = (z-\alpha_1) r_{0,d_0-1}(z), \quad r_{0,d_0-1} \in \mathcal{M}^1_{d_0-1}.$$

Then, we choose both Q and q as multiple of r_{0,d_0-1} (recall that $d_0 \leq n-2$) and write

$$Q(z) = r_{0,d_0-1} Q^*(z), \qquad q(z) = r_{0,d_0-1} q^*(z),$$

where Q^* and q^* belong to $\mathcal{M}^1_{n-d_0+1}$. Since r_{0,d_0-1} divides q and is prime with $\widetilde{r}_1 \widetilde{Q}$, we deduce from (5.1) that it also divides A. Similarly, since $\widetilde{r}_{0,d_0-1}$ divides $\widetilde{Q}$ and is prime with $r_1 q$, it divides B. Hence, performing simplification in (5.1) leads to

$$z(z-\alpha_1)(1-\alpha_1 z)\widetilde{P} - \widetilde{r}_1 \widetilde{Q}^* A^* = r_1 q^* B^*, \tag{5.23}$$

with $\deg A^* = 1$ and $\deg B^* = 1$. Equation (5.23) entails, on the one hand, that the first and last coefficients of the polynomial $\widetilde{r}_1 \widetilde{Q}^* A^* + r_1 q^* B^*$ of degree $m+4$ vanish and, on the other hand, that it vanishes at α_1 and $1/\alpha_1$. With $r_1(z) = z^{d_1} + c$,

$$A^*(z) = A_0 + A_1 z, \qquad B^*(z) = B_0 + B_1 z,$$

and Q_0^* and q_0^* denoting respectively the constant coefficients of Q^* and q^*, we get four linear equations which can be written in matrix form as

$$\begin{bmatrix} 1 & 0 & cq_0^* & 0 \\ \widetilde{r}_1\widetilde{Q}^*(\alpha_1) & \alpha_1\widetilde{r}_1\widetilde{Q}^*(\alpha_1) & r_1q^*(\alpha_1) & \alpha_1r_1q^*(\alpha_1) \\ \alpha_1r_1Q^*(\alpha_1) & r_1Q^*(\alpha_1) & \alpha_1\widetilde{r}_1\widetilde{q}^*(\alpha_1) & \widetilde{r}_1\widetilde{q}^*(\alpha_1) \\ 0 & cQ_0^* & 0 & 1 \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ B_0 \\ B_1 \end{bmatrix} = 0. \tag{5.24}$$

Here, one may remark that the first and fourth lines of the matrix are similar to those met in the matrix (5.7) corresponding to the case $l \geq 2$, while the second and third lines are similar to those met in (5.17) corresponding to the case $l \leq 0$. As in the previous cases, we show that the determinant of this system can happen to vanish.

We first suppose that $r_1(z) = z^{d_1} + 1 = \widetilde{r}_1(z)$. Again, r_1 can be factorized in (5.24) and we proceed without it. We set $n' = n - d_0 + 1 \geq 3$ and choose Q^* and q^* of the special forms

$$Q^*(z) = (z + Q_0)^{n'}, \qquad q^*(z) = z^{n'} + q_0^*, \tag{5.25}$$

where Q_0 and q_0^* are some real numbers in $(-1, 1)$. Then, the 4×4 matrix in (5.24) rewrites as

$$\begin{bmatrix} 1 & 0 & q_0^* & 0 \\ (1 + Q_0\alpha_1)^{n'} & \alpha_1(1 + Q_0\alpha_1)^{n'} & (\alpha_1^{n'} + q_0^*) & \alpha_1(\alpha_1^{n'} + q_0^*) \\ \alpha_1(\alpha_1 + Q_0)^{n'} & (\alpha_1 + Q_0)^{n'} & \alpha_1(1 + q_0^*\alpha_1^{n'}) & 1 + q_0^*\alpha_1^{n'} \\ 0 & Q_0^{n'} & 0 & 1 \end{bmatrix}. \tag{5.26}$$

We compute the previous determinant in the limit case where Q_0 and q_0^* are of modulus 1. For instance, we get for $Q_0 = 1$ and $q_0^* = 1$,

$$(1 - \alpha_1^2)(1 + \alpha_1^{n'} - (1 + \alpha_1)^{n'})^2,$$

which is positive as α_1 is of modulus less than 1, and for $Q_0 = 1$ and $q_0^* = -1$,

$$(1 + \alpha_1)\left((1 - \alpha_1)(1 - \alpha_1^{n'}) - (1 + \alpha_1)^{n'+1}\right)$$
$$\times \left((1 - \alpha_1)(1 + \alpha_1)^{n'-1} - (1 - \alpha_1^{n'})\right).$$

Denoting respectively by $D_1(\alpha_1)$ and $D_2(\alpha_1)$ these two expressions, we have

$$D_2(0) = 0, \quad D_2'(0) = 0, \quad D_2''(0) = -2(n' + 2)(n' - 2) < 0,$$

from which we deduce that D_1 and D_2 are of opposite signs as α_1 comes close to zero. By continuity, this is still true when $Q_0 < 1$ remains in a neighbourhood of 1. Thus, for such a value of Q_0, there exists q_0^* in $(-1,1)$ for which the determinant of (5.26) vanishes. Consequently, we get a nonzero solution to the system (5.24) or, equivalently, two polynomials A^* and B^* of degree at most 1 with $A^* \neq 0$ or $B^* \neq 0$, such that

$$\widetilde{Q}^* A^* + q^* B^* = z(z-\alpha_1)(1-\alpha_1 z)P^*, \tag{5.27}$$

where P^* is some polynomial of degree m. Observe, from the previous equation that, actually, both A^* and B^* are distinct from zero. Also, P^* cannot identically vanish, since, otherwise, q^* would divide $\widetilde{Q}^* A^*$, hence A^*. But this is impossible as q^* has degree $n' \geq 3$ larger than $\deg A^* \leq 1$. In view of (5.23), we set $P = \widetilde{P}^*$ and obtain that $0 = z^{n-m-1}(0/q)$ is a critical point of $\mathrm{Pb}^r(m,n,d_0,d_1)$ when approximating the nonzero rational $z^{n-m-1}P/Q$.

Again, the irreducibility of the fraction P/Q can be obtained by using a perturbation argument as in the two previous cases. The case of a nontrivial weight $r_1(z) = z^{d_1} + c$, with c close to 1, can still be obtained by continuity from the limit situation where $c = 1$. □

Proof of Theorem 5.3. First, observe that the proof of Theorem 5.1 is based on the vanishing of a determinant involving the coefficients of the denominators q and Q. Assume that Q is fixed. By the identification of $\mathcal{M}_n^1$ with an open subset of $\mathbb{R}^n$ that was described at the beginning of Sect. 3, this determinant equals zero if and only if the point $\mathcal{M}$ whose coordinates are the coefficients of q belongs to some hypersurface $\mathcal{H}$ in $\mathbb{R}^n$. It is shown in the proof of Theorem 5.1 that, if any one of the conditions (i), (ii) or (iii) is satisfied, then there exist polynomials q and Q in $\mathcal{M}_n^1$ such that the point $\mathcal{M}$ belongs to $\mathcal{H}$. Hence, in this case, the hypersurface $\mathcal{H}$ intersects $\mathcal{M}_n^1$. The numerator P of the function f was then subsequently obtained from the critical points equations (5.1) or (5.2). When $m = 0$, this P reduces to a real number and since it is only determined up to a constant multiplicative factor, one can set $P(z) = 1$ as $q \in \mathcal{M}_n^1$ describes $\mathcal{H}$. Consequently, $\mathrm{Pb}^r(0,n,d_0,d_1)$ admits a whole set of reducible critical points $\{z^{n-1}p/q,\ p = 0,\ q \in \mathcal{M}_n^1 \cap \mathcal{H}\}$ when approximating the unique function $f(z) = z^{n-1}/Q(z)$.

Second, since $p = 0$ when $q \in \mathcal{M}_n^1 \cap \mathcal{H}$, the criterion $\psi_{0,n}^{f,\mu}$ is constant on $\mathcal{M}_n^1 \cap \mathcal{H}$ and equal to its maximum value, namely $\|f\|_{2,\mu}^2$. On the boundary of $\mathcal{M}_n^1$, it has the same value, since we are assuming $m = 0$ (see Corollary 3.3).

Third, the set $\mathcal{M}_n^1$, when identified to an open subset of $\mathbb{R}^n$, is homeomorphic to the open unit ball of $\mathbb{R}^n$ (cf. [4]). Since the hypersurface $\mathcal{H}$ intersects $\mathcal{M}_n^1$, we obtain a partition of the closure of $\mathcal{M}_n^1$ into at least two compact subsets U_1 and U_2, with nonempty interiors, separated by $\mathcal{H}$. Since $\psi_{0,n}^{f,\mu}$ takes its maximum value on the boundaries of these two subsets, we can conclude, as Söderström did in the simpler case when $n = 2$ (cf [26]), that $\psi_{0,n}^{f,\mu}$ must have a minimum in U_1 as well as in U_2. This shows the existence of at least two relative minima and finishes the proof of the theorem. $\square$

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