The Sensitivity of Least Squares Polynomial Approximation

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Abstract. Given integers $N \geq n \geq 0$, we consider the least squares problem of finding the vector of coefficients $\vec{P}$ with respect to a polynomial basis $\{p_0, ..., p_n\}$, $\deg p_j = j$, of a polynomial P, $\deg P \leq n$, which is of best approximation to a given function f with respect to some weighted discrete norm, i.e., which minimizes $\sum_{j=0}^{N} w_n(z_j)^2 |f(z_j) - P(z_j)|^2$. Here a perturbation of the values $f(z_j)$ leads to some perturbation of the coefficient vector $\vec{P}$. We denote by κ_n the maximal magnification of relative errors, i.e., the Euclidean condition number of the underlying weighted Vandermonde–like matrix.

For the basis of monomials ($p_j(z) = z^j$), the quantity κ_n equals one when the abscissas are the roots of unity; however, it is known that κ_n increases exponentially in the case of real abscissas. Here we investigate the nth–root behavior of κ_n for some fixed basis and a fixed distribution of (complex) abscissas. An estimate for the nth–root limit of κ_n is given in terms of the solution to a weighted constrained energy problem in complex potential theory.

Key words: Least squares polynomial approximation, Condition number, Vandermonde matrices, complex potential theory.
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Introduction

Given a sequence of polynomials $(p_j)_{j\geq 0}$, $\deg p_j = j$ for all j, some integers $N \geq n \geq 0$, $E_n := \{z_0, ..., z_N\} \subset \mathbb{C}$, and a weight function w_n taking only positive values on E_n, the corresponding *weighted Vandermonde–like matrix* $V_n(w_n, E_n)$ of size $(N+1) \times (n+1)$ is defined by

$$\begin{pmatrix} w_n(z_0)p_0(z_0) & w_n(z_0)p_1(z_0) & w_n(z_0)p_2(z_0) & \cdots & w_n(z_0)p_n(z_0) \\ w_n(z_1)p_0(z_1) & w_n(z_1)p_1(z_1) & w_n(z_1)p_2(z_1) & \cdots & w_n(z_1)p_n(z_1) \\ & & & & \\ w_n(z_N)p_0(z_N & w_n(z_N)p_1(z_N) & w_n(z_N)p_2(z_N & & w_n(z_N)p_n(z_N \end{pmatrix}$$

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In this paper we will be interested in determining the size of the Euclidean condition number $\kappa(V_n(w_n, E_n))$ of $V_n(w_n, E_n)$, which equals the square root of the ratio of the largest and the smallest eigenvalue of the Hermitian positive definite matrix $V_n(w_n, E_n)^H \cdot V_n(w_n, E_n)$, where $V_n(w_n, E_n)^H$ denotes the Hermitian counterpart of $V_n(w_n, E_n)$. It is well known that the Euclidean condition number is a measure for the relative distance to matrices not having full rank [GoVL93, p.80]. Moreover, writing $||\cdot||$ for the Euclidean vector and matrix norm, and $V = V_n(w_n, E_n)$, we have [GoVL93, Subsection 2.7.2]

$$\kappa(V) = \max_{y\neq 0} \frac{||y||}{||Vy||} \cdot \max_{x\neq 0} \frac{||Vx||}{||x||} = ||V|| \cdot ||V^+||, \tag{1}$$

where $V^+ = (V^H V)^{-1} V^H$ denotes the pseudoinverse of V. The study of the condition number of weighted Vandermonde–like matrices is very much related to a study of the condition number of more general classes of structured matrices such as (modified) Gram matrices, positive definite Hankel matrices, or (modified) Krylov matrices (see, e.g., [Tay78, Tyr94, Bec96, Bec97, BeSt98]).

The quantity $\kappa(V_n(w_n, E_n))$ may serve to measure the sensitivity of least squares polynomial approximation: Given some function f defined on E_n, consider the problem of finding a polynomial P of degree at most n with minimal deviation from f with respect to some discrete L_2 norm. Writing $\vec{P} = (a_0, ..., a_n)^T$ for the polynomial $P = \sum_{j=0}^n a_j p_j$, we are left with the problem of determining $a_0, ..., a_n \in \mathbb{C}$ minimizing the expression

$$\sum_{k=0}^{N} w_n(z_k)^2 |f(z_k) - \sum_{j=0}^{n} a_j p_j(z_k)|^2,$$

with its unique solution given by

$$\vec{P} = V_n(w_n, E_n)^+ (w_n(z_0) f(z_0), ..., w_n(z_N) f(z_N))^T.$$

Suppose now that the vector $b = (w_n(z_0)f(z_0), ..., w_n(z_N)f(z_N))^T$ is perturbed slightly; what happens to the vector of coefficients of the corresponding best approximant? The factor of magnification of the corresponding relative errors is given by

$$\frac{||V_n(w_n, E_n)^+ (b + \Delta b) - V_n(w_n, E_n)^+ b||}{||V_n(w_n, E_n)^+ b||} \cdot \left[\frac{||\Delta b||}{||b||}\right]^{-1}$$

for some vectors of weighted data values $b, \Delta b$. Suppose now that b is the vector of weighted data values resulting from some polynomial of degree at most n, i.e., $b = V_n(w_n, E_n) \cdot \vec{P}$ with $\deg P \leq n$, which is perturbed by an arbitrary $\Delta b \in \mathbb{C}^{N+1}$. From (1) we may conclude that the maximal factor of magnification of relative errors is just given by $\kappa(V_n(w_n, E_n))$.

For given bases of polynomials and a given sequence of weights and of abscissas, one observes quite often that the quantity $\kappa(V_n(w_n, E_n))$ grows exponentially in n. In the present paper we provide a lower bound of the nth–root limit in terms of complex potential theory, and describe necessary and sufficient conditions for the data in order to insure subexponential growth. In the case of square

Vandermonde–like matrices, such a study may be based on properties of an underlying weighted Lebesgue function (see, e.g., [Bec96, Appendix B]). Here we will be interested in the more involved case where $\#E_n/n$ tends to some constant larger than one, a rather typical situation for least squares approximation.

We conclude this section by describing our assumptions on the input data (or for some subsequence) which are appropriate for describing the nth–root behavior of $\kappa(V_n(w_n, E_n))$:

(i) E is some compact subset of the complex plane $\mathbb{C}$, and $E_n \subset E$ for all $n \geq 0$. Furthermore, there exists some finite positive Borel measure σ with finite logarithmic energy, $\mathrm{supp}(\sigma) = E$, such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{z\in E_n} f(z) = \int f(z)\, d\sigma(z)$$

for all functions f continuous on E.

(ii) $w_n : E \to (0, \infty)$, and $(w_n^{1/n})_{n\geq 0}$ converges to some positive continuous function w uniformly in E.

(iii) Let $\rho_n(z) := \sqrt{|p_0(z)|^2 + |p_1(z)|^2 + \ldots + |p_n(z)|^2}$, then $(\rho_n^{1/n})_{n\geq 0}$ tends to some function ρ uniformly on compact subsets of $\mathbb{C}$.

Examples of polynomials satisfying assumption (iii) include monomials, Chebyshev polynomials or other suitable sequences of orthonormal polynomials, see also the more detailed discussion in Section 4.

The rest of this paper is organized as follows: We recall in Section 2 some related recent estimates for special structured matrices. Subsequently, we state our main results: on the one hand we obtain in Theorem 2.1 exponentially increasing condition numbers for most of the configurations described in (i)–(iii). In contrast, we give in Theorem 2.2 necessary and sufficient conditions on the data for insuring subexponential growth. The proofs of these properties are given in Section 3. Section 4 is devoted to studying an illustrating example.

2. Statement of our main results and related estimates

The numerical condition of (weighted) Vandermonde(–like) matrices has received much attention in the past 25 years. In a number of papers [Gau75a, Gau75b, Gau90, GaIn88], Gautschi investigated the condition number of square Vandermonde matrices (i.e., $p_j(z) = z^j$, $w_n = 1$) with real abscissas, showing that $\kappa_n(V_n(1, E_n))$ is bounded from below by some function increasing exponentially in n. Further results in this area have been given in [Tay78, Tyr94, Bec96]; it is shown in [Bec97, Theorem 4.1] that for all $n \geq 2$

$$\kappa(V_n(1, E_n)) \geq \sqrt{\frac{2}{n+1}}\ (1+\sqrt{2})^{n-1} \text{ if } E_n \subset \mathbb{R}, \#E_n = n+ \qquad (2)$$

and that this bound may be improved at most by a factor $(n+1)^{3/2}$. Similar results may be stated for nonnegative abscissas [Bec97, Theorem 4.1]. It is important to notice that the choice of real abscissas is not appropriate for the basis of monomials: If E_n is the set of $(N+1)$st roots of unity, then obviously $\kappa(V_n(1,E_n)) = 1$. Similarly, if E_n results from a Van der Corput enumeration of particular roots of unity, then $\kappa(V_n(1,E_n)) \leq \sqrt{2(n+1)}$ [CGR90, Corollary 3].

The condition number of (unweighted) Vandermonde–like matrices has also been investigated for other bases of polynomials, e.g., the basis of Newton polynomials [FiRe89, Rei90], the basis of Faber polynomials of some ellipses [ReOp91], or the basis of a family of orthogonal polynomials [Gau90, ReOp91]. Here, in general, subexponential growth of $(\kappa(V_n(1,E_n)))_{n\geq 0}$ is established; however, each time the choice of the abscissas was motivated by asymptotic properties of the corresponding basis.

One might expect to be able to decrease the condition number by allowing for an additional weight function – such a (in general unknown) weight occurs naturally in the context of Krylov or Gram matrices (see [Bec96, Bec97, BeSt98]). However, for the basis of monomials it is shown in [Bec97, Theorem 3.6] that for all $n \geq 2$

$$\kappa(V_n(w_n,E_n)) \geq \frac{\gamma^n}{4\sqrt{n+1}}, \; \gamma := \exp\left(\frac{2\text{Catalan}}{\pi}\right) \approx .792 \text{ if } E_n \subset \mathbb{R}, \qquad (3)$$

and that this bound may be improved at most by a factor $\gamma \cdot (8n+8)^{1/2}$. Similar results are given in [Bec97, Corollary 3.2, Remark 3.4 and Remark 3.5] for the case of abscissas located in some real interval.

The numerical condition of weighted Vandermonde–like matrices for arbitrary "admissible" bases $(p_j)_{j\geq 0}$ (see assumption (iii) above) is discussed in [BeSt98, Theorem 1.2 and Theorem 1.3]. In order to describe their findings and the findings of the present paper, we will need some facts from complex potential theory. Here we follow [SaTo97]; however, for ease of presentation, we will impose some quite strong regularity assumptions which simplify some of the characterization statements.

For an arbitrary finite Borel measure μ with compact support supp(μ), the logarithmic potential of μ is defined by

$$U^\mu(z) = \int \log \frac{1}{|z-t|}\, d\mu(t).$$

Let E and w satisfy assumptions (i),(ii), and suppose in addition that E is regular, i.e., the connected components of $\mathbb{C} \setminus E$ are regular with respect to the Dirichlet problem. We denote by $\mathcal{M}(E)$ the set of all positive unit Borel measures, and define for $\mu \in \mathcal{M}(E)$ the weighted energy integral

$$I_w(\mu) := \int\int \log \frac{1}{|z-t| w(z) w(t)}\, d\mu(t)\, d\mu(z).$$

Then there exists a unique *extremal measure* $\lambda_w \in \mathcal{M}(E)$ with $I_w(\lambda_w) = \inf\{I_w(\mu)$ $\mu \in \mathcal{M}(E)\}$ (see [SaTo97, Theorem I.1.3]). According to our regularity assumptions, we know from [SaTo97, Theorem I.4.8 and Theorem I.5.1] that the potential U^{λ_w} is continuous in $\mathbb{C}$. Moreover, by [SaTo97, Theorem I.1.3] there exists a constant $F =: F_w$ such that for $\mu = \lambda_w$

$$U^\mu(z) - \log w(z) \begin{cases} \geq F & \text{for all } z \in E, \\ = F & \text{for all } z \in \operatorname{supp}(\mu) \end{cases} \tag{4}$$

In addition [SaTo97, Theorem I.3.1], if $\mu \in \mathcal{M}(E)$ satisfies (4), then necessarily (μ, F) coincides with (λ_w, F_w). We refer the reader to [SaTo97] for various applications of the weighted energy problem. For regular compact sets E and for admissible bases, it is shown in [BeSt98, Theorem 1.2 and Theorem 1.3] that

$$\begin{aligned} &\liminf_{n\to\infty} \log \inf\{\kappa(V_n(w_n, E_n)) : E_n \subset E,\ w_n \text{ positive on } E_n\} \\ &\geq \sup_{z\in\operatorname{supp}(\lambda_{1/\rho})} \left[U^{\lambda_{1/\rho}}(z) + \log\rho(z)\right] - \inf_{z\in\mathbb{C}}\left[U^{\lambda_{1/\rho}}(z) + \log\rho(z)\right] \end{aligned} \tag{5}$$

with ρ as in (iii). Also, various bases are discussed where actually equality holds (e.g., the basis of monomials, compare with (3)). Notice that the term on the right–hand side is necessarily nonnegative by (4), and that only a careful choice of a basis in terms of E (as well as of w_n and E_n) will enable us to obtain subexponential growth of $(\kappa(V_n(w_n, E_n))_{n\geq 0}$.

In the present paper we derive similar estimates for the case where the basis as well as a configuration of abscissas (and possibly the weights) are given. We will show that the constrained weighted energy problem plays an important role [Rak96, DrSa97]: here one tries to minimize $I_w(\mu)$ with respect to all $\mu \in \mathcal{M}(E)$ satisfying the additional constraint $\mu \leq \sigma$. Recently, this energy problem has been introduced by Rakhmanov and further studied by several other authors [DaSa98, KuRa98, KuVA98] for describing the asymptotic behavior of so–called ray sequences of orthonormal polynomials with respect to some discrete measure (such as discrete Chebyshev or Krawtchouk polynomials). Let E, w, and σ be as in assumptions (i),(ii), with $\sigma(\mathbb{C}) > 1$ (the case $\sigma(\mathbb{C}) = 1$ is also allowed but trivial). We denote by $\mathcal{M}^\sigma$ the set of all positive unit Borel measures satisfying the additional constraint $\mu \leq \sigma$, that is, $\sigma - \mu$ is a positive Borel measure. Then there exists again a unique *constrained extremal measure* $\lambda_w^\sigma \in \mathcal{M}^\sigma$ with $I_w(\lambda_w^\sigma) = \inf\{I_w(\mu) : \mu \in \mathcal{M}^\sigma\}$ (see [DrSa97, Theorem 2.1]).

We shall prove the following:

Theorem 2.1. *Let $E_n, E, \sigma, p_n, \rho$ be as in* (i),(iii). *Then*

$$\begin{aligned} &\liminf_{n\to\infty} \log \kappa(V_n(w_n, E_n))^{1/n} \geq \\ &\sup_{z\in\operatorname{supp}(\lambda)} \left[U^\lambda(z) + \log\rho(z)\right] - \inf_{z\in\mathbb{C}}\left[U^\lambda (z) + \log\rho(z)\right] \end{aligned} \tag{6}$$

with $\lambda = \lambda_1^\sigma$ *If in addition* w_n, w *are as in* (ii), *then*

$$\liminf_{n\to\infty} \log \kappa(V_n(w_n, E_n))^{1/n} \geq \sup_{z\in \mathrm{supp}(\lambda)} \left[U^\lambda(z) - \log w(z)\right] \qquad (7)$$
$$- \inf_{z\in\mathbb{C}}\left[U^\lambda(z) + \log\rho(z)\right] + \max_{z\in E}\left[\log w(z) + \log\rho(z)\right]$$

with $\lambda = \lambda_w^\sigma$.

Conversely, provided that the additional assumptions (iv),(v) *described below hold, and* $\mathrm{supp}(\lambda_w^\sigma) \cap \mathrm{supp}(\sigma - \lambda_w^\sigma)$ *is nonempty, the limit of the sequence on the left-hand side of* (7) *exists and coincides with the right-hand side of* (7).

The proof of Theorem 2.1 is based on several observations: first we notice that for a polynomial $P = a_0p_0 + \cdots + a_np_n$ we have $\vec{P} = (a_0, ..., a_n)^T$, and from the Cauchy–Schwarz inequality we get $|P(z)| \leq ||\vec{P}|| \cdot \rho_n(z)$ for all $z \in \mathbb{C}$. This inequality enables us to relate the norm of the pseudoinverse of $V_n(w_n, E_n)$ to some polynomial extremal problem as described in Lemma 3.1 below. However, we may only expect to have equality in (7) provided that

(iv) $\qquad \lim_{n\to\infty} \epsilon_n(H)^{1/n} = 1$ for some $H \subset \mathbb{C}$ being compact,

$$\epsilon_n(H) := \max_{\deg P \leq n} \frac{||\vec{P}||}{||P/\rho_n||_H} \geq 1,$$

where $||\cdot||_H$ denotes the usual supremum norm on H.

A second key observation is that $||V_n(w_n, E_n)^+||$ approximately equals the norm of the inverse of the square submatrix of order $n+1$ which has maximal determinant. This submatrix is given by $V_n(w_n, E_n^*)$, where E_n^* is the set of the $n+1$ weighted *Fekete points* out of E_n. Asymptotic properties of weighted Fekete points formed from discrete sets have been described in [Bec98]. Here one necessarily requires an additional separation property for the sets E_n: we denote the *scaled counting measure* of some finite $A \subset \mathbb{C}$ by

$$\nu_n(A) := \frac{1}{n}\sum_{a\in A} \delta_a,$$

where δ_a denotes the usual Dirac measure at a. Notice that $(\nu_n(E_n))_{n\geq 0}$ has the weak* limit σ by assumption (i). Here, following [DrSa97], we will have to add the property

(v) $\qquad$ for any sequence $(\zeta_n)_{n\geq 0}$, $\zeta_n \in E_n$, with limit ζ, there holds

$$\lim_{n\to\infty} U^{\nu_n(E_n\setminus\{\zeta_n\})}(\zeta_n) = U^\sigma(\zeta) < \infty.$$

Sequences of sets E_n as described in (i) satisfying condition (v) are described in [DrSa97, Lemma 3.2]; examples are equidistant nodes, or sets of zeros of polynomials orthogonal with respect to some measure in the class **Reg** on an interval.[1] One

[1] Abscissas satisfying both (i),(v) are for instance given by $E_n = \{f(j/N_n) : j = 0, \cdots, N_n := \#E_n - 1\}$, where $f : [0,1] \to \mathbb{C}$ is continuous and injective, and $f^{-1} \in \mathrm{Lip}_\alpha$ for some $0 < \alpha \leq 1$.

may easily construct[2] sets E_n satisfying (i) even with continuous U^σ, where (v) is violated (and the last part of Theorem 2.1 does not remain valid). We should however mention that we may relax (v) by allowing for an exceptional set of capacity zero (see [DaSa98, KuVA98]). Also, following [Bec98], it is possible to replace condition (v) by the regularity assumption

(v') U^σ is continuous, and

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{x,y\in E_n, x\neq y} \log\frac{1}{|x-y|} = \int\int \log\frac{1}{|x-y|}\, d\sigma(x)\, d\sigma(y),$$

as conjectured by Rakhmanov [KuRa98] to be sufficient for establishing the asymptotic behavior of related discrete orthogonal polynomials. Note that condition (v') has an interpretation in terms of the asymptotic behavior of the determinants of square Vandermonde matrices of order $\#E_n$.

Theorem 2.1 is of some theoretical interest for detecting configurations of data where the condition number grows exponentially in n. Of more practical interest, however, is the case where we observe subexponential growth. Here we have the following result.

Theorem 2.2. *Let $E_n, E, \sigma, w_n, n, p_n, \rho$ be as in* (i),(ii),(iii),(v), *with* $\mathrm{supp}(\lambda_w^\sigma) \cap \mathrm{supp}(\sigma - \lambda_w^\sigma)$ *being nonempty. We have subexponential growth of* $(\kappa(V_n(w_n, E_n)))_{n\geq 0}$ *if and only if*

(a) $\log[w(z)\cdot\rho(z)] = \max_{t\in E}\log[w(t)\cdot\rho(t)] =: F'$ *for all* $z \in \mathrm{supp}(\lambda_w^\sigma)$;

(b) $U^{\lambda_w^\sigma} + \log\rho$ *equals some constant* F^* *in* $\mathbb{C}$;

(c) *condition* (iv) *holds.*

In this case, we have the implications

(d) $\lambda_w^\sigma = \lambda_w$ *and* $F^* - F' = F_w$ *(and thus* $\lambda_w \leq \sigma$*)*;

(e) *For the leading coefficient* b_j *of* p_j, *there holds* $\lim_{n\to\infty} |b_n|^{1/n} = \exp(F^*)$;

(f) $\lim_{n\to\infty}\left[\max_{z\in E}|w(z)^n \cdot P_n(z)|\right]^{1/n} = \exp(-F_w), \quad P_n := p_n/b_n.$

For numerical reasons, in general one also wants that $||V_n(w_n, E_n)||^{1/n} \to 1$, and hence the constant F' in Theorem 2.2(a) should be equal to 0.

Following [SaTo97, Chapter III.4], we may conclude from Theorem 2.2(f) that P_n are asymptotically extremal monic polynomials. For such polynomials, many results about zero distributions in terms of $\lambda_w^\sigma = \lambda_w$ are known.

To conclude this section, we mention that many of the above results remain valid in the case of an unbounded set $E = \mathrm{supp}(\sigma)$. Here one requires a particular decay rate of w at infinity, and a suitable reformulation of assumptions (i),(ii),(v') to insure that the corresponding weighted Fekete points remain uniformly bounded. For further details we refer the reader to [KuVA98, KuRa98, DaSa98, Bec98].

[2]Take, e.g., $E_n = \{j/n, j/n - \delta_n : j = 1, 2, \ldots, n\}$, where δ_n tends rapidly to zero.

3. Proofs

In all of this section we will assume that $E_n, E, \sigma, p_n, \rho$ are as in (i),(iii), and $w_n : E \to (0, \infty)$. Further assumptions will be mentioned explicitly. A basic observation in our proofs of Theorems 2.1 and 2.2 is that condition numbers of weighted Vandermonde–like matrices are closely related to some weighted extremal problems for polynomials, as shown in the following result.

Lemma 3.1. *Let $H \subset \mathbb{C}$ be compact, and define*

$$\delta_n(w_n, E_n, H) := \max_{\deg P \leq n} \frac{||P/\rho_n||_H}{||w_n P||_{E_n}}.$$

Then, with $N +$ $:= \# E_n$ and $\epsilon_n(H)$ as defined in (iv), *there holds*

$$\kappa(V_n(w_n, E_n))\sqrt{N+1} \geq ||w_n \rho_n||_{E_n} \delta_n(w_n, E_n, H) \geq \frac{\kappa(V_n(w_n, E_n))}{\epsilon_n(H)\sqrt{N+1}}, \quad (8)$$

$$||w_n \rho_n||_{E_n} \delta_n(w_n, E_n, H) \geq \delta_n(1/\rho_n, E_n, H). \quad (9)$$

Furthermore, if we denote by $E_n^* \subset E_n$ a set of weighted Fekete nodes, i.e., a set where the maximum is attained in $\max\{|\det V_n(w_n, E_n')| : E_n' \subset E_n, \# E_n' = n+1\}$, *then*

$$(n+1) \cdot \delta_n(w_n, E_n, H) \geq \delta_n(w_n, E_n^*, H) \geq \delta_n(w_n, E_n, H). \quad (10)$$

Proof: Writing $E_n = \{z_0, ..., z_N\}$, we first notice that $w_n(z_j) \cdot \rho_n(z_j)$ equals the Euclidean norm of the jth row of $V_n(w_n, E_n)$. Consequently,

$$||w_n \rho_n||_{E_n} \leq ||V_n(w_n, E_n)|| \leq \Big[\sum_{j=0}^{N} w_n(z_j)^2 \rho_n(z_j)^2\Big]^{1/2} \leq \sqrt{N+1} \cdot ||w_n \rho_n||_{E_n}. \quad (11)$$

Also, for any polynomial P of degree at most n and for any $z \in \mathbb{C}$ there holds

$$|P(z)| \leq ||\vec{P}|| \cdot \rho_n(z), \quad ||w_n P||_{E_n} \leq ||V_n(w_n, E_n)\vec{P}|| \leq \sqrt{N+1} \cdot ||w_n P||_{E_n}.$$

Thus we get, using (1) and (11), that

$$\kappa(V_n(w_n, E_n)) \geq ||w_n \rho_n||_{E_n} \cdot \frac{||\vec{P}||}{||V_n(w_n, E_n)\vec{P}||} \geq \frac{||w_n \rho_n||_{E_n}}{\sqrt{N+1}} \cdot \frac{|P(z)/\rho_n(z)|}{||w_n P||_{E_n}},$$

as required for the first part of (8). The other part follows by observing that

$$\epsilon_n(H) \cdot \delta_n(w_n, E_n, H) \geq \max_{\deg P \leq n} \frac{||\vec{P}||}{||w_n P||_{E_n}} \geq \frac{\kappa(V_n(w_n, E_n))}{\sqrt{N+1} \cdot ||w_n \rho_n||_{E_n}},$$

where for the second step we have applied (1) and (11). The inequality (9) is trivial. It remains to show (10). Here the inequality $\delta_n(w_n, E_n^*, H) \geq \delta_n(w_n, E_n, H)$ is a trivial consequence of the fact that $E_n^* \subset E_n$. In order to obtain the other inequality of (10), let $E^* := \{x_0, ..., x_n\}$, and consider the corresponding Lagrange polynomials

$$\ell_j(z) = \prod_{k \neq j} \frac{z - x_k}{x_j - x_k} \qquad j = 0, \quad n.$$

By the triangle inequality

$$\max_{\deg P \leq n} \frac{||w_n P||_{E_n}}{||w_n P||_{E_n^*}} = \max_{z \in E_n} \sum_{j=0}^{n} \frac{w_n(z)}{w_n(x_j)} \cdot |\ell_j(z)|,$$

and it is sufficient to show that the right-hand side is $\leq n+1$. This, however, is an immediate consequence of the construction of E_n^* since for any $z \in E_n$ and for any $0 \leq j \leq n$ there holds

$$\left|\frac{w_n(z)}{w_n(x_j)} \ell_j(z)\right| = \frac{|\det V_n(w_n, \{x_0, ..., x_{j-1}, z, x_{j+1}, ..., x_n\})|}{|\det V_n(w_n, \{x_0, ..., x_{j-1}, x_j, x_{j+1}, ..., x_n\})|} \leq 1 \tag{12}$$

□

We see from Lemma 3.1 that – at least for bases satisfying (iv) — the asymptotic behavior of $(\kappa(V_n(w_n, E_n))^{1/n})_{n\in\Lambda}$ is completely determined by that of $(\delta_n(w_n, E_n, H)^{1/n})_{n\in\Lambda}$ (or of $(\delta_n(w_n, E_n^*, H)^{1/n})_{n\in\Lambda}$) since the asymptotic behavior of $(||w_n\rho_n||_{E_n}^{1/n})_{n\in\Lambda}$ is known according to assumptions (i)–(iii). Also, though the determination of a weight w_n minimizing $\kappa(V_n(w_n, E_n))$ in general is a nontrivial task (see, e.g., [Bau63]), the simpler expression $||w_n\rho_n||_{E_n} \cdot \delta_n(w_n, E_n, H)$ is clearly minimized for the choice $w_n = 1/\rho_n$ by (9). Finally, the occurrence of weighted Fekete points is quite natural since, as in the proof of Lemma 3.1, one shows that

$$||V_n(w_n, E_n)^+|| \leq ||V_n(w_n, E_n^*)^{-1}|| \leq (N+1) \cdot ||V_n(w_n, E_n)^+||.$$

In other words, the sensitivity of polynomial least squares approximation is closely related to the sensitivity of polynomial interpolation at a suitable subset of abscissas.

Taking into account Lemma 3.1, it remains to discuss the asymptotic behavior of $(\delta_n(w_n, E_n^*, H)^{1/n})_{n\geq 0}$. Here the constrained energy problem with external field plays an important role.

Lemma 3.2. *Let $R > 0$ and $\Delta_R := \{z \in \mathbb{C} : |z| \leq R\}$. Furthermore, suppose that assumption* (ii) *holds, and denote by $\lambda = \lambda_w^\sigma$ the extremal measure of the constrained weighted energy problem as introduced before Theorem 2.1. Then*

$$\liminf_{n\to\infty} \log \delta_n(w_n, E_n^*, \Delta_R)^{1/n} \geq \sup_{z\in \mathrm{supp}(\lambda)} \left[U^\lambda(z) - \log w(z)\right] - \min_{z\in\Delta_R} \left[U^\lambda(z) + \log \rho(z)\right].$$

Proof: We write more explicitly $E_n^* = \{x_{0,n}, ..., x_{n,n}\}$ for the set of weighted Fekete points as introduced in Lemma 3.1, denote by $\ell_{0,n}, .., \ell_{n,n}$ the corresponding Lagrange polynomials, and consider the measures $\mu_{j,n} := \nu_n(E_n^* \setminus \{x_{j,n}\})$, $\mu_n := \nu_n(E_n^*)$, $0 \leq j \leq n$. First, as in the proof of Lemma 3.1, one shows that

$$\delta_n(w_n, E_n^*, \Delta_R) = \max_{z\in\Delta_R} \sum_{j=0}^{n} \frac{|\ell_{j,n}(z)|/\rho_n(z)}{w_n(z_j)}.$$

Taking into account that $\log|\ell_{j,n}(z)|^{1/n} = U^{\mu_{j,n}}(x_{j,n}) - U^{\mu_{j,n}}(z)$ and writing $a_{j,n} := U^{\mu_{j,n}}(x_{j,n}) - \log w_n(x_{j,n})^{1/n}$, $0 \le j \le n$, $a_n := \max_{0\le j\le n} a_{j,n}$, we obtain

$$
\begin{aligned}
&\frac{1}{n}\log\delta_n(w_n, E_n^*, \Delta_R)\\
&= \frac{1}{n}\cdot\log\max_{z\in\Delta_R}\sum_{j=0}^{n}\exp\Big(n\cdot\big(a_{j,n} - U^{\mu_{j,n}}(z) - \log\rho_n(z)^{1/n}\big)\Big)\\
&= \frac{\log\eta_n}{n} + \max_{0\le j\le n}\max_{z\in\Delta_R}\big(a_{j,n} - U^{\mu_{j,n}}(z) - \log\rho_n(z)^{1/n}\big)
\end{aligned}
\tag{13}
$$

for some $\eta_n \in [1,(n+1)]$.

Since $\mathrm{supp}(\mu_n) \subset E$, Helly's theorem asserts that, given some infinite set Λ_0 of integers, we may find $\Lambda_1 \subset \Lambda_0$ such that $(\mu_n)_{n\in\Lambda_1}$ converges weak* to a probability measure μ. One easily verifies, using assumption (i), that $\mu \in \mathcal{M}^\sigma$. Moreover, for any $0 \le j_n \le n$, the sequence $(\mu_{j_n,n})_{n\in\Lambda_1}$ also has the weak* limit μ. By construction, for any $z \in \mathrm{supp}(\mu)$ we may find a sequence $(x_{j_n,n})_{n\in\Lambda_1}$ tending to z, and therefore

$$U^\mu(z) - \log w(z) \le \liminf_{n\to\infty, n\in\Lambda_1} a_{j_n,n} \le \liminf_{n\to\infty, n\in\Lambda_1} a_n \tag{14}$$

for all $z \in \mathrm{supp}(\mu)$ by the principle of descent (see, e.g., [SaTo97, Theorem I.6.8]) and assumption (ii). Also, the closed disk Δ_R has the K-property, and thus

$$\lim_{n\to\infty, n\in\Lambda_1}\min_{z\in\Delta_R}\big[U^{\mu_{j_n,n}}(z) + \log\rho_n(z)^{1/n}\big] = \min_{z\in\Delta_R}\big[U^\mu(z) + \log\rho(z)\big] \tag{15}$$

for any $0 \le j_n \le n$ according to [NiSo88, Theorem V.4.3, p.182] and assumption (iii). Combining (13),(14), and (15), we obtain

$$
\begin{aligned}
&\liminf_{n\to\infty, n\in\Lambda_1}\frac{1}{n}\log\delta_n(w_n, E_n^*, \Delta_R)\\
&\ge \sup_{z\in\mathrm{supp}(\mu)}\big[U^\mu(z) - \log w(z)\big] - \min_{z\in\Delta_R}\big[U^\mu(z) + \log\rho(z)\big].
\end{aligned}
$$

Since the choice of Λ_0 was arbitrary, the assertion of Lemma 3.2 now follows by showing that for all $z \in \mathbb{C}$ we have

$$U^\mu(z) \le \sup_{t\in\mathrm{supp}(\mu)}\big[U^\mu(t) - \log w(t)\big] + U^\lambda(z) - \sup_{t\in\mathrm{supp}(\lambda)}\big[U^\lambda(t) - \log w(t)\big]. \tag{16}$$

In fact, from [DrSa97, Theorem 2.1.(c)] we know that there exists a set $K \subset E$ with $(\sigma - \lambda)(K) = 0$ such that

$$U^\lambda(z) - \log w(z) \ge \sup_{t\in\mathrm{supp}(\lambda)}\big[U^\lambda(t) - \log w(t)\big], \quad z \in \mathrm{supp}(\sigma-\lambda)\setminus K \tag{17}$$

Denote by μ' the restriction of μ to $\mathrm{supp}(\sigma-\lambda)$. If $\mu' = 0$ then necessarily $\lambda = \mu$ (cf. [DrSa97, Lemma 5.1]), and thus (16) trivially holds. Also, (16) trivially is true if $\sup\{[U^\mu(t) - \log w(t)] : t \in \mathrm{supp}(\mu)\} = +\infty$, the latter being equivalent to $\sup\{U^\mu(t) : t \in \mathbb{C}\} = +\infty$ by the maximum principle for potentials [SaTo97, Corollary II.3.3]. It therefore remains to discuss the case where $\mu \neq 0$ has a finite potential, and thus $\mu' \neq 0$ has both a finite potential and finite logarithmic energy. Then one verifies using Fubini's Theorem that also $\mu'(K) = 0$. Furthermore, from (17) we obtain for $z \in \text{supp}(\mu') \setminus K \subset \text{supp}(\mu)$

$$\begin{aligned} U^{\mu'}(z) &\leq \sup_{t \in \text{supp}(\mu)} \left[U^\mu(t) - \log w(t)\right] + \log w(z) - U^{\mu-\mu'}(z) \\ &\leq \sup_{t \in \text{supp}(\mu)} \left[U^\mu(t) - \log w(t)\right] - \sup_{t \in \text{supp}(\lambda)} \left[U^\lambda(t) - \log w(t)\right] \\ &\quad + U^\lambda(z) - U^{\mu-\mu'}(z). \end{aligned} \tag{18}$$

Also, with $S := E \setminus \text{supp}(\sigma - \lambda)$ we have by construction $\mu - \mu' = \mu|_S \leq \sigma|_S = \lambda|_S$, showing that $\lambda - \mu + \mu'$ is a nonnegative finite Borel measure with compact support, with its total mass not exceeding that of μ'. From the principle of domination [SaTo97, Theorem II.3.2] we may conclude that (18) holds for all $z \in \mathbb{C}$, as claimed in (16). □

Observe that, for $z \to \infty$, inequality (16) provides a new characterization of the extremal measure λ_w^σ in the case of compact $\text{supp}(\sigma)$ and continuous w which is complementary to [DrSa97, Theorem 2.1.(e)]. Here the uniqueness result follows from the unicity theorem [SaTo97, Theorem 2.1] and the maximum principle for subharmonic functions.

Corollary 3.3. *We have for $\lambda = \lambda_w^\sigma$*

$$\sup_{t \in \text{supp}(\lambda)} \left[U^\lambda(t) - \log w(t)\right] = \min_{\mu \in \mathcal{M}^\sigma} \sup_{t \in \text{supp}(\mu)} \left[U^\mu(t) - \log w(t)\right]. \tag{19}$$

If, in addition, the polynomial convex hull of $\text{supp}(\lambda_w^\sigma)$ *is of two-dimensional Lebesgue measure zero, then any measure* $\lambda \in \mathcal{M}^\sigma$ *satisfying* (19) *necessarily coincides with* λ_w^σ.

For the second part of Theorem 2.1 we need a sharper version of Lemma 3.2 which is attainable if we add some separation property such as assumption (v) or (v'). In fact, it follows from [Bec98, Theorem 1.5(a),(c)] that equality holds in Lemma 3.2 provided that $E = \text{supp}(\sigma)$ is connected and (v') holds, and that this result may even be generalized for measures σ with unbounded support. Here we will restrict ourselves to the simpler condition (v)

Lemma 3.4. *Under the assumptions of Lemma 3.2, suppose in addition that* (v) *holds, and that* $\text{supp}(\lambda) \cap \text{supp}(\sigma - \lambda)$ *is nonempty, where* $\lambda = \lambda_w^\sigma$. *Then*[3]

$$\lim_{n\to\infty} \log \delta_n(w_n, E_n^*, \Delta_R)^{1/n} = \sup_{z \in \text{supp}(\lambda)} \left[U^\lambda(z) - \log w(z)\right] - \min_{z \in \Delta_R} \left[U^\lambda(z) + \log \rho(z)\right].$$

[3]The set Δ_R in Lemma 3.2 and Lemma 3.4 may be replaced, e.g., by any compact set having an empty intersection with E.

Proof: Let $\mu_n, \mu_{j,n}, a_n, a_{j,n}, \Lambda_0, \Lambda_1, \mu$ be as in the proof of Lemma 3.2. We choose $k_n \in \{0, ..., n\}$, $\zeta \in \text{supp}(\sigma - \mu)$, and $\Lambda_2 \subset \Lambda_1$ with

$$\limsup_{n\to\infty, n\in\Lambda_1} a_n = \lim_{n\to\infty, n\in\Lambda_2} a_{k_n,n},$$
$$\text{and} \quad \min_{t\in\text{supp}(\sigma-\mu)} [U^\mu(t) - \log w(t)] = U^\mu(\zeta) - \log w(\zeta).$$

By assumption (i), we may find $\zeta_n \in E_n \setminus E_n^*$ with $(\zeta_n)_{n\in\Lambda_2}$ tending to ζ. Set $\sigma_n' := \mu_{k_n,n}$, $\sigma_n'' := \nu_n(E_n \setminus \{\zeta_n\}) - \sigma_n'$. Then the sequence $(\sigma_n'')_{n\in\Lambda_1}$ has the weak* limit $\sigma - \mu \geq 0$. Applying twice the principle of descent and assumption (v), we obtain

$$\begin{aligned} U^\mu(\zeta) &\leq \liminf_{n\to\infty, n\in\Lambda_2} U^{\sigma_n'}(\zeta_n) \\ &\leq \limsup_{n\to\infty, n\in\Lambda_2} U^{\sigma_n'}(\zeta_n) = U^\sigma(\zeta) - \liminf_{n\to\infty, n\in\Lambda_2} U^{\sigma_n''}(\zeta_n) \leq U^\mu(\zeta) \end{aligned}$$

With (12) and assumption (ii) taken into account, it follows that

$$U^\mu(\zeta) - \log w(\zeta) = \lim_{n\to\infty, n\in\Lambda_2} [U^{\sigma_n'}(\zeta_n) - \log w_n(\zeta_n)^{1/n}] \geq \lim_{n\to\infty, n\in\Lambda_2} a_{k_n,n},$$

and a combination with (14) leads to

$$\begin{aligned} \min_{t\in\text{supp}(\sigma-\mu)} [U^\mu(t) - \log w(t)] &\geq \limsup_{n\to\infty, n\in\Lambda_1} a_n \geq \liminf_{n\to\infty, n\in\Lambda_1} a_n \\ &\geq \max_{t\in\text{supp}(\mu)} [U^\mu(t) - \log w(t)]. \end{aligned} \tag{20}$$

In particular, the equilibrium condition (17) holds for the measure $\mu \in \mathcal{M}^\sigma$ with K being empty. From the uniqueness result [DrSa97, Theorem 2.1(d)] we may conclude that $\mu = \lambda$. Recalling that the set Λ_0 was arbitrary, we may conclude that the sequence of normalized counting measures of Fekete points $(\mu_n)_{n\geq 0}$ has the weak* limit λ (see also [Bec98, Theorem 1.5(a)]). Also, since $\text{supp}(\lambda) \cap \text{supp}(\sigma - \lambda)$ is nonempty, we obtain from (20) the convergence[4] of $(a_n)_{n\geq 0}$, with limit described in (20). Finally, the assertion of Lemma 3.4 now follows from (13) together with (15). □

We are now prepared to establish our main theorems.

Proof of Theorem 2.1: In order to establish (6), recall from (8), (9), and (10) that

$$\kappa(V_n(w_n, E_n)) \geq \frac{1}{\sqrt{\# E_n}} \cdot \delta_n(1/\rho_n, E_n, \Delta_R) \geq \frac{1}{(n+1) \cdot \sqrt{\# E_n}} \cdot \delta_n(1/\rho_n, E_n^{**}, \Delta_R)$$

for every $R > 0$, where E_n^{**} is an $(n+1)$-point Fekete set for the weight $1/\rho_n$. Thus, it just remains to apply Lemma 3.2 with $w_n = 1/\rho_n$. Similarly, for a proof

[4]Using [DrSa97, Example 2.4] one may construct examples where $\mu = \lambda$, but $(a_n)_{n\geq 0}$ does not necessarily converge.

of (7) we apply (8), (10), and Lemma 3.2, and observe that

$$\lim_{n\to\infty} \frac{1}{n} \log ||w_n\rho_n||_{E_n} = \max_{z\in E}[\log w(z) + \log \rho(z)]$$

by assumptions (i), (ii), and (iii).

Now let (iv) and (v) hold, and let $\mathrm{supp}(\lambda_w^\sigma) \cap \mathrm{supp}(\sigma - \lambda_w^\sigma)$ be nonempty. We choose a sufficiently large $R > 0$ such that the set H of assumption (iv) is contained in the disk Δ_R. Since $\epsilon_n(H) \geq \epsilon_n(\Delta_R) \geq 1$ by construction, we may conclude that (iv) is also true for $H = \Delta_R$. Thus we obtain from (8) and (10)

$$\begin{aligned}&\limsup_{n\to\infty} \log \kappa(V_n(w_n, E_n))^{1/n} \\ &\leq \max_{z\in E}[\log w(z) + \log \rho(z)] + \limsup_{n\to\infty} \log \delta_n(w_n, E_n^*, H)^{1/n},\end{aligned}$$

with the right-hand side being computed in Lemma 3.4. Letting $R \to \infty$ and combining with (7) yields the final claim of Theorem 2.1. □

Proof of Theorem 2.2: In the first part of the proof we assume that there is subexponential growth, and write $\lambda = \lambda_w^\sigma$. First, recalling the inequalities of Lemma 3.1, we have

$$\begin{aligned} 1 \leq \epsilon_n(E) \leq \epsilon_n(E_n) &= \max_{\deg P\leq n} \frac{||\vec{P}|| \cdot ||w_n\rho_n||_{E_n}}{||P/\rho_n||_{E_n} \cdot ||w_n\rho_n||_{E_n}} \\ &\leq \max_{\deg P\leq n} \frac{||\vec{P}|| \cdot ||w_n\rho_n||_{E_n}}{||w_nP||_{E_n}} \leq \sqrt{\#E_n} \cdot \kappa(V_n(w_n, E_n)),\end{aligned}$$

and thus (iv) holds with $H = E$, as claimed in part (c). Moreover, because of the subexponential growth, the right-hand side of (6) has to be ≤ 0. In particular, the function $f(z) := U^\lambda(z) + \log \rho(z)$ has to be equal to some constant F^* on $\mathrm{supp}(\lambda)$, and $f(z) \geq F^*$ for $z \in \mathbb{C} \setminus \mathrm{supp}(\lambda)$. One verifies (see, e.g., [BeSt98, Lemma 2.1]), using assumption (iii), that $\log \rho$ is continuous and subharmonic in $\mathbb{C}$, and $\log \rho(z) - \log|z|$ is bounded above around infinity. It follows from the principle of continuity [SaTo97, Theorem II.3.5] that U^λ is continuous in $\mathbb{C}$. Consequently, f is subharmonic in $\mathbb{C}\setminus\mathrm{supp}(\lambda)$, continuous in $\mathbb{C}$, bounded above by F^* on $\mathrm{supp}(\lambda)$, and bounded above around infinity. From the maximum principle for subharmonic functions it follows that $f(z) \leq F^*$, and thus $f(z) = F^*$ for all $z \in \mathbb{C}$, which yields property (b). In addition, since the right-hand side of (7) has to be ≤ 0, we get

$$\sup_{z\in\mathrm{supp}(\lambda)} [U^\lambda(z) - \log w(z)] + \max_{z\in E}[\log w(z) - U^\lambda(z)] \leq 0,$$

that is, $U^\lambda(z) - \log w(z) - F^* = -\log w(z) - \log \rho(z)$ is equal to some constant $-F'$ in $\mathrm{supp}(\lambda)$, and $\geq -F'$ in $E \setminus \mathrm{supp}(\lambda)$, as claimed in part (a).

Conversely, if (a),(b) and (c) hold, then subexponential growth follows from the second part of Theorem 2.1. Part (d) now is an immediate consequence of (a),(b), and [SaTo97, Theorem I.3.1] (see equation (4)), with $F^* - F' = F_w$. In order to show part (e), define $\tilde{\rho}_n(z) := \max_{0\leq j\leq n} |p_j(z)|$. Then $\log \tilde{\rho}_n^{1/n}$ tends to $\log \rho$

uniformly on compact subsets of $\mathbb{C}$ by assumption (iii). Furthermore, $\log \tilde{\rho}_n^{1/n} + U^\lambda$ is subharmonic in $(\mathbb{C} \cup \{\infty\}) \cap \{|z| \geq R\}$, for R large enough. Consequently, we get from the maximum principle for subharmonic functions

$$\log |b_n|^{1/n} = \lim_{|z|\to\infty} \left[\log \tilde{\rho}_n(z)^{1/n} + U^\lambda(z)\right] \leq \max_{|z|=R} \left[\log \tilde{\rho}_n(z)^{1/n} + U^\lambda(z)\right],$$

with the right-hand side tending to F^* for $n \to \infty$. Thus, $\limsup_{n\to\infty} \log |b_n|^{1/n} \leq F^*$. On the other hand, we may conclude from assumptions (i)–(iii) and (a) that $\log ||V_n(w_n, E_n)||^{1/n}$ tends to F' (see (11)), and thus $\log ||V_n(w_n, E_n)^+||^{1/n}$ tends to $-F'$. Furthermore, for any monic polynomial P of degree n there holds $|b_n| \cdot ||\vec{P}|| \geq 1$, and by [SaTo97, Theorem III.3.1]

$$\lim_{n\to\infty} \left[\min_{P(z)=z^n+\dots} ||w^n P||_E\right]^{1/n} = \exp(-F_w). \tag{21}$$

Therefore, using again assumption (ii), we obtain

$$\begin{aligned} F_w &\leq \liminf_{n\to\infty} \frac{1}{n} \log \max_{\deg P=n} \frac{|b_n| \cdot ||\vec{P}||}{||w^n P||_E} \leq \liminf_{n\to\infty} \frac{1}{n} \log \max_{\deg P=n} \frac{|b_n| \cdot ||\vec{P}||}{||w_n P||_{E_n}} \\ &\leq \liminf_{n\to\infty} \left[\log |b_n|^{1/n} + \log ||V_n(w_n, E_n)^+||^{1/n}\right] = \liminf_{n\to\infty} \log |b_n|^{1/n} - F', \end{aligned}$$

showing that $\liminf_{n\to\infty} \log |b_n|^{1/n} \geq F_w + F' = F^*$, as required for part (e).

In order to establish part (f), notice first that $|P_n(z)| \leq \rho_n(z)/|b_n|$, and thus by (21), assumption (ii), part (a), and part (e),

$$e^{-F_w} \leq \liminf_{n\to\infty} ||w^n P_n||_E^{1/n} \leq \liminf_{n\to\infty} |b_n|^{-1/n} \cdot ||w_n \rho_n||_E^{1/n} = e^{-F^*+F'} = e^{-F_w}.$$

□

We complete this section by discussing two special cases of Theorem 2.2 which are of major interest for applications: let $w = 1$ and denote by H the polynomial convex hull of E. From Theorem 2.2(d) we may conclude that $\lambda_w^\sigma = \lambda_1 =: \omega_E$, the equilibrium measure of E, and Theorem 2.2(b) implies that $\log \rho$ is the Green function for the unbounded component of $\mathbb{C} \setminus E$ (or of $\mathbb{C} \setminus H$), with pole at infinity, denoted by $g_E(z, \infty)$. Conversely, these data satisfy parts (a),(b) of Theorem 2.2 (provided of course that $\omega_E \leq \sigma$). As a second case, suppose that $\log \rho(z) = g_H(z, \infty)$ for some compact set H of positive logarithmic capacity having a connected complement. Then Theorem 2.2(b) may be equivalently written as $\lambda_w^\sigma = \omega_H$. We summarize our findings in

Corollary 3.5. *Let $E_n, E, \sigma, w_n, n, p_n, \rho$ be as in* (i),(ii),(iii),(v), *with* $\text{supp}(\lambda_w^\sigma) \cap \text{supp}(\sigma - \lambda_w^\sigma)$ *being nonempty.*
(a) *In the case $w = 1$ (e.g., $w_n = 1$ for all $n \geq 0$), we have subexponential growth of $(\kappa(V_n(w_n, E_n)))_{n\geq 0}$ if and only if $\omega_E \leq \sigma$, $\log \rho = g_E(\cdot, \infty)$, and condition* (iv) *holds.*
(b) *Let $\log \rho(z) = g_H(z, \infty)$ with some compact set H of positive logarithmic capacity having a connected complement. Then we have subexponential growth of $(\kappa(V_n(w_n, E_n)))_{n\geq 0}$ if and only if $\omega_H \leq \sigma$, condition* (iv) *holds, w equals some*

constant $\exp(F')$ *on the boundary* ∂H *of* H *and* $w \cdot \rho$ *is less than or equal to* $\exp(F')$ *in* $E \setminus \partial H$. □

4. Examples

In order to illustrate our main results, we will restrict ourselves to the case of an asymptotically trivial weight $w = 1$, and consider only real abscissas with $E = \mathrm{supp}(\sigma) = [-1, 1]$. Furthermore, in order to be able to give some integral representation for the constant occurring in Theorem 2.1, we wish to restrict ourselves to measures σ having a even potential which is concave on $(-1, 1)$.

Let us first describe how to obtain the corresponding equilibrium measure $\lambda = \lambda_1^\sigma$ of the constrained energy problem as described before Theorem 2.1. According to [DrSa97, Corollary 2.15], the probability measure $\tau := (\sigma - \lambda)/(\sigma(E) - 1)$ is the solution to the (unconstrained) weighted energy problem on E with weight $v(z) := \exp(U^\sigma(z)/(\sigma(E) - 1))$. Since $\log(1/v)$ is convex and even, by assumption, we may conclude from [SaTo97, Theorem IV.1.10] that $\mathrm{supp}(\tau)$ is an interval[5] of the form $[-r, r]$. Having determined the shape of the support, we may find the corresponding parameter r by maximizing the Mhaskar–Saff functional [SaTo97, Theorem IV.1.5]

$$F(r) = -\log(\mathrm{cap}([-r, r]) - \int g_{[-r,r]}(t, \infty)\, d\sigma(t).$$

The equation $F'(r) = 0$ allows us to determine r as the unique solution of

$$1 = 2 \int_r^1 \frac{t}{\sqrt{t^2 - r^2}}\, d\sigma(t). \tag{22}$$

According to [DrSa97, Corollary 2.15], we have $\sigma - \lambda = \hat{\sigma} - \omega_{[-r,r]}$, where $\hat{\sigma}$ is the balayage measure of σ onto $[-r, r]$. By [SaTo97, Section II.4, Eqn. (4.47)], $\hat{\sigma}$ may be rewritten as $\hat{\sigma} = \sigma|_{[-r,r]} + \mu$, with

$$\frac{d\mu}{dx}(x) = \frac{2}{\pi} \int_r^1 \frac{t\sqrt{t^2 - r^2}}{(t^2 - x^2) \cdot \sqrt{r^2 - x^2}}\, d\sigma(t), \quad x \in [-r, r],$$

and thus

$$\frac{d\lambda}{dx}(x) = \frac{d(\omega_{[-r,r]} - \mu)}{dx}(x) = \frac{1}{\pi} \int_r^1 \frac{2t}{t^2 - x^2} \frac{\sqrt{r^2 - x^2}}{\sqrt{t^2 - r^2}}\, d\sigma(t), \quad x \in [-r, r], \tag{23}$$

whereas $d\lambda(x) = d\sigma(x)$ for $x \in E \setminus [-r, r]$.

We now turn our attention to suitable bases of polynomials. Let H be some compact set with connected complement and positive logarithmic capacity. It is shown in [Rei90] that the basis $(p_j)_{n \geq 0}$ of Newton polynomials at Leja points of H satisfies both assumptions (iii) and (iv) provided H has capacity 1, with $\rho =$

[5] This property can also be derived under weaker assumptions on U^σ; see, e.g., [SaTo97, Theorem IV.1.10], [SaTo97, Corollary IV.1.10] or [Rak96, Theorem 4].

$g_H(\cdot, \infty)$ (this result remains valid in the case of capacity different from one if one divides the jth Newton polynomial by its maximum norm on H). More generally [Bec96, Theorem 2.11], provided H is regular with respect to the Dirichlet problem, we may replace Leja points by any other sequence of points if the weak* limit of the corresponding sequence of normalized zero counting measures coincides with the equilibrium measure of H, e.g., we may take a Van der Corput enumeration of Fejer points [FiRe89]. Another family of polynomials satisfying (iii),(iv) with $\log \rho = g_H(\cdot, \infty)$ for some domain is given by the corresponding sequence of Faber polynomials [Bec96, Section 2.4.3].

As a final class of polynomials let μ be some positive Borel measure with compact support, and denote by p_n the corresponding nth orthonormal polynomial. Then condition (iv) holds with H being the polynomial convex hull of supp(μ). To see this, notice that for a polynomial P of degree at most n there holds

$$||P||^2 = \int |P(z)|^2 \, d\mu(z) \leq ||P/p_n||_H^2 \int p_n(z)^2 \, d\mu(z) \leq (n+1) \cdot ||P/p_n||_H^2$$

and thus $\epsilon_n(H) \leq \sqrt{n+1}$. Now if H is regular, then assumption (iii) holds with $\log \rho = g_H(\cdot, \infty)$ iff $\mu \in$ **Reg** (see [StTo92, Theorem 3.2.3] where further equivalent descriptions are given). Examples include the sequences of monomials ($H = \mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$) and the sequence of Chebyshev polynomials ($H = [-1, 1]$).

Let us now determine the constants occurring in Theorem 2.1 for the special case $w = 1$ and $\log \rho = g_H$ for some compact set $H \supset E = [-1, 1]$. First notice that $\log w + \log \rho$ equals zero on E. Also, $U^\lambda + g_H$ is superharmonic in $(\mathbb{C} \cup \{\infty\}) \setminus \partial H$, and thus

$$\begin{aligned} \Gamma(\sigma, H) := & \sup_{z \in \mathrm{supp}(\lambda)} [U^\lambda(z) - \log w(z)] \\ & \inf_{z \in \mathbb{C}} [U^\lambda(z) + \log \rho(z)] + \max_{z \in E} [\log w(z) + \log \rho(z)] \\ & \sup_{z \in \mathrm{supp}(\lambda)} U^\lambda(z) - \inf_{z \in \partial H} U^\lambda(z), \end{aligned}$$

in accordance with the observations of Corollary 3.5. Recall from (23) that we have at our disposal an integral representation for the potential U^λ. From (4) we know that that $U^{\sigma-\lambda}$ is constant on $[-r, r]$. Furthermore, $U^{\sigma-\lambda}$ is convex outside the support of $\sigma - \lambda$. Using the representation $U^\lambda = U^\sigma - U^{\sigma-\lambda}$, we may conclude that U^λ is concave on $[-1, 1]$, decreasing on $[1, +\infty)$, and even. Consequently, $\Gamma(\sigma, H) = U^\lambda(0) - U^\lambda(c)$, with $c = \max(|a|, |b|)$ in the case of a real interval $H = [a, b] \supset E$, and $c = i$ in the case $H = \mathbb{D}$.

To be more concrete, consider the case $\sigma = \alpha \cdot \omega_E + \beta \cdot \tau$ with α, β some nonnegative real constants, and $d\tau(x) = dx$ on $[-1, 1]$, with $\sigma(E) = \alpha + 2\beta > 1$. One easily verifies that U^τ is concave on E, and so is U^σ. In the case $\alpha \geq 1$ we have $\omega_E \leq \sigma$, and thus $\Gamma(\sigma, H) = g_E(c, \infty)$, which is obviously equal to 0 in the case $H = [-1, 1]$ and equal to $\log(1 + \sqrt{2})$ in the case $H = \mathbb{D}$. Otherwise, we may determine r via (22), and obtain the equation

$$1 = 2\int_r^1 \left[\frac{\alpha t}{\pi\sqrt{t^2-r^2}\sqrt{1-t^2}} + \frac{\beta t}{\sqrt{t^2-r^2}}\right] dt = \alpha + 2\beta \cdot \sqrt{1-r^2},$$

that is, $r = \sqrt{1-((1-\alpha)/(2\beta))^2}$. The integral occurring in (23) may be calculated explicitly; for the sake of simplicity we will restrict ourselves to the case $\alpha = 0$. Then for $x \in [-r, r]$ (compare with [DrSa97, Example 4.1])

$$\frac{d\lambda}{dx}(x) = \begin{cases} \beta, & x \in E \setminus [-r,r], \\ \frac{2\beta}{\pi}\arctan(\frac{\sqrt{1-r^2}}{\sqrt{r^2-x^2}}), & x \in [-r,r], \end{cases}$$

leading to

$$\Gamma(\sigma, H) = U^\lambda(0) - U^\lambda(c) = \beta \cdot \int_r^1 \log\left|1 - \frac{c^2}{x^2}\right| dx$$
$$+\frac{2\beta}{\pi} \cdot \int_0^r \log\left|1-\frac{c^2}{x^2}\right| \cdot \arctan\left(\frac{\sqrt{1-r^2}}{\sqrt{r^2-x^2}}\right) dx.$$

In the case $H = [-1, 1]$ (i.e., $c = 1$) of, e.g., Chebyshev polynomials, we obtain a function decreasing in β (see Figure 1), which for $\beta \to \infty$ tends to zero (then we obtain Fekete nodes which approximately are distributed like the arcsin measure), and for $\beta \to 1/2$ (the case of square Vandermonde–Chebyshev matrices with equidistant nodes) tends to $\log 2 \approx 0.693$, describing the classical behavior of the Lebesgue constant for equidistant nodes on $[-1, 1]$.

In contrast, in the case $H = \mathbb{D}$ (i.e., $c = i$) of, e.g., monomials we obtain a function decreasing in β (see Figure 1), which for $\beta \to \infty$ tends to $\log(1+\sqrt{2}) \approx 0.881$ (Vandermonde–matrices with optimal choice of abscissas in $[-1, 1]$ being distributed like the arcsin measure [Bec96, Bec97]), and for $\beta \to 1/2$ (the case of square Vandermonde matrices with equidistant nodes) tends to $\log(\sqrt{2}) + \pi/4 \approx 1.132$, confirming a result of Gautschi [Gau90, Example 3.3].

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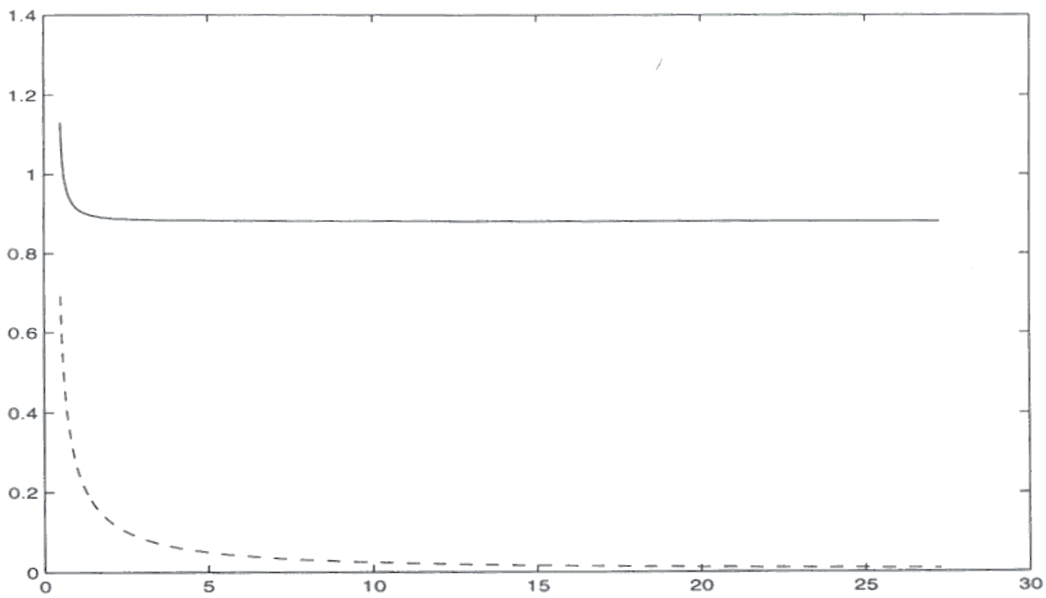

FIGURE 1. THE QUANTITIES $\Gamma(\beta \cdot \tau, \mathbb{D})$ (SOLID) AND $\Gamma(\beta \cdot \tau, [-1,1])$ (DASHED) FOR THE MEASURE $d\tau(x) = dx$ ON $[-1,1]$ OF EQUIDISTANT NODES IN $[-1,1]$ FOR $\beta \in [0.5, 27]$.

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