Behavior of Convolution Sequences of a Family of Probability Measures on [0, ∞)

A. MUKHERJEA & E. B. SAFF

Communicated by MURRAY ROSENBLATT

Let S be a locally compact Hausdorff semigroup (*i.e.*, an algebraic semigroup with locally compact Hausdorff topology and jointly continuous multiplication). Let μ and ν be two regular probability measures on the Borel sets (generated by the open sets) of S. Then it is well-known that the mapping $x \to \mu(Bx^{-1})$, $Bx^{-1} = \{y \in S \mid yx \in B\}$, is a Borel-measurable function for every Borel set B, and $\lambda(B) = \int \mu(Bx^{-1})\, \nu(dx)$ defines a regular probability measure called the *convolution* of μ and ν which is denoted by $\mu * \nu$. This convolution operation is associative. When $\mu = \nu$, we write $\mu^2 = \mu * \mu$, $\mu^3 = \mu^2 * \mu$, *etc.*

The following result is frequently useful in the theory of random walks and probability measures on semigroups and has been proven by Rosenblatt [R, p. 141].

Theorem (Rosenblatt). *Let μ be a regular probability measure on a compact semigroup S which is generated by the support of μ. Then given any open set $\mathcal{O}$ containing an ideal of S, $\mu^n(\mathcal{O})$ converges to 1 as $n \to \infty$.*

This result also holds for any discrete countable semigroup S and has been used by Martin–löff in his study of probability theory on discrete semigroups [M]. Furthermore by a careful examination of Rosenblatt's proof it is not difficult to see that Rosenblatt's result holds on any locally compact semigroup if the ideal I contains an interior point.

To extend certain results on random walks from compact to locally compact semigroups, it is natural to consider the question of the behavior of μ^n as $n \to \infty$, around an ideal I. In this note, we present an example of an interesting family of probability measures on $[0, \infty)$, the non-negative reals with multiplication as the semigroup operation and the usual topology of the real line, which sheds some light on the above question and shows that Rosenblatt's theorem cannot be extended to a general locally compact semigroup.

Indiana University Mathematics Journal, ©, Vol. 24, No. 3 (1974)

Example. *Let $S = [0, \infty)$ (as described above) and μ be the normalized Lebesgue measure with support $F = [0, a]$, $a > 1$.* Then F generates S. Since $\{0\}$ is the smallest ideal of S, we wish to determine, among other things, the behavior of $\mu^n([0, \alpha])$, $0 < \alpha$.

We claim the following:

(i) *If $a < e$, then for $0 < \alpha$, $\mu^n([0, \alpha])$ converges to 1 as $n \to \infty$.*
(ii) *If $a > e$, then for $0 < \alpha$, $\mu^n([0, \alpha])$ converges to 0 as $n \to \infty$.*
(iii) *If $a = e$, then for $0 < \alpha$, $\mu^n([0, \alpha])$ converges to 1/2 as $n \to \infty$.*
(iv) *For all $a > 1$, $\mu^n(K)$ converges to 0 as $n \to \infty$, whenever K is compact and doesn't contain 0.*

Actually the proof of the first three assertions, which utilizes the Central Limit Theorem of probability theory, shows that it is possible to obtain a more general class of measures (not necessarily Lebesgue measures) for which Rosenblatt's theorem fails to hold on general locally compact semigroups. After the proofs are given we indicate how the above assertions are related to certain limiting properties of the sections of the Taylor expansion of the exponential function.

Proof. Let $X_1, X_2, \cdots$ be a sequence of independent, identically distributed random variables on some probability space, with values in $[0, \infty)$ and with distribution μ, *i.e.*, $P[X_i \in B] = \mu(B)$, where P is the probability measure and B is any Borel set on $[0, \infty)$.

Let $Z_n = X_1 \cdot X_2 \cdots X_n$. Since the X_i's are independent, $P(Z_n \in B) = \mu^n(B)$. Consider the sequence $\log Z_n$, which is clearly defined with probability 1. Now

$$\log Z_n = \sum_{i=1}^{n} \log X_i = \sum_{i=1}^{n} Y_i ,$$

where $Y_i = \log X_i$. We wish to apply the Central Limit Theorem to the sequence $Y_1, Y_2, \cdots$. To do so, we must check if the mean $m = E(Y_i)$ and the variance $\sigma^2 = E(Y_i^2) - m^2$ are finite.

Clearly,

$$m = \frac{1}{a}\int_0^a \log x \, dx$$
$$= \log a - 1,$$

and

$$\sigma^2 + m^2 = \frac{1}{a}\int_0^a (\log x)^2 \, dx$$
$$= (\log a - 1)^2 + 1,$$

so that both m and σ^2 are finite. Hence by the Central Limit Theorem, the distributions of

$$\frac{\sum_{i=1}^{n} Y_i - n \cdot m}{n^{1/2}\sigma}$$

converge to $N(0, 1)$, the normal distribution with mean 0 and variance 1, as $n \to \infty$. We wish to find $\lim_{n\to\infty} \mu^n([0, \alpha])$, *i.e.*, the limit of $P(Z_n \varepsilon [0, \alpha])$ as $n \to \infty$. To do this, we have to find sets $A_n \subset (-\infty, \infty)$ such that the following set equation holds

$$n^{1/2}\sigma A_n + n\cdot m = \log (0, \alpha),$$

i.e.

$$A_n = \left(-\infty, n^{1/2}\cdot\frac{-m}{\sigma} + \frac{1}{n^{1/2}\sigma}\log \alpha\right). \tag{1}$$

Case (i): $a < e$. In this case, $m = \log a - 1 < 0$. Since $-m > 0$, it is clear from (1) that given any positive integer p, we can find N_p such that $n > N_p$ implies $A_n \supset (-\infty, p)$. Then, we have for $n > N_p$,

$$\begin{aligned}\mu^n([0, \alpha]) = P(Z_n \varepsilon [0, \alpha]) &= P(\log Z_n \varepsilon \log (0, \alpha))\\ &= P\left[\frac{\sum_{i=1}^{n} Y_i - n\cdot m}{n^{1/2}\sigma} \varepsilon A_n\right]\\ &\geqq P\left[\frac{\sum_{i=1}^{n} Y_i - n\cdot m}{n^{1/2}\sigma} \varepsilon (-\infty, p)\right],\end{aligned}$$

where the last term converges, as $n \to \infty$, to

$$\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{p} e^{-x^2}\,dx,$$

by the Central Limit Theorem. Since

$$\lim_{p\to\infty}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{p} e^{-x^2}\,dx = 1,$$

it follows that $\lim_{n\to\infty} \mu^n([0, \alpha]) = 1$.

Case (ii): $a > e$. In this case, $m = \log a - 1 > 0$. Since $m > 0$, we see from (1) that given any positive integer p, there exists an N_p such that $n > N_p$ implies $A_n \subset (-\infty, -p)$. Now for $n > N_p$, we have

$$\mu^n([0, \alpha]) = P\left[\frac{\sum_{i=1}^{n} Y_i - n\cdot m}{n^{1/2}\sigma} \varepsilon A_n\right] \leqq P\left[\frac{\sum_{i=1}^{n} Y_i - n\cdot m}{n^{1/2}\sigma} \varepsilon (-\infty, -p)\right]$$

which converges, as $n \to \infty$, to

$$\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{-p} e^{-x^2}\,dx.$$

Since

$$\int_{-\infty}^{-p} e^{-x^2}\, dx \to 0 \quad \text{as} \quad p \to \infty,$$

it is clear that $\lim_{n\to\infty} \mu^n([0, \alpha]) = 0$.

Case (iii): $a = e$. In this case, $m = \log a - 1 = 0$. Therefore, from (1)

$$A_n = \left(-\infty, \frac{1}{n^{1/2}\sigma} \log \alpha\right).$$

Since, as $n \to \infty$, $(\log \alpha)/n^{1/2}\sigma \to 0$, and

$$\frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{0} e^{-x^2}\, dx = 1/2,$$

by similar arguments as in previous cases, it follows easily that

$$\lim_{n\to\infty} \mu^n([0, \alpha]) = 1/2.$$

Remark. Clearly, from the case $a > e$, one can see that if we take *any* probability measure (not necessarily a Lebesgue measure) whose support generates $[0, \infty)$ such that $m > 0$ and the mean and the variance of Y_i are finite, then $\lim_{n\to\infty} \lambda^n[0, \alpha] = 0$ for every $\alpha > 0$.

Finally, to prove (iv), we notice from (i), (ii), and (iii), that there is an open set V such that for all $a > 1$, $\mu^n(V) \to 0$ as $n \to \infty$. [For instance, take $V = (1/2, 1)$.] Now let $0 < x < \infty$. Suppose for the moment that for every neighborhood $N(x)$ of x, $\mu^n(N(x))$ does not converge to 0 as $n \to \infty$. Let $z = 3/4x$. Then $x \cdot z = 3/4 \in V = (1/2, 1)$. Clearly there are open neighborhoods $N(x)$ and $N(z)$ of x and z respectively such that $N(x)N(z) \subset V$. Then since $(0, \infty) \subset \bigcup_{k=1}^{\infty} F^k$, there is a k_0 such that $N(z) \cap F^{k_0} \neq \varnothing$. Hence $\mu^{k_0}(N(z)) > 0$. But

$$\begin{aligned} \mu^{n+k_0}(V) &= \int \mu^n(Vy^{-1})\mu^{k_0}(dy) \\ &\geqq \int_{N(z)} \mu^n(Vy^{-1})\mu^{k_0}(dy) \\ &\geqq \mu^n(N(x))\mu^{k_0}(N(z)), \end{aligned}$$

which doesn't go to 0 as $n \to \infty$. This is a contradiction. It follows that for $0 < x < \infty$, there is an open neighborhood $N(x)$ of x such that $\mu^n(N(x))$ converges to 0 as $n \to \infty$. Since every compact set K in $(0, \infty)$ can be covered by finitely many such $N(x)$, it follows that $\mu^n(K)$ converges to 0 as $n \to \infty$. This proves (iv).

We remark that (iv) could have been deduced immediately from (i), (ii) and (iii); however, the above proof shows more generally that if S is a locally compact Hausdorff topological group and K is any compact subset, then $\mu^n(K) \to 0 \Leftrightarrow \mu^n(V) \to 0$ for some open set V.

We now show that the assertions (i)–(iii) are actually equivalent to certain statements concerning the sections

$$S_n(x) = \sum_{j=0}^{n} \frac{x^j}{j!}$$

of the Taylor expansion of e^x. To see this we first notice that for $0 < \alpha \leqq a^n$, $n > 2$, we have

$$\begin{aligned}\mu^n([0, \alpha]) &= \frac{1}{a}\int_0^a \mu^{n-1}([0, \alpha/x])\, dx \\ &= \frac{1}{a}\int_0^\tau dx + \frac{1}{a}\int_\tau^a \mu^{n-1}([0, \alpha/x])\, dx,\end{aligned}$$

where $\tau \equiv \alpha/a^{n-1}$. By using induction it then follows that

$$\mu^n([0, \alpha]) = \frac{\alpha}{a^n}\sum_{j=0}^{n-1}\left(\log\frac{a^n}{\alpha}\right)^j \cdot \frac{1}{j!}.$$

On setting $b \equiv \log a$ (>0) and $\gamma \equiv -\log \alpha$ this last equation becomes

$$\begin{aligned}\mu^n([0, \alpha]) &= \frac{1}{e^{nb+\gamma}}\sum_{j=0}^{n-1}(nb+\gamma)^j \cdot \frac{1}{j!} \\ &= \frac{1}{e^{nb+\gamma}}\left[S_n(nb+\gamma) - \frac{(nb+\gamma)^n}{n!}\right].\end{aligned} \tag{2}$$

Now by applying Stirling's formula and the fact that $eb/e^b \leqq 1$, it is easy to see that

$$\lim_{n\to\infty}\frac{(nb+\gamma)^n}{n!\, e^{nb+\gamma}} = 0, \tag{3}$$

and so the assertions (i), (ii), and (iii) are equivalent to the conditions that

$$\lim_{n\to\infty}\frac{S_n(nb+\gamma)}{e^{nb+\gamma}} = \begin{cases} 1\,, & \text{if } 0 < b < 1, \\ 0\,, & \text{if } b > 1, \\ 1/2, & \text{if } b = 1,\end{cases} \tag{4}$$

for each fixed $\gamma \geqq 0$.

We remark that it is possible to give an independent proof of (4) which does not utilize the Central Limit Theorem. Indeed, if $0 < b < 1$, we have

$$\begin{aligned}1 - \frac{S_n(nb+\gamma)}{e^{nb+\gamma}} &= \frac{1}{e^{nb+\gamma}}[e^{nb+\gamma} - S_n(nb+\gamma)] \\ &= \frac{(nb+\gamma)^{n+1}}{(n+1)!\, e^{nb+\gamma}}\left[1 + \frac{nb+\gamma}{n+2} + \frac{(nb+\gamma)^2}{(n+3)(n+2)} + \cdots\right].\end{aligned}$$

Since $b < 1$, there exists a constant y so that $(nb + \gamma)/(n + 2) < y < 1$ for all n sufficiently large. Hence, for such n, we have

$$0 \leqq 1 - \frac{S_n(nb + \gamma)}{e^{nb+\gamma}} \leqq \frac{(nb + \gamma)^{n+1}}{(n + 1)!\, e^{nb+\gamma}} \sum_{i=0}^{\infty} y^i$$
$$= \frac{(nb + \gamma)^{n+1}}{(n + 1)!\, e^{nb+\gamma}} \cdot \frac{1}{1 - y},$$

and by (3) the last expression approaches zero as $n \to \infty$. Thus

$$\lim_{n\to\infty} \frac{S_n(nb + \gamma)}{e^{nb+\gamma}} = 1, \quad \text{if} \quad 0 < b < 1.$$

For the cases $b > 1$ and $b = 1$ we can utilize the technique illustrated in [N]. It is shown in this reference that for any real number w

$$\frac{S_n(n + wn^{1/2})}{e^{n+wn^{1/2}}} = \frac{c_n}{(2\pi)^{1/2}} \int_w^\infty f_n(x)\, dx,$$

where

$$c_n \equiv \frac{(2\pi n)^{1/2} \cdot e^{-n} n^n}{n!}, \quad \text{and} \quad f_n(x) \equiv \left(1 + \frac{x}{n^{1/2}}\right)^n e^{-x \cdot n^{1/2}}.$$

Setting $w_n \equiv (b - 1)n^{1/2} + \gamma/n^{1/2}$ we therefore have

$$\frac{S_n(nb + \gamma)}{e^{nb+\gamma}} = \frac{c_n}{(2\pi)^{1/2}} \int_{w_n}^\infty f_n(x)\, dx. \tag{5}$$

Using (5), Stirling's formula, and the inequality ([N], p. 407)

$$f_n(x) \leqq e^{4/3} e^{-x}, \quad x \geqq 2, \quad n > 5,$$

it is not difficult to show from Lebesgue integration theory that the stated limits hold as $n \to \infty$.

The authors wish to thank Professor Rosenblatt for his helpful suggestions.

References

[M] P. Martin-Löff, *Probability theory on discrete semigroups*, Z. Wahrscheinlichkeitstheorie und Verw. **4** (1965), 78–102.

[N] D. J. Newman, & T. J. Rivlin, *The zeros of the partial sums of the experimential function*, J. Approx. Theory **5** (1972), 405–412.

[R] M. Rosenblatt, *Markov processes. Structure and asymptotic behavior*, Springer-Verlag, New York, 1971.

The work of the second author was partially supported by the Air Force Office of Scientific Research, Grant AFOSR-73-2503.

University of South Florida

Date communicated: September 7, 1973