Ray Sequences of Best Rational Approximants to Entire Functions

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Abstract. This article is devoted to results relating to the theory of rational approximation of entire functions. An analysis is made of the rate of decrease of the best approximation $\rho_{n,m}$ of an entire function by rational functions of type (n, m) in the uniform metric. It is assumed that the indices (n, m) progress along a ray sequence of the Walsh table, i.e. the sequence of indices (n, m) satisfies $m/n \to \theta \in [0, 1]$ as $m + n \to \infty$.

§1. Introduction

Let E be an arbitrary compact set, $E \subset \mathbb{C}$, and let f be an entire function. For any nonnegative integers n and m denote by $\mathcal{R}_{n,m}$ the class of all rational functions with complex coefficients of order (n, m):

$$\mathcal{R}_{n,m} = \{r : r = p/q, \deg p \leq n, \deg q \leq m, q \not\equiv 0\}$$

The deviation of f from $\mathcal{R}_{n,m}$ (in the uniform metric on E) is denoted by $\rho_{n,m}$:

$$\rho_{n,m} = \rho_{n,m}(f, E) = \inf_{r \in \mathcal{R}_{n,m}} ||f - r||_E,$$

where $\|\cdot\|_E$ is the supremum norm on E.

We assume that $m = m(n)$ and the sequence of positive integers $\{m(n)\}$, $m(n) \leq n, n = 0, 1, 2 \dots,$ tending to infinity satisfies the following conditions:

$$m(n-1) \leq m(n) \leq m(n-1) + 1, \; n = 1, 2, \tag{1}$$

and

$$\lim_{n \to \infty} \frac{m(n)}{n} = \theta, \; 0 \leq \theta \leq 1 \tag{2}$$

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One of the main results of this paper is Theorem 1 characterizing the rate of decrease of the ray sequence $\{\rho_{n,m(n)}\}_{n=0}^{\infty}$ of the Walsh table $\{\rho_{n,m}\}_{n,m=0}^{\infty}$ of the best rational approximations of an entire function of finite order.

The case of polynomial approximation ($m(n) = 0,\ n = 0, 1, 2, \ldots$) of entire functions has been thoroughly investigated. We mention works of Varga [18], Shah [15], and Winiarski [20] relating to this direction. The methods of the theory interpolation by polynomials and Walsh inequality (see [19]) give us, in terms related to the degree of decrease of the values $\rho_{n,0}$, necessary and sufficient conditions for a continuous function on E to admit a continuation to an entire function of finite order. More precisely the following result is known.

Suppose that f is continuous on E, where E is a compact set with the positive logarithmic capacity cap(E). *Then f can be prolonged to an entire function of order $\sigma \geq 0$ if and only if*

$$\limsup_{n\to\infty} \frac{\ln \rho_{n,0}}{n \ln n} = -\frac{1}{\sigma}.$$

It is to be noted that the condition

$$\limsup_{n\to\infty} \rho_{n,0}^{\sigma/n} n = \sigma e \tau (\mathrm{cap}(E))^{\sigma}$$

allows us to describe the class of entire functions of finite order $\sigma > 0$ and finite type $\tau > 0$.

We now point out the following estimates characterizing the behavior of the ray sequence $\{\rho_{n,m(n)}\}_{n=0}^{\infty}$ and following immediately from the results in polynomial approximation.

If f is an entire function of finite order $\sigma \geq 0$, then

$$\limsup_{n\to\infty} \frac{\ln \rho_{n,m(n)}}{n \ln n} \leq -\frac{1}{\sigma}; \tag{3}$$

and if f is an entire function of finite order $\sigma > 0$ and finite type $\tau > 0$, then

$$\limsup_{n\to\infty} \rho_{n,m(n)}^{\sigma/n} n \leq \sigma e \tau (\mathrm{cap}(E))^{\sigma}.$$

An important role in the theory of rational approximation of analytic functions is played by methods of rational interpolation of analytic functions, especially Padé approximants (interpolation sequences of rational functions with free poles) (see, for example, [5, 7, 16]). In addition to constructive methods, the theory of Hankel operators has been widely used in recent years in studying the degree of rational approximation of analytic functions (see [8, 9, 10, 13, 14]). These methods are based on the Adamyan-Arov-Kreĭn theorem [1] (see also [12]). This theorem makes it possible to reduce the investigation of the degree of rational approximation of analytic functions to an investigation of the rate of decrease of the sequence $\{s_n\}, n = 0, 1, \ldots,$ of singular numbers of the Hankel operator constructed from the function to be approximated.

Application of the methods of the theory of Hankel operators allowed one of the authors (see [13]) to estimate the rate of decrease of the diagonal sequence (in this case $m(n) = n,\ n = 0, 1, 2, \ldots$) of the best rational approximations of an entire function of finite order $\sigma \geq 0$. The following upper estimate was established:

$$\liminf_{n\to\infty} \frac{\ln \rho_{n,n}}{n \ln n} \leq -\frac{2}{\sigma}.$$

In connection with the last inequality we point out that the limit in this relation exists for certain entire functions, for example, $f(z) = e^z$ (see [2]):

$$\lim_{n\to\infty} \frac{\ln \rho_{n,n}}{n \ln n} = -\frac{2}{\sigma}.$$

In the present article we use methods employing the theory of Hankel operators and a generalization of the Adamyan– Arov–Kreĭn theorem to prove Theorem 1 concerning the ray sequence $\{\rho_{n,m(n)}\}_{n=0}^{\infty}$ of the deviations in best rational approximations. In this theorem the rate of convergence of the product $\prod_{i=0}^{m(n)} \rho_{n-i,m(n)-i}$ to zero is estimated.

Theorem 1. *If E is an arbitrary compact set in $\mathbb{C}$ and f is an entire function of finite order $\sigma \geq 0$, then*

$$\limsup_{n\to\infty} \frac{\ln(\rho_{n,m(n)}\rho_{n-1,m(n)-1}\cdots\rho_{n-m(n),0})}{nm(n)\ln n} \leq -\frac{1}{\sigma} \tag{4}$$

The next assertion, which follows from Theorem 1, gives an estimate for $\liminf_{n\to\infty} \ln \rho_{n,m(n)}/n \ln n$.

Corollary 1.

$$\liminf_{n\to\infty} \frac{\ln \rho_{n,m(n)}}{n \ln n} \leq -\frac{2}{(2-\theta)\sigma}.$$

The next assertion enables us to characterize the behavior of the ray sequence $\{\rho_{n,m(n)}\}_{n=0}^{\infty}$ for functions for which equality is attained in (3).

Corollary 2. *If*

$$\limsup_{n\to\infty} \frac{\ln \rho_{n,m(n)}}{n \ln n} \qquad \frac{1}{\sigma}$$

then

$$\liminf_{n\to\infty} \frac{\ln \rho_{n,m(n)}}{n \ln n} < -\frac{1}{(1-\theta)\sigma}$$

An investigation of the asymptotic behavior of the singular numbers of the Hankel operator constructed from the function being approximated enables us to also prove other results in the theory of rational approximation of entire functions.

The following theorem relates to the case when f is an entire function of finite order $\sigma > 0$ and finite type $\tau > 0$.

Theorem 2. *Suppose that E is an arbitrary compact set in $\mathbf{C}$, and f is an entire function of finite order $\sigma > 0$ and finite type $\tau > 0$. Then*

$$\limsup_{n\to\infty}(\rho_{n,m(n)}\rho_{n-1,m(n)-1}\cdots\rho_{n-m(n),0})^{\sigma/n(m(n)+1)}n \le \sigma e\tau(\mathrm{cap}(E))^{\sigma}, \quad (5)$$

where $\mathrm{cap}(E)$ is the logarithmic capacity of E

We now state corollaries of this theorem.

Corollary 3. *If*

$$\lim_{n\to\infty}(m(n)/n-\theta)\ln n = 0,$$

$$\liminf_{n\to\infty}\rho_{n,m(n)}^{\sigma(2-\theta)/2n}n \le \sigma e^{1/2}\tau(\mathrm{cap}(E))^{\sigma}e^{-\frac{(1-\theta)^2}{\theta(2-\theta)}\ln(1-\theta)} \quad (6)$$

Corollary 4. *If*

$$\limsup_{n\to\infty}\rho_{n,m(n)}^{\sigma/n}n = \sigma e\tau(\mathrm{cap}(E))^{\sigma},$$

then for any λ with $1-\theta < \lambda \le 1$

$$\liminf_{n\to\infty}\rho_{n,m(n)}^{\lambda\sigma/n}n \le \lambda\sigma e\tau(\mathrm{cap}(E))^{\sigma}.$$

The outline of this paper is as follows. Results needed below from the theory of Hankel operators are presented in Section 2. In Section 3 we investigate the degree of rational approximation of functions having $s \ge 1$ essential singularities of finite order. Theorem 1 is a consequence of the results obtained there. In Section 4 the proof of Theorem 2 is given.

§2. Some Results from the Theory of Hankel Operators

2.1. A Generalization of the Adamyan-Arov-Kreĭn Theorem

Let G be a bounded domain whose boundary Γ consists of N disjoint closed analytic Jordan curves. It will be assumed that Γ is positively oriented with respect to G, $0 \in G$.

Denote by $E_p(G)$, $1 \le p \le \infty$, the Smirnov class of analytic functions on G. We note that the condition

$$\int_\Gamma \frac{\varphi(\xi)\,d\xi}{\xi - z} = 0 \qquad \text{for all } \ z \in \overline{\mathbf{C}} \setminus \overline{G} \quad (7)$$

is necessary and sufficient for a function $\varphi(\xi)$, $\xi \in \Gamma$, belonging to $L_p(\Gamma)$, to be the boundary value of a function in the Smirnov class $E_p(G)$ (see [3, 6, 11, 17] for more details about the classes $E_p(G)$).

Fix a nonnegative integer l. Denote by $H_l = H_l(G)$ the class of functions q representable in the form $q = \varphi/\xi^l$, where $\varphi \in E_2(G)$.

Denote by $L_{p,l}(\Gamma)$, $1 \le p < \infty$, the Lebesgue space of functions φ measurable on Γ such that

$$\|\varphi\|_{p,l} = \left(\int_\Gamma |\varphi(\xi)|^p |\xi|^l |d\xi|\right)^{1/p} < \infty.$$

The inner product in the Hilbert space $L_{2,l}(\Gamma)$ is denoted by

$$(\varphi, \psi)_{2,l} = \int_\Gamma (\varphi\overline{\psi})(\xi)|\xi|^l |d\xi|, \quad \varphi, \psi \in L_{2,l}(\Gamma).$$

For $l = 0$ we will write $L_{p,l}(\Gamma) = L_p(\Gamma)$ and $\|\cdot\|_{p,l} = \|\cdot\|_p$.

$L_\infty(\Gamma)$ is the space of essentially bounded functions, with the norm

$$\|\varphi\|_\infty = \operatorname*{ess\,sup}_\Gamma |\varphi(\xi)|.$$

Let $C(\Gamma)$ be the space of continuous functions on Γ, with the norm

$$\|\varphi\|_\Gamma = \max_{\xi \in \Gamma} |\varphi(\xi)|.$$

We represent $L_{2,l}(\Gamma)$ as direct sum $L_{2,l}(\Gamma) = H_l \oplus H_l^\perp$ of subspaces, where $H_l^\perp$ is the orthogonal complement of H_l in $L_{2,l}(\Gamma)$. Here and in what follows we will consider H_l and $E_2(G)$ as subspaces of $L_{2,l}(\Gamma)$.

Assume that f is continuous on Γ. The Hankel operator $A_f : E_2(G) \to H_l^\perp$ is defined as follows. For any function $q \in E_2(G)$ let $A_f q = \mathbb{P}_-(qf)$, where $\mathbb{P}_-$ is the orthogonal projection of $L_{2,l}(\Gamma)$ onto $H_l^\perp$. It is not hard to see that A_f is a compact operator.

Denote by $\{s_{n,l}\}$, $s_{n,l} = s_{n,l}(f; G)$, $n = 0, 1, 2, \ldots$, the sequence of singular numbers (counting multiplicity) of the operator A_f ($s_{n,l}$ is an eigenvalue of the operator $(A_f{}^* A_f)^{1/2}$, where $A_f{}^* : H_l^\perp \to E_2(G)$ is the adjoint operator of A_f). Assume that $s_{0,l} \ge s_{1,l} \ge \ldots \ge s_{n,l} \ge \ldots$ (for the properties of singular numbers see [4]).

For any nonnegative integer n denote by $\mathcal{M}_{n+l,n} = \mathcal{M}_{n+l,n}(G)$ the class functions representable in the form $h = p/q\xi^l$, where $p \in E_\infty(G)$ and q is a polynomial of degree at most n, $q \not\equiv 0$. We remark that $h \in \mathcal{M}_{n+l,n}$ has no more than $n + l$ poles and no more than n free poles. The deviation of f from the class $\mathcal{M}_{n+l,n}$ in the space $L_\infty(\Gamma)$ is denoted by

$$\Delta_{n+l,n} = \Delta_{n+l,n}(f; G) = \inf_{h \in \mathcal{M}_{n+l,n}} \|f - h\|_\infty.$$

Using the same arguments as in [12] it is not hard to prove a theorem establishing a connection between the singular numbers of the Hankel operator and the quantities $\Delta_{n+l,n}$.

Let G be a bounded domain whose boundary Γ consists of N disjoint closed analytic Jordan curves, and let f be a continuous function on Γ. Then

$$\Delta_{n+N-1+l,n+N-1} \le s_{n,l} \le \Delta_{n+l,n} \tag{8}$$

for all integers $n \ge N - 1$.

This theorem is a generalization of the well-known Adamyan-Arov-Kreĭn theorem, which relates to the case when $G = \{z : |z| < 1\}$, $l = 0$. We then have $s_{n,0} = \Delta_{n,n}$, $n = 0, 1, 2, \ldots$ (see [1] and [12] for more details).

2.2. Auxiliary Results

In this subsection we point out some useful relations having to do with singular numbers of a Hankel operator.

We first prove a lemma giving necessary and sufficient conditions for a function u belonging to the space $L_{2,l}(\Gamma)$ to be an element of the subspace $E_2^{\perp}(G)$, where $E_2^{\perp}(G)$ is the orthogonal complement of $E_2(G)$ in $L_{2,l}(\Gamma)$.

Lemma 1. *Suppose that $u \in L_{2,l}(\Gamma)$. Then $u \in E_2^{\perp}(G)$ if and only if there exists a function $v \in E_2(G)$ such that*

$$\overline{u}(\xi)|\xi|^l\,|d\xi| = v(\xi)\,d\xi. \tag{9}$$

almost everywhere on Γ.

Proof: Assume the relation (9) for the function $u \in L_{2,l}(\Gamma)$, where v is some function in $E_2(G)$. We show that

$$(q, u)_{2,l} = 0 \quad \text{for any } q \in E_2(G), \tag{10}$$

where $(q, u)_{2,l}$ is the inner product in $L_{2,l}(\Gamma)$. We have that

$$(q, u)_{2,l} = \int_\Gamma q(\xi)\overline{u}(\xi)|\xi|^l\,|d\xi| = \int_\Gamma q(\xi)v(\xi)\,d\xi = 0.$$

The last equality in this relation follows from the fact that both functions q and v belong to $E_2(G)$. It remains to see that by (10), $u \in E_2^{\perp}(G)$.

Assume now that $u \in E_2^{\perp}(G)$. Then for any function $q \in E_2(G)$

$$(q, u)_{2,l} = \int_\Gamma q(\xi)\overline{u}(\xi)\,|\xi|^l\,|d\xi| = 0.$$

In particular, for $q(\xi) = 1/(\xi - z)$, (z is an arbitrary point in $\overline{\mathbb{C}} \setminus \overline{G}$,) we get

$$\int_\Gamma \frac{1}{\xi - z}\,\overline{u}(\xi)\,|\xi|^l\,|d\xi| = 0.$$

This implies (see (7)) that there exists a function $v \in E_2(G)$ such that (9) holds. □

The next lemma is established along the same lines presented above. This lemma gives us necessary and sufficient conditions for a function u belonging to the space $L_{2,l}(\Gamma)$ to be an element of subspace $H_l^{\perp}$.

Lemma 2. *Suppose that $u \in L_{2,l}(\Gamma)$. Then $u \in H_l^\perp$ if and only if there exists a function $v \in E_2(G)$ such that*

$$\overline{u}(\xi)|\xi|^l|d\xi| = v(\xi)\xi^l d\xi \tag{11}$$

almost everywhere on Γ.

Let $\{q_{n,l}\}$, $n = 0, 1, 2, \ldots$, be an orthonormal system of eigenfunctions of the operator $(A_f^* A_f)^{1/2}$, corresponding to the sequence of singular numbers $\{s_{n,l}\}$, $n = 0, 1, 2, \ldots$. We fix a nonnegative integer n. Since $s_{n,l}$ is an eigenvalue of $(A_f^* A_f)^{1/2}$,

$$A_f q_{n,l} = s_{n,l} u_{n,l}, \tag{12}$$

$$A_f^* u_{n,l} = s_{n,l} q_{n,l}, \tag{13}$$

where $u_{n,l} \in H_l^\perp(G)$.

Let us write these relations in another form. For this we note first that since $A_f q_{n,l} = \mathbb{P}_-(q_{n,l} f)$, there exists a unique function $p_{n,l} \in E_2(G)$ such that $A_f q_{n,l} = q_{n,l} f - p_{n,l}/\xi^l$; therefore, by (12),

$$q_{n,l} f - p_{n,l}/\xi^l = s_{n,l} u_{n,l}. \tag{14}$$

Second, it follows from the definition of A_f that the adjoint operator A_f^* of A_f is the composition of the operator of multiplication by the function $\bar{f}$ and the orthogonal projection $\mathbb{P}_+$ of $L_{2,l}(\Gamma)$ onto $E_2(G)$; namely, for any function $u \in H_l^\perp$ we have $A_f^* u = \mathbb{P}_+(u\bar{f})$. Therefore, $A_f^* u_{n,l}$ can be represented in the form $A_f^* u_{n,l} = u_{n,l}\bar{f} - v_{n,l}$, where $v_{n,l} \in E_2^\perp(G)$. The equality (13) thus implies that

$$u_{n,l}\bar{f} - v_{n,l} = s_{n,l} q_{n,l}. \tag{15}$$

We now use the fact that the functions $u_{n,l} \in H_l^\perp$ and $v_{n,l} \in E_2^\perp(G)$. Then we can assert (see (9) and (11)) that there exist functions $\alpha_{n,l}, \beta_{n,l} \in E_2(G)$ such that

$$\bar{u}_{n,l}(\xi)|\xi|^l|d\xi| = \alpha_{n,l}(\xi)\xi^l d\xi$$

and

$$\bar{v}_{n,l}(\xi)|\xi|^l|d\xi| = \beta_{n,l}(\xi) d\xi$$

almost everywhere on Γ. Therefore, by (14) and (15) we get

$$(q_{n,l} f - p_{n,l}/\xi^l)(\xi)\xi^l d\xi = s_{n,l}\bar{\alpha}_{n,l}(\xi)|\xi|^l|d\xi|, \tag{16}$$

$$(\alpha_{n,l} f - \beta_{n,l}/\xi^l)(\xi)\xi^l d\xi = s_{n,l}\bar{q}_{n,l}(\xi)|\xi|^l|d\xi| \tag{17}$$

almost everywhere on Γ.

The system of functions $\{q_{n,l}\}$, $n = 0, 1, 2, \ldots$, is an orthonormal system of functions; therefore, by (17),

$$\int_\Gamma (q_{i,l}\alpha_{j,l} f)(\xi)\xi^l d\xi = s_{j,l}\delta_{i,j}, \qquad i, j = 0, 1, 2, \ldots,$$

where $\delta_{i,j}$ is the Kronecker symbol.

Thus, the following formula holds for the product of singular numbers

$$s_{0,l}s_{1,l}\cdots s_{k,l} = \left| \int_\Gamma (q_{i,l}\alpha_{j,l}f)(\xi)\xi^l d\xi \right|_{i,j=0}^{k} \qquad k = 0,1,2,\ldots, \tag{18}$$

(the right-hand side is a determinant of order $k+1$).

We mention also that the relations (16) and (17), together with $||q_{n,l}||_{2,l} = 1$, $n = 0,1,2,\ldots$, imply that $||\alpha_{n,l}||_{2,l} = 1$ for all $n = 0,1,\ldots$.

§3. Rational Approximation of Functions Having Finitely Many Essential Singularities

The Statement of Theorem 3

In this section we introduce and study a situation more general than that presented above. Here it will be assumed that the function to be approximated has $s \geq 1$ essential singularities of finite order in the extended complex plane.

Theorem 3. *Suppose that E is an arbitrary compact set in $\mathbf{C}$, f is a holomorphic function on $\bar{\mathbf{C}}\backslash\{a_1,\ldots,a_s\}$, $a_i \in \mathbf{C}\backslash E$, $i = 1,\ldots,s-1$, $a_s \in \overline{\mathbf{C}}\backslash E$, and the point a_i, $i = 1,\ldots,s$, is an essential singularity of f of finite order $\sigma_i \geq 0$. Then*

(i)

$$\limsup_{n\to\infty} \frac{\ln(\rho_{n,m(n)}\rho_{n-1,m(n)-1}\cdots\rho_{n-m(n),0})}{} \leq -\frac{(1+\theta)^2}{} \qquad \frac{}{4\theta\sigma_s}$$

for $a_s = \infty$ and $\sigma_s/(\sigma_1+\cdots+\sigma_s) > (1-\theta)/(1+\theta)$;

(ii)

$$\limsup_{n\to\infty} \frac{\ln(\rho_{n,m(n)}\rho_{n-1,m(n)-1}\cdots\rho_{n-m(n),0})}{nm(n)\ln n} \leq -\frac{\theta}{\sigma_1+\cdots+\sigma_{s-1}}$$

(iii)

$$\limsup_{n\to\infty} \frac{\ln(\rho_{n,m(n)}\rho_{n-1,m(n)-1}\cdots\rho_{n-m(n),0})}{nm(n)\ln n} \leq -\frac{\theta}{\sigma_1+\cdots+\sigma_s}$$

Here and what follows we will use the following notation. Denote by $1/\sigma$ the expression

$$\frac{}{4\theta(\sigma_1+\cdots+\sigma_s)} \qquad \frac{(1-\theta)^2}{4\theta\sigma_s}$$

for $a_s \quad \infty$ and $\sigma_s/(\sigma_1 + \cdots + \sigma_s) > (1 \quad \theta)/(1+\theta)$,

$$\frac{\theta}{\sigma_1 + \cdots + \sigma_{s-1}}$$

for $a_s \quad \infty$ and $\sigma_s/(\sigma_1 + \cdots + \sigma_s) \leq (1-\theta)/(1+\theta)$, and

$$\frac{\theta}{\sigma_1 + \cdots + \sigma_s}$$

for $a_s \neq \infty$.

We mention that according to Theorem 3

$$\limsup_{n\to\infty} \frac{\ln(\rho_{n,m(n)}\rho_{n-1,m(n)-1}\cdots\rho_{n-m(n),0})}{nm(n)\ln n} \leq \quad \frac{1}{\sigma}.$$

We have the following consequence to the theorem.

Corollary 5.

$$\liminf_{n\to\infty} \frac{\ln \rho_{n,m(n)}}{n \ln n} < \quad \frac{2}{(2 \quad \theta)\sigma}$$

From Theorem 3 and the fact that the sequence $\{\rho_{n,m(n)}\}$, $n \quad 0, 1, \ldots,$ is nonincreasing, we immediately have the following.

Corollary 6.

$$\limsup_{n\to\infty} \frac{\ln \rho_{n,m(n)}}{} < \quad \frac{1}{\sigma}$$

The next corollary concerns functions for which equality holds in the last relation.

Corollary 7. *If*

$$\limsup_{n\to\infty} \frac{\ln \rho_{n,m(n)}}{n \ln n} \qquad \frac{1}{\sigma}$$

then

$$\liminf_{n\to\infty} \frac{\ln \rho_{n,m(n)}}{n \ln n} \leq - \frac{1}{(1-\theta)\sigma}$$

It is not hard to see from the proof of Theorem 3 that this theorem is valid under more general assumptions on f. Namely, it can be assumed that the point $a_i, i = 1, \ldots, s$, is an essential singularity of f of finite order no greater then σ_i.

Let us mention the scheme of proof of Theorem 3. It will be assumed that $\sigma_i > 0$ for $i = 1, \ldots, s$. The general case can be obtained from this case with help of the corresponding limit transition $\sigma_i \to 0$. In Subsections 3.2 and 3.3 we consider the situation when the function to be approximated has exactly one essential singularity of finite order at infinity, and by this we actually prove Theorem 1. In Subsection 3.4 we apply the results obtained in Subsections 3.2 and 3.3 to consider the general case.

Before proving Theorem 3 we note that the diagonal case when $m(n) = n, n = 0, 1, 2, \ldots$, was investigated in the paper [13]. In this case the following estimate is valid:

$$\limsup_{n\to\infty} \frac{\ln(\rho_{n,n}\rho_{n-1,n-1}\cdots\rho_{0,0})}{n^2 \ln n} \leq -\frac{1}{\sigma_1 + \cdots + \sigma_s} \tag{19}$$

3.2. The Case When f is an Entire Function of Finite Order

In this subsection we shall start with the assumption that E is an arbitrary compact set in $\mathbf{C}$, and f is an entire function of finite order σ.

In order to continue with the proof of the corresponding assertion, let us begin with some remarks.

First, seeing that $\rho_{n,m} = \rho_{n,m}(f;E)$ is nondecreasing as the compact set E expands ($\rho_{n,m}(f;E) \leq \rho_{n,m}(f;E')$ for $E \subseteq E'$), we can assume that the complement G of E is connected, and E is bounded by finitely many disjoint closed analytic Jordan curves Γ.

Second, with help of an appropriate linear fractional transformation we can reduce the original theorem to the situation when E contains ∞ and f has exactly one essential singularity $a = 0$, $0 \in G = \bar{\mathbf{C}} \setminus E$, of finite order σ.

Precisely, for this situation we prove that

$$\limsup_{n\to\infty} \frac{\ln(\rho^*_{n,m(n)}\rho^*_{n-1,m(n)-1}\cdots\rho^*_{n-m(n),0})}{nm(n)\ln n} \leq -\frac{1}{\sigma}, \tag{20}$$

where

$$\rho^*_{n-j,m(n)-j} = \rho^*_{n-j,m(n)-j}(f;E) = \inf_{r\in\mathcal{R}^*_{n-j,m(n)-j}} \|f - r\|_E,$$
$$j = 0, 1, \ldots, m(n).$$

Here and what follows we will use the following notation:

$$\mathcal{R}^*_{n-j,m(n)-j} = \{r : r = p/qz^{n-m(n)}, \deg p \leq n-j, \deg q \leq m(n) - j, q \not\equiv 0\}.$$

In this subsection we prove the inequality

$$\limsup_{n\to\infty} \frac{\ln(\Delta_{n,m(n)}\Delta_{n-1,m(n)-1}\cdots\Delta_{n-m(n),0})}{nm(n)\ln n} \leq -\frac{1}{\sigma}, \tag{21}$$

where

$$\Delta_{n-j,m(n)-j} = \Delta_{n-j,m(n)-j}(f;G) = \inf_h \|f - h\|_\infty, \quad j = 0, 1, \ldots, m(n),$$

is the best approximation of f in $L_\infty(\Gamma)$ in the class $\mathcal{M}_{n-j,m(n)-j}$ of functions h such that $h = p/qz^{n-m(n)}$, $p \in E_\infty(G)$ and q is a polynomial of degree at most $m(n) - j$, $q \not\equiv 0$.

For this it suffices to show (see the estimate (8)) that

$$\limsup_{n\to\infty} \frac{\ln(s_{0,n-m(n)}s_{1,n-m(n)}\cdots s_{m(n),n-m(n)})}{nm(n)\ln n} \le -\frac{1}{\sigma}, \tag{22}$$

where $\{s_{k,n-m(n)}\}$, $s_{k,n-m(n)} = s_{k,n-m(n)}(f;G)$, $k = 0,1,2,\ldots$, is the sequence of singular numbers of the Hankel operator $A_f : E_2(G) \to H^{\perp}_{n-m(n)}$, constructed from the function f, and $H_{n-m(n)}$ is the class of functions q representable in the form $q = \varphi/\xi^{n-m(n)}$, where $\varphi \in E_2(G)$.

It is not difficult to pass from the estimate (21) to (20) (see Subsection 3.3); therefore, we now restrict ourselves to proving the inequality (22).

We fix an arbitrary domain G_1, $\overline{G}_1 \subset G$, $0 \in G_1$, bounded by a finite number of closed analytic Jordan curves, and $\sigma' > \sigma$. Let $t_n = n^{-1/\sigma'}$. Denote by l_n the circle of radius t_n with the center at 0. It will be assumed that l_n is positively oriented with respect to the open disk of radius t_n about 0 and n is a sufficiently large positive integer, $n \ge n_0$, such that the closed disk of radius t_n with center at 0 belongs to G_1.

Let us use (18), with $k = m(n)$, $l = n - m(n)$. Since the functions $q_{i,n-m(n)}, \alpha_{j,n-m(n)}$, $i,j = 0,1,2,\ldots$, belong to $E_2(G)$ and f is holomorphic on and outside the circle l_n, the relation

$$\begin{aligned} &s_{0,n-m(n)}s_{1,n-m(n)}\cdots s_{m(n),n-m(n)} \\ &\qquad = \left| \int_{l_n} (q_{i,n-m(n)}\alpha_{j,n-m(n)}f)(\xi)\xi^{n-m(n)}\,d\xi \right|_{i,j=0}^{m(n)} \end{aligned}$$

can be written for the product of singular numbers. From the last relation

$$\begin{aligned} &(m(n)+1)!s_{0,n-m(n)}s_{1,n-m(n)}\cdots s_{m(n),n-m(n)} \\ &\quad = \int_{l_n}\cdots\int_{l_n} f(\xi_0)\cdots f(\xi_{m(n)})B_1(\xi_0, \quad ,\xi_{m(n)})B_2(\xi_0,\ldots,\xi_{m(n)}) \\ &\qquad \times \xi_0^{n-m(n)}\cdots\xi_{m(n)}^{n-m(n)}\,d\xi_0\cdots d\xi_{m(n)}, \end{aligned} \tag{23}$$

where

$$B_1(\xi_0,\xi_1,\ldots,\xi_{m(n)}) = \left|\alpha_{j,n-m(n)}(\xi_i)\right|_{i,j=0}^{m(n)} \tag{24}$$

and

$$B_2(\xi_0,\xi_1,\ldots,\xi_{m(n)}) = \left|q_{j,n-m(n)}(\xi_i)\right|_{i,j=0}^{m(n)}. \tag{25}$$

We estimate the determinants B_1 and B_2. By the Cauchy formula,

$$\alpha^2_{j,n-m(n)}(\xi)\xi^{n-m(n)} = \frac{1}{2\pi i}\int_{\Gamma} \frac{\alpha^2_{j,n-m(n)}(t)t^{n-m(n)}}{t-\xi}\,dt,\ \xi \in \overline{G}_1,$$
$$j = 0,1,2,\ldots$$

Since

$$\int_{\Gamma} |\alpha_{j,n-m(n)}(t)|^2 |t|^{n-m(n)} |dt| = 1, \quad j = 0, 1, 2, \ldots,$$

it follows from the last formula that

$$\max_{\xi \in \overline{G}_1} |\alpha^2_{j,n-m(n)}(\xi) \xi^{n-m(n)}| \leq C, \qquad j = 0, 1, 2, \ldots \tag{26}$$

(here and in what follows $C, C_1, C_2, \ldots$ will denote positive quantities not depending on n).

Similarly, since

$$\int_{\Gamma} |q_{j,n-m(n)}(t)|^2 |t|^{n-m(n)} |dt| = 1, \quad j = 0, 1, 2, .$$

it follows that

$$\max_{\xi_i \in \overline{G}_1} |q^2_{j,n-m(n)}(\xi) \xi^{n-m(n)}| \leq C, \quad j = 0, 1, 2, \ldots. \tag{27}$$

Using the inequalities (26) and (27), we can write

$$\begin{aligned} \max_{\xi_i \in \overline{G}_1} &|B_1(\xi_0, \ldots, \xi_{m(n)}) \times B_2(\xi_0, \ldots, \xi_{m(n)}) \xi_0^{n-m(n)} \cdots \xi_{m(n)}^{n-m(n)}| \\ &\leq ((m(n)+1)!)^2 C^{m(n)+1} \end{aligned} \tag{28}$$

For $\xi \in G_1$ let $g(z, \xi)$ be the Green's function of the domain G_1 with singularity at ξ. We estimate the product $B_1 B_2 \xi_0^{n-m(n)} \cdots \xi_n^{n-m(n)}$, in the case when the variables ξ_i, $i = 0, \ldots, m(n)$, belong to l_n. The next equality easily follows from (24) and (25):

$$\begin{aligned} &D(\xi_0, \ldots, \xi_{m(n)}) \\ &\quad := B_1(\xi_0, \ldots, \xi_{m(n)}) B_2(\xi_0, \ldots, \xi_{m(n)}) \xi_0^{n-m(n)} \cdots \xi_{m(n)}^{n-m(n)} \\ &\quad = \prod_{0 \leq i < j \leq m(n)} (\xi_i - \xi_j)^2 \cdot \Psi(\xi_0, \xi_1, \ldots, \xi_{m(n)}) \xi_0^{n-m(n)} \cdots \xi_{m(n)}^{n-m(n)} \end{aligned}$$

where the function $\Psi(\xi_0, \xi_1, \ldots, \xi_{m(n)})$ is a holomorphic function of $m(n)+1$ complex variables in the domain $G \times \cdots \times G$ (with $m(n)+1$ factors in the Cartesian product).

Consider now the function

$$\ln |D(\xi_0, \xi_1, \ldots, \xi_{m(n)})| + 2 \sum_{0 \leq i < j \leq m(n)} g(\xi_i, \xi_j) + (n - m(n)) \sum_{i=0}^{m(n)} g(\xi_i, 0).$$

This function is subharmonic in the domain G_1 with respect to the variable ξ_i, $i = 0, \ldots, m(n)$, when the remaining variables $\xi_j \in G_1$, $j \neq i$, $j \in \{0, 1, \ldots, m(n)\}$, are fixed.

We now employ the maximum principle for subharmonic functions successively with respect to each variable, together with the inequality (28), to get

$$\begin{aligned}\ln |D(\xi_0, \xi_1, \ldots, \xi_{m(n)})| + 2 \sum_{0 \le i < j \le m(n)} g(\xi_i, \xi_j) \\ + (n - m(n)) \sum_{i=0}^{m(n)} g(\xi_i; 0) \\ \le \ln(((m(n)+1)!)^2 C^{m(n)+1}),\end{aligned}$$

where $\xi_i \in l_n$, $i = 0, 1, \ldots, m(n)$.

In view of the formula for a product of singular numbers (see (23)), the last inequality implies

$$\begin{aligned}& s_{0,n-m(n)} s_{1,n-m(n)} \cdots s_{m(n),n-m(n)} \\ & \le (m(n)+1)! C_1^{m(n)} \left(\max_{\xi \in l_n} |f(\xi)| \right)^{m(n)+1} \exp(-w_n),\end{aligned} \tag{29}$$

where

$$w_n = \min_{\xi_i \in l_n} \left(2 \sum_{0 \le i < j \le m(n)} g(\xi_i, \xi_j) + (n - m(n)) \sum_{i=0}^{m(n)} g(\xi_i, 0) \right) \tag{30}$$

We now use the following representation of the Green's function

$$g(z, \xi) = \ln \frac{1}{|z - \xi|} + d(\xi) + u(z, \xi), \qquad z, \xi \in G_1. \tag{31}$$

where $d(\xi)$ is a quantity dependent on ξ, and $u(z, \xi)$ is a function harmonic in G_1 with respect to z, $u(\xi, \xi) = 0$, to establish a lower estimate of w_n. It can be shown that

$$d(\xi) \to d(\zeta) \quad \text{as} \quad \xi \to \zeta, \quad \zeta \in G_1,$$

and

$$u(z, \xi) \to 0,$$

if both the variables z and ξ tend to a point $\zeta \in G_1$.

We will consider the points $\xi_{0,n}, \xi_{1,n}, \ldots, \xi_{m(n),n}$, where the expression (30) takes its minimum on l_n. It is not hard to see that all the points $\xi_{i,n}$ are distinct. Taking into account (31), we get

$$
\begin{aligned}
w_n &= 2 \sum_{0 \le i < j \le m(n)} g(\xi_{i,n}, \xi_{j,n}) + (n - m(n)) \sum_{i=0}^{m(n)} g(\xi_{i,n}, 0) \\
&= 2 \sum_{0 \le i < j \le m(n)} \ln \frac{1}{|\xi_{i,n} - \xi_{j,n}|} + (n - m(n)) \sum_{i=0}^{m(n)} \ln \left| \frac{1}{\xi_{i,n}} \right| \qquad (32) \\
&\quad m(n) \\
&\quad + 2 \sum_{0 \le i < j \le m(n)} u(\xi_{i,n}, \xi_{j,n}) + (n - m(n)) \sum_{i=0}^{m(n)} u(\xi_{i,n}, 0).
\end{aligned}
$$

Next we investigate how the last expression in (32) changes as $n \to \infty$. We know that the radius t_n tends to zero; therefore

$$
\frac{1}{n(m(n)+1)} \left(2 \sum_{i=1}^{m(n)} id(\xi_{i,k}) + (n - m(n))(m(n)+1) d(0) \right. \to d(0)
$$

and

$$
\frac{1}{n(m(n)+1)} \left(2 \sum_{0 \le i < j \le m(n)} u(\xi_{i,n}, \xi_{j,n}) + (n - m(n)) \sum_{i=0}^{n} u(\xi_{i,n}, 0) \right) \to 0
$$

$$
w_n = 2 \sum_{0 \le i < j \le m(n)} \ln \frac{1}{|\xi_{i,n} - \xi_{j,n}|} + (n - m(n)) \sum_{i=0}^{m(n)} \ln \frac{1}{|\xi_{i,n}|} + n(m(n)+1) d_n,
$$

where $d_n \to d(0)$ as $n \to \infty$. From this it follows that

$$
\begin{aligned}
w_n \ge \min_{\xi_i \in l_n} 2 \sum_{0 \le i < j \le m(n)} \ln \frac{1}{|\xi_i - \xi_j|} + (n - m(n)) \sum_{i=0}^{m(n)} \ln \frac{1}{|\xi_{i,n}|} \\
+ n(m(n)+1) d_n.
\end{aligned}
$$

The estimate of the first term on the right-hand side of (33) proceeds in several steps. First, from the representations $\xi_i = t_n x_i$ for any point $\xi_i \in l_n$, where $|x_i| = 1$, we compute that

$$
\begin{aligned}
\min_{\xi_i \in l_n} \sum_{0 \le i < j \le m(n)} \ln \frac{1}{|\xi_i - \xi_j|} &= \frac{m(n)(m(n)+1)}{2} \ln \frac{1}{|t_n|} \\
&\quad + \min_{|x_i|=1} \sum_{0 \le i < j \le m(n)} \ln \frac{1}{|x_i -
\end{aligned}
$$

Second, taking into account of

$$\min_{|x_i|=1} \sum_{0\le i<j\le m(n)} \ln \frac{1}{|x_i - x_j|} \ge \ln \frac{1}{(m(n)+1)!},$$

we obtain

$$w_n \ge n(m(n)+1)\ln\frac{1}{|t_n|} + n(m(n)+1)d_n + 2\ln\cdot\frac{1}{(m(n)+1)!} \tag{34}$$

Thus, the estimate (see (29), (34))

$$\begin{aligned} &s_{0,n-m(n)}s_{1,n-m(n)}\cdots s_{m(n),n-m(n)} \\ &\quad\le ((m(n)+1)!)^3 C_1^{m(n)} \left(\max_{\xi\in l_n}|f(\xi)|\right)^{m(n)+1} \\ &\quad\times \exp(-n(m(n)+1)d_n)|t_n|^{n(m(n)+1)} \end{aligned} \tag{35}$$

holds for $n \ge n_0$.

Recall that we have assumed that f has an essential singularity at a of order σ; this implies that

$$\max_{\xi\in l_n}|f(\xi)| \le e^{1/t_n^{\sigma'}} = e^n \tag{36}$$

for sufficiently large n, $n \ge n_1$.

Finally, we obtain from (35), (36), and the equality $t_n = n^{-1/\sigma'}$, that

$$\limsup_{n\to\infty} \frac{\ln(s_{0,n-m(n)}s_{1,n-m(n)}\cdots s_{m(n),n-m(n)})}{nm(n)\ln n} \le -\frac{1}{\sigma'}. \tag{37}$$

Letting $\sigma' \to \sigma$ on the right-hand side of the inequality (37) we obtain (22), which implies the inequality (21).

3.3. Proof of the Inequality (20)

We now apply the estimate (21) to get the inequality (20).

Fix an arbitrary domain G_1, $\bar{G}_1 \subset G$, $0 \in G_1$. We assume that the boundary Γ_1 of the domain G_1 consists of disjoint closed analytic Jordan curves; moreover, we assume that Γ_1 is positively oriented with respect to G_1.

Fix also nonnegative integers n and j, $0\le j\le m(n)$. Using the Cauchy formula for an arbitrary function h representable in the form $h = p/(qz^{n-m(n)})$, where $p \in E_\infty(G_1)$, q is a polynomial of degree at most $m(n) - j$, with zeros outside Γ_1, $q \not\equiv 0$, we obtain

$$(r' - f)(z) + f(\infty) = \frac{1}{2\pi i}\int_{\Gamma_1} \frac{(f-h)(\xi)d\xi}{\xi - z} \qquad z \in E, \tag{38}$$

where r' is the sum of the principal parts of h corresponding to poles of h lying in G_1.

Consider the integral on the left-hand side. We have that

$$||f - f(\infty) - r'||_E \leq C||f - h||_\infty, \tag{39}$$

where the positive quantity C is independent of h, n , and j, and $||\cdot||_\infty$ is the norm in the space $L_\infty(\Gamma_1)$.

From the definition of the quantity $\rho^*_{n-j,m(n)-j}$ and the fact that the rational function $r' + f(\infty)$ belongs to the class $\mathcal{R}^*_{n-j,m(n)-j}$, the inequality (39) becomes

$$\rho^*_{n-j,m(n)-j} \leq C||f - h||_\infty.$$

Now, using the fact that h is an arbitrary function in $\mathcal{M}_{n-j,m(n)-j}(G_1)$ we obtain

$$\rho^*_{n-j,m(n)-j} \leq C \inf_{h \in \mathcal{M}_{n-j,m(n)-j}} ||f - h||_\infty = C\Delta_{n-j,m(n)-j}(f; G_1).$$

Following the results in Subsection 3.2 (see the relation (21)), applied to the region G_1, we get

$$\limsup_{n\to\infty} \frac{\ln(\rho^*_{n,m(n)}\rho^*_{n-1,m(n)-1}\cdots\rho^*_{n-m(n),0})}{nm(n)\ln n} \leq -\frac{1}{\sigma}.$$

Thus, Theorem 3 has been proved for the case when the function being approximated is an entire function of finite order.

3.4. The Case When the Number of Singularities $s \geq 2$

Assume that the number s of singularities is ≥ 2. For $i = 1, \ldots, s-1$, let

$$f_i(z) = \varphi_i(z), \qquad z \in \bar{\mathbb{C}} \backslash \{a_i\},$$

where φ_i is the principal part of the Laurent expansion of f in a neighborhood of a_i; the function $f - (\varphi_1 + \cdots + \varphi_{s-1})$ can be extended to an analytic function f_s on $\bar{\mathbb{C}}\backslash\{a_s\}$. It is not hard to see that $f = f_1 + f_2 + \cdots + f_s$, where each of the functions f_i is holomorphic in $\bar{\mathbb{C}}\backslash\{a_i\}$, and a_i is an essential singularity of order σ_i for f_i, $i = 1, \ldots, s$.

It will be assumed that $\theta > 0$. For $\theta = 0$ the corresponding assertion is obvious.

First of all we consider the case when $a_s = \infty$.

The main goal of this subsection is to obtain an upper estimate of the product

$$\prod_{j=0}^{m(n)} \rho_{n-j,m(n)-j}(f; E)$$

by means of the product of the quantities $\rho_{j,j}(f_i; E)$, $i = 1, 2, \ldots, s-1$, and $\rho_{n-m(n)+j,j}(f_s; E)$.

Lemma 3. *For any nonnegative integer n, there exist nonnegative integers $\chi_i(n)$, $i = 1, 2, \ldots, s$, such that $\chi_1(n) + \chi_2(n) + \cdots + \chi_s(n) = m(n) + 1$ and*

$$\prod_{j=0}^{m(n)} \rho_{n-j,m(n)-j}(f;E) \le s^{m(n)+1} \prod_{i=1}^{s-1} \prod_{j=0}^{\chi_i(n)-1} \rho_{j,j}(f_i;E) \times \prod_{j=0}^{\chi_s(n)-1} \rho_{n-m(n)+j,j}(f_s;E), \tag{40}$$

(let $\prod_{j=0}^{\chi_i(n)-1} \rho_{j,j}(f_i;E) = 1$, in the product (40) if $\chi_i(n) = 0$ and respectively $\prod_{j=0}^{\chi_s(n)-1} \rho_{n-m(n)+j,j}(f_s;E) = 1$ if $\chi_s(n) = 0$).

Before proving the lemma, for any nonnegative integer n we define the quantities

$$\mu^*_{n-m(n)+j,j} = \min_{k_i} \left(\max \left(\max_{1 \le i \le s-1} \rho_{k_i,k_i}(f_i;E), \rho_{n-m(n)+k_s,k_s}(f_s;E) \right) \right) \quad j = 0, 1, 2, \ldots,$$

where the minimum is over all tuples of nonnegative integers k_i, $i = 1, \ldots, s$, such that $k_1 + k_2 + \cdots + k_s \le j$.

Proof of Lemma 3: Let us show that for any nonnegative integers n and j,

$$\rho_{n-m(n)+j,j}(f;E) \le s\mu^*_{n-m(n)+j,j}. \tag{41}$$

For this purpose, choose arbitrary nonnegative integers k_i, $i = 1, \ldots, s$, such that $k_1 + k_2 + \cdots + k_s \le j$. Also, for each $i \in \{1, \ldots, s-1\}$ choose an arbitrary rational function r_i in $\mathcal{R}_{k_i,k_i}$. Let r_s be an arbitrary rational function in $\mathcal{R}_{n-m(n)+k_s,k_s}$ and $r = r_1 + r_2 + \cdots + r_s$. Since $k_1 + k_2 + \cdots + k_s \le j$, $r \in \mathcal{R}_{n-m(n)+j,j}$. We have

$$\rho_{n-m(n)+j,j}(f;E) \le \|f - r\|_E \le \sum_{i=1}^{s} \|f_i - r_i\|_E \tag{42}$$

We now use the fact that r_i, $i = 1, 2, \ldots, s-1$, is an arbitrary rational function in $\mathcal{R}_{k_i,k_i}$ and r_s is an arbitrary rational function in $\mathcal{R}_{n-m(n)+k_s,k_s}$. From (42),

$$\rho_{n-m(n)+j,j}(f;E) \le \sum_{i=1}^{s-1} \rho_{k_i,k_i}(f_i;E) + \rho_{n-m(n)+k_s,k_s}(f_s;E).$$

In turn, we obtain the inequality (41) from the last inequality and the fact that $k_1, \ldots, k_s$ are arbitrary nonnegative integers with $k_1 + k_2 + \cdots + k_s \le j$.

We remark that, from (41),

$$\prod_{j=0}^{m(n)} \rho_{n-m(n)+j,j}(f;E) \le s^{m(n)+1} \prod_{j=0}^{m(n)} \mu^*_{n-m(n)+j,j}. \tag{43}$$

The desired estimate (40) is an immediate consequence of the inequality (43) and an assertion about numerical sequences. We state the corresponding assertion (see [13]).

Suppose that we are given s nonincreasing sequences of nonnegative numbers $\{a_{j,i}\}_{j=0}^{\infty}$, $\lim_{j\to\infty} a_{j,i} = 0$, $i = 1, 2, \ldots, s$. Let

$$a_j^* = \min_{k_i} \left(\max_{1\le i\le s} (a_{k_i,i}) \right) \qquad j = 0, 1, 2, \ldots,$$

where min is taken over all possible tuples of nonnegative integers k_i, i $1, \ldots, s$, such that $k_1 + k_2 + \cdots + k_s \le j$.

Lemma 4. *For any nonnegative integer k there exist nonnegative integers $\chi_i(n)$, $i = 1, 2, \ldots, s$, such that $\chi_1(n) + \chi_2(n) + \cdots + \chi_s(n) = k + 1$ and*

$$a_0^* a_1^* \cdots a_k^* \le \prod_{i=1}^{s} \prod_{j=0}^{\chi_i(n)-1} a_{j,i}$$

(let $\prod_{j=0}^{\chi_i(n)-1} a_{j,i} = 1$ in the product if $\chi_i(n) = 0$).

Fix a nonnegative integer n. Using Lemma 4, with nonincreasing sequences $\{\rho_{j,j}(f_i; E)\}_{j=0}^{\infty}$, $i = 1, \ldots, s-1$, $\{\rho_{n-m(n)+j,j}(f_s; E)\}_{j=0}^{\infty}$, and $k = m(n)$, by (43), we get (40). □

According to Lemma 3,

$$\prod_{j=0}^{m(n)} \rho_{n-j,m(n)-j}(f; E) \le s^{m(n)+1} \prod_{i=0}^{s-1} \prod_{j=0}^{\chi_i(n)-1} \rho_{j,j}(f_i; E) \times \prod_{j=0}^{\chi_s(n)-1} \rho_{n-m(n)+j,j}(f_s; E), \tag{44}$$

where $\chi_1(n) + \cdots + \chi_s(n) = m(n) + 1$.

We take a sequence Λ of positive integers such that

$$\lim_{n\to\infty, n\in\Lambda} \frac{\ln \prod_{j=0}^{m(n)} \rho_{n-j,m(n)-j}}{nm(n)\ln n} = \limsup_{n\to\infty} \frac{\ln \prod_{j=0}^{m(n)} \rho_{n-j,m(n)-j}}{nm(n)\ln n} \tag{45}$$

and

$$\lim_{n\to\infty, n\in\Lambda} \frac{\chi_i(n)}{m(n)} = \omega_i, \qquad i = 1, 2, \ldots, s. \tag{46}$$

We note that $\omega_i \ge 0$ and $\omega_1 + \omega_2 + \cdots + \omega_s = 1$.

Since for all $i \in \{1, \ldots, s-1\}$, a_i is an essential singularity of f_i of order σ_i, it follows from (19), with $s = 1$, that

$$\limsup_{n\to\infty, n\in\Lambda} \frac{\ln(\prod_{j=0}^{\chi_i(n)-1} \rho_{j,j}(f_i; E))}{nm(n)\ln n} \le -\frac{\omega_i^2 \theta}{\sigma_i}, \qquad i = 1, 2, \ldots, s-1$$

By the relations (2), (4), and (46), we have

$$\lim_{n\to\infty, n\in\Lambda} \frac{\ln \prod_{j=0}^{\chi_s(n)-1} \rho_{n-m(n)+j,j}(f_s;E)}{nm(n)\ln n} \le -\frac{\omega_s(1-\theta+\omega_s\theta)}{\sigma_s}.$$

From the last inequalities with the aid of (44) and (45) we obtain that

$$\begin{aligned}\limsup_{n\to\infty} \frac{\ln \prod_{j=0}^{m(n)} \rho_{n-j,m(n)-j}}{nm(n)\ln n} &< -\left(\sum_{i=1}^{s-1} \frac{\omega_i^2\theta}{\sigma_i} + \frac{\omega_s(1-\theta+\omega_s\theta)}{\sigma_s}\right) \\ &= -\left(\sum_{i=1}^{s} \frac{\omega_i^2\theta}{\sigma_i} + \frac{\omega_s(1-\theta)}{\sigma_s}\right).\end{aligned} \tag{47}$$

To estimate the last expression we use the following simple assertion. This assertion can be obtained, for example, by the method of Lagrange multipliers.

Lemma 5. *If $\theta_i \ge 0$, $i=1,2,\dots,s$, and $\theta_s/(\theta_1+\dots+\theta_s) > (1-\theta)/(1+\theta)$, then*

$$\frac{(1+\theta)^2}{4\theta^2\sum_{i=1}^{s}\theta_i} - \frac{(1-\theta)^2}{4\theta^2\theta_s} \le \sum_{i=1}^{s}\frac{\omega_i^2}{\theta_i} + \frac{\omega_s(1-\theta)}{\theta\theta_s}$$

for all $\omega_i \ge 0$, $i=1,2,\dots,s$, $\sum_{i=1}^{s}\omega_i = 1$. If $\theta_i \ge 0$ $i=1,2,\dots,s$ and $\theta_s/(\theta_1+\dots+\theta_s) \le (1-\theta)/(1+\theta)$, then

$$\frac{1}{\sum_{i=1}^{s-1}\theta_i} \le \sum_{i=1}^{s}\frac{\omega_i^2}{\theta_i} + \frac{\omega_s(1-\theta)}{\theta\theta_s}$$

for all $\omega_i \ge 0$, $i=1,2,\dots,s$, $\sum_{i=1}^{s}\omega_i = 1$.

It remains to employ Lemma 5 with σ_i/θ $(i=1,\dots,s)$ instead of θ_i, use the inequality (47), and get the required relation

$$\limsup_{n\to\infty} \frac{\ln(\rho_{n,m(n)}\rho_{n-1,m(n)-1}\cdots\rho_{n-m(n),0})}{nm(n)\ln n} \le -\frac{(1+\theta)^2}{4\theta(\sigma_1+\quad+\sigma_s)} + \frac{(1-\theta)^2}{4\theta\sigma_s}$$

for the case when $\sigma_s/(\sigma_1+\dots+\sigma_s) > (1-\theta)/(1+\theta)$.

In the situation when $\sigma_s/(\sigma_1+\dots+\sigma_s) \le (1-\theta)/(1+\theta)$ we have

$$\limsup_{n\to\infty} \frac{\ln(\rho_{n,m(n)}\rho_{n-1,m(n)-1}\cdots\rho_{n-m(n),0})}{nm(n)\ln n} \le -\frac{\theta}{\sigma_1+\dots+\sigma_{s-1}}$$

We now consider the case $a_s \ne \infty$. Since

$$\prod_{j=0}^{m(n)} \rho_{n-j,m(n)-j} \le \prod_{j=0}^{m(n)} \rho_{m(n)-j,m(n)-j} = \prod_{j=0}^{m(n)} \rho_{j,j},$$

we get with help of (2) and (19),

$$\limsup_{n\to\infty} \frac{\ln(\prod_{j=0}^{m(n)} \rho_{n-j,m(n)-j})}{nm(n)\ln n} < -\frac{\theta}{\sigma_1+\ldots+\sigma_s}$$

Theorem 3 is proved.

Proof of the Corollaries to Theorem 3

We start with the proofs of the corollaries to Theorem 3. Here

$$\frac{1}{\sigma} = \frac{(1+\theta)^2}{4\theta(\sigma_1 + \ldots + \sigma_s)} \quad \frac{(1-\theta)^2}{4\theta\sigma_s}$$

for $a_s = \infty$ and $\sigma_s/(\sigma_1 + \cdots + \sigma_s) > (1-\theta)/(1+\theta)$,

$$\frac{1}{\sigma} = \frac{\theta}{\sigma_1 + \cdots + \sigma_{s-1}}$$

for $a_s = \infty$, $\sigma_s/(\sigma_1 + \cdots + \sigma_s) \leq (1-\theta)/(1+\theta)$, and

$$\frac{1}{\sigma} = \frac{\theta}{\sigma_1 + \ldots + \sigma_s}$$

for $a_s \neq \infty$. It will be assumed that $\theta > 0$ and $\sigma > 0$. For $\theta = 0$ and $\sigma = 0$ the corresponding assertions are obvious.

To prove Corollary 5, suppose to the contrary that

$$\liminf_{n\to\infty} \frac{\ln \rho_{n,m(n)}}{n \ln n} \geq -\frac{2}{2-\theta} \cdot \frac{\lambda}{\sigma} \tag{48}$$

where $0 < \lambda < 1$.

It follows from the relation (1) that

$$m(n) - j \leq m(n-j), \quad j = 0, 1, \quad m(n)$$

Therefore, we have

$$\rho_{n-j,m(n-j)} \leq \rho_{n-j,m(n)-j}, \quad j = 0, \quad \ldots, m(n)$$

and

$$\prod_{k=n-m(n)}^{n} \rho_{k,m(k)} = \prod_{j=0}^{m(n)} \rho_{n-j,m(n-j)} \leq \prod_{j=0}^{m(n)} \rho_{n-j,m(n)-j}. \tag{49}$$

It follows from Theorem 3 that

$$\limsup_{n\to\infty} \frac{\ln \prod_{j=0}^{m(n)} \rho_{n-j,m(n)-j}}{n m(n) \ln n} \leq -\frac{1}{\sigma} \tag{50}$$

By the relations (48), (49) and

$$\lim_{n\to\infty} \frac{\sum_{k=n-m(n)}^{n} k \ln k}{n m(n) \ln n} = \frac{2-\theta}{2}$$

we get

$$\liminf_{n\to\infty} \frac{\ln \prod_{j=0}^{m(n)} \rho_{n-j,m(n)-j}}{nm(n)\ln n} \geq -\frac{\lambda}{\sigma} > -\frac{1}{\sigma},$$

which contradicts the inequality (50). □

We now prove Corollary 7. Let Λ be a sequence of positive integers such that

$$\lim_{n\to\infty, n\in\Lambda} \frac{\ln \rho_{n,m(n)}}{n\ln n} = -\frac{1}{\sigma}. \tag{51}$$

Fix an arbitrary $1-\theta < \lambda \leq 1$. Denote by $\{k_n\}$, $n = 1, 2, \ldots$, the sequence of integers such that $n - m(n) \leq k_n \leq n$ and $k_n/n \to \lambda$ as $n \to \infty$. Since the sequence $\{\rho_{n,m(n)}\}$, $n = 1, 2, \ldots$, is nonincreasing,

$$\rho_{n,m(n)}^{m(n)+1} \leq \rho_{k_n,m(k_n)}^{k_n-n+m(n)+1} \rho_{n,m(n)}^{n-k_n} \leq \prod_{k=n-m(n)}^{n} \rho_{k,m(k)}$$

From this and from the relations (49), (50), and (51), we get

$$\lim_{n\to\infty, n\in\Lambda} \frac{(k_n - n + m(n) + 1)\ln \rho_{k_n,m(k_n)}}{nm(n)\ln n} = -\frac{1}{\sigma} + \frac{1}{\sigma}\left(\frac{1-\lambda}{\theta}\right)$$

which implies that

$$\lim_{n\to\infty, n\in\Lambda} \frac{\ln \rho_{k_n,m(k_n)}}{k_n \ln k_n} = -\frac{1}{\sigma\lambda}$$

It follows from the last relation that

$$\liminf_{n\to\infty} \frac{\ln \rho_{n,m(n)}}{n \ln n} \leq -\frac{1}{\sigma\lambda}$$

Finally, we let λ tend to $(1-\theta)$ and obtain

$$\liminf_{n\to\infty} \frac{\ln \rho_{n,m(n)}}{n\ln n} \leq -\frac{1}{\sigma(1-\theta)} \qquad \square$$

§4. Proof of Theorem 2

4.1. Proof of Theorem 2

The specific setting of our problem is this. Using the fact that the quantity $\rho_{n,m(n)}(f;E)$ does not decrease with the widening of the compact set E, we assume that the complement of the compact set E is connected (we can replace the compact set E by the compact set $E_1 = \bar{\mathbb{C}} \backslash U$, where U is a the connected component of $\bar{\mathbb{C}} \backslash E$ containing ∞). Moreover, it can be assumed that E is bounded by a finite number of closed analytic Jordan curves Γ. The transition from such compact sets to arbitrary compact sets is not difficult (cf. Subsection 3.3). With help of a corresponding fractional-linear transformation, we reduce the theorem to the situation when E is a compact set in $\overline{\mathbb{C}}$ containing ∞, with the function f having one essential singularity at $0 \in G = \overline{\mathbb{C}} \backslash E$ of finite order $\sigma > 0$ and finite type $\tau > 0$. We must show that

$$\limsup_{n\to\infty} (\rho^*_{n-m(n),0}\rho^*_{n-m(n)+1,1}\cdots\rho^*_{n,m(n)})^{\sigma/n(m(n)+1)} n \leq \sigma e \tau e^{-d(0)\sigma},$$

where

$$\rho^*_{n-j,m(n)-j} = \rho^*_{n-j,m(n)-j}(f;E) = \inf_{r\in\mathcal{R}^*_{n-j,m(n)-j}} \|f-r\|_E,\ j = 0,1,\ldots,m(n),$$

and $d(0)$ is determined from the representation of the Green's function (see (31) with 0 in place of ξ). To do this it suffices to show that

$$\begin{aligned}\limsup_{n\to\infty} ((s_{0,n-m(n)}s_{1,n-m(n)}\quad s_{m(n),n-m(n)})^{\sigma/n(m(n)+1)} n) \\ \leq \sigma e \tau e^{-d(0)\sigma}\end{aligned} \tag{52}$$

where $\{s_{k,n-m(n)}\}$, $s_{k,n-m(n)} = s_{k,n-m(n)}(f;G)$, $k = 0,1,2,\ldots$, is the sequence of singular numbers of the Hankel operator $A_f : E_2(G) \to H^\perp_{n-m(n)}$ constructed from f, $H_{n-m(n)}$ is the space of the function q representable in the form $q = \varphi/\xi^{n-m(n)}$, where $\varphi \in E_2(G)$.

To prove Theorem 2 we employ the same arguments as in the proof of Theorem 1. We only sketch the proof.

Fix an arbitrary domain G_1, $\overline{G}_1 \subset G, 0 \in G_1$, bounded by a finite number of closed Jordan curves, and $\tau' > \tau$. Let $t_n = (\sigma\tau')^{1/\sigma}n^{-1/\sigma}$. Denote by l_n the circle of radius t_n with center at 0. It will be assumed that l_n is positively oriented with respect to the open disk of radius t_n about 0, and n is a sufficiently large positive integer, $n \geq n_0$, such that the closed disk of radius t_n with center at 0 belongs to G_1. The rest of the arguments are analogous to the corresponding arguments in Subsection 3.2. For sufficiently large n it immediately follows from the inequality (35) that

$$\begin{aligned} & s_{0,n-m(n)}s_{1,n-m(n)}\cdots s_{m(n),n-m(n)} \\ & \quad \leq ((m(n)+1)!)^3 C_1^{m(n)} \|f\|_{l_n}^{m(n)+1} \\ & \qquad \times \exp(-n(m(n)+1)d_n)|t_n|^{n(m(n)+1)}\end{aligned} \tag{53}$$

where $d_n \to d(0)$ as $n \to \infty$.

The fact that f has at 0 an essential singularity of order σ and type τ implies

$$\|f\|_{l_n} \leq e^{\tau'/t_n^\sigma} = e^{n/\sigma}$$

for $n \geq n_1$. This allows us to use the formula $t_n = (\sigma\tau')^{1/\sigma}n^{-1/\sigma}$ for the radius, to obtain from (53)

$$\limsup_{n\to\infty} (s_{0,n-m(n)}s_{1,n-m(n)}\cdots s_{m(n),n-m(n)})^{\sigma/n(m(n)+1)} n \leq \sigma e \tau' e^{-d(0)\sigma}$$

It remains to pass to the limit as $\tau' \to \tau$ on the right-hand side of the last inequality and obtain (52). □

2. Proof of the Corollaries to Theorem 2

e now turn our attention to the corollaries to Theorem 2. We assume that p$(E) > 0$ and $\theta > 0$. For $\mathrm{cap}(E) = 0$ and $\theta = 0$ the corresponding assertions e obvious. We first show that the inequality (6) holds. For simplicity we note

$$\sigma e^{1/2}\tau(\mathrm{cap}(E))^{\sigma} e^{-\frac{(1-\theta)^2}{\theta(2-\theta)}\ln(1-\theta)}$$

w. Assume that

$$\liminf_{n\to\infty} \rho_{n,m(n)}^{\sigma(2-\theta)/2n} n > \lambda w, \quad \lambda > 1,$$

order to reach a contradiction. Therefore, for sufficiently large positive egers n, $n \geq n_0$, we have

$$\ln \rho_{n,m(n)} \geq \frac{2n}{\sigma(2-\theta)}\ln(\lambda w) - \frac{2n}{\sigma(2-\theta)}\ln n,$$

ich implies (see (49))

$$\begin{aligned}\ln \prod_{j=0}^{m(n)} \rho_{n-j,m(n)-j} &\geq \ln \prod_{k=n-m(n)}^{n} \ln \rho_{k,m(k)} \\ &> \frac{2}{\sigma(2-\theta)}\ln(\lambda w) \sum_{k=n-m(n)}^{n} k \\ &\quad \frac{2}{\sigma(2-\theta)} \sum_{k=n-m(n)}^{n} k\ln k, \quad n - m(n) \geq n_0,\end{aligned}$$

d

$$\begin{aligned}&\ln \prod_{j=0}^{m(n)} \rho_{n-j,m(n)-j} \\ &\geq \frac{(2n-m(n))(m(n)+1)}{\sigma(2-\theta)}\ln(\lambda w) \\ &\quad - \frac{(2n-m(n))(m(n)+1)}{\sigma(2-\theta)}\ln n \\ &\quad - \frac{n(m(n)+1)}{\sigma(2-\theta)}\left(-\frac{(1-\theta)^2}{\theta}\ln(1 \quad \theta) - 1 + \frac{\theta}{2} + \delta_n\right)\end{aligned}$$

ere $\delta_n \to 0$ as $n \to \infty$. From this we obtain

$$\begin{aligned}&\ln \prod_{j=0}^{m(n)} \rho_{n-j,m(n)-j}^{\sigma/n(m(n)+1)} n \\ &\geq \ln(\lambda w)\frac{2-m(n)/n}{2-\theta} + \ln n \frac{m(n)/n-\theta}{2-\theta} \\ &\quad \frac{1}{2-\theta}\left(-\frac{(1-\theta)^2}{\theta}\ln(1-\theta) - 1 + \frac{\theta}{2} + \delta_n\right)\end{aligned}$$

By the relation

$$\lim_{n\to\infty}(m(n)/n-\theta)\ln n=0,$$

we now get the the inequality

$$\liminf_{n\to\infty}\prod_{j=0}^{m(n)}\rho_{n-j,m(n)-j}^{\sigma/n(m(n)+1)}n\ge\lambda\sigma e\tau(\operatorname{cap}(E))^{\sigma}$$

which contradicts the relation (5). □

We now prove Corollary 4. Denote by S the following expression

$$S=\sigma e\tau(\operatorname{cap}(E))^{\sigma}.$$

Let Λ be a sequence of positive integers such that

$$\lim_{n\to\infty,n\in\Lambda}\rho_{n,m(n)}^{\sigma/n}n=S. \tag{54}$$

Fix $1-\theta<\lambda\le 1$. Denote by $\{k_n\}$, $n=1,2,\dots$, the sequence of integers $k_n=[n\lambda]$.

Since the sequence $\{\rho_{n,m(n)}\}$, $n=1,2,\dots$, is nonincreasing, for sufficiently large n we get

$$\rho_{n,m(n)}^{m(n)+1}\le\rho_{k_n,m(k_n)}^{k_n-n+m(n)+1}\rho_{n,m(n)}^{n-k_n}\le\prod_{k=n-m(n)}^{n}\rho_{k,m(k)}$$

From this and from the relation (54), it follows that

$$\lim_{n\to\infty,n\in\Lambda}\left(\rho_{k_n,m(k_n)}^{k_n-n+m(n)+1}\rho_{n,m(n)}^{n-k_n}\right)^{\frac{\sigma}{n(m(n)+1)}}n=S$$

$$\lim_{n\to\infty,n\in\Lambda}\rho_{k_n,m(k_n)}^{\frac{\sigma(k_n-n+m(n)+1)}{n(m(n)+1)}}\,n^{\frac{k_n-n+m(n)+1}{m(n)+1}}=S^{\frac{\theta-1+\lambda}{\theta}}$$

$$\lim_{n\to\infty,n\in\Lambda}\left(\rho_{k_n,m(k_n)}^{\sigma/n}k_n\right)^{\frac{k_n-n+m(n)+1}{m(n)+1}}\left(\frac{n}{\ }\right)^{\frac{k_n-n+m(n)+1}{m(n)+1}}=S^{\frac{\theta-1+\lambda}{\theta}}$$

$$\liminf_{n\to\infty,n\in\Lambda}\rho_{k_n,m(k_n)}^{\lambda\sigma/k_n}k_n\le\lambda S.$$

Then, using the last inequality, we deduce that

$$\liminf_{n\to\infty}\rho_{n,m(n)}^{\lambda\sigma/n}n\le\lambda\sigma e\tau(\operatorname{cap}(E))^{\sigma}\quad\square$$

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