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How can the meromorphic approximation help to solve some 2D inverse problems for the Laplacian?

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Abstract. We exhibit new links between approximation theory in the complex domain and a family of inverse problems for the 2D Laplacian related to non-destructive testing.

1. Introduction

Our aim is to describe a method related to the approximation by analytic and meromorphic functions that allows us to detect, from boundary data, the presence of cracks in a planar domain and to provide information about their location.

Existing procedures for solving non-destructive control problems from either thermal, electric, acoustic, or elastic measurements classically rely on multiple iterative integrations of the involved partial differential equation (PDE); hence, they are highly time consuming and very sensitive to initial guesses. Existing identifiability results and reconstruction algorithms are effective only when *complete overdetermined* data are available, and under strong *a priori* information, for instance when the crack is known to lie on some line, see [2, 8, 13] and the bibliographies therein. This is restrictive in two respects: from a physical viewpoint since one may not be able to get measurements on the whole outer surface of a material and also because the prior knowledge that the crack lies on some curve is not realistic. It is thus necessary to develop methods that would be able to give information about the crack without *a priori* assumptions and from *incomplete* data.

We stick here to the two-dimensional situation, where D is assumed to be an open, bounded, simply connected subset of $\mathbb{C}$ with Hölder smooth ($C^{1,\alpha}$, $0 < \alpha < 1$) boundary Γ and where we permit the existence of an unknown one-dimensional crack consisting of a smooth ($C^{1,\alpha}$) Jordan arc $\gamma \subset D$ with distinct endpoints γ_0, γ_1. Therefore, neighbourhoods of interior points to γ are a union of their intersection with γ and of two disjoint regions which are entirely contained on one side of γ or the other. Functions on γ will then be superscripted by + or $-$, depending on the side of γ from which they are to be considered as non-tangential limits (see figure 1). We consider the steady-state system whose behaviour is described by the

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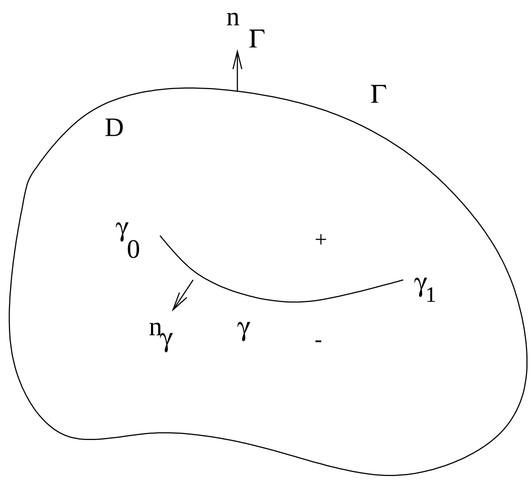

Figure 1. The domain D.

Neumann boundary value equation:

$$\begin{cases} \Delta u = 0 & \text{in } D \setminus \gamma = D_\gamma \\ \dfrac{\partial u}{\partial n_\Gamma} = \Phi & \text{on } \Gamma \\ \dfrac{\partial u^\pm}{\partial n_\gamma} = 0 & \text{on } \gamma^o = \gamma \setminus \{\gamma_0, \gamma_1\} \end{cases} \tag{1}$$

where n_Γ denotes the outer unit normal vector to Γ, n_γ one of the two unit normal vectors to γ, and $\Phi \in L^2(\Gamma)$ is, in the thermal framework, a prescribed heat flux along Γ. Whenever it satisfies

$$\int_\Gamma \Phi(z)\, \mathrm{d}s(z) = 0 \tag{2}$$

for the arc length parameter s on Γ, problem (1) admits a solution unique up to an additive constant. For a C^2 crack, this follows from [17] and, for $\Phi \in C(\Gamma)$, from [15, section II.4.4, theorem 3]; our slightly more refined version, where $\Phi \in L^2(\Gamma)$ and we have a Hölder smooth crack, can be derived from complex analysis and conformal mapping, see the proof of theorem 1 later. Uniqueness can be ensured by imposing some normalization.

It will also follow from the proof of theorem 1 that u is defined and smooth up to Γ and, from the + or − side, up to γ, as $u^\pm$. The normal derivative in (1) may be given at least three different meanings. The usual one in the literature on PDEs is to regard it almost everywhere as the trace function that makes the Green formula work:

$$\int_\Gamma \left(\Phi v - u \frac{\partial v}{\partial n_\Gamma}\right) \mathrm{d}s = -\int_\gamma \left[\left(\frac{\partial u^+}{\partial n_\gamma} - \frac{\partial u^-}{\partial n_\gamma}\right) v - \frac{\partial v}{\partial n_\gamma}(u^+ - u^-)\right] \mathrm{d}s$$

for any smooth harmonic function v. From the point of view of function theory, the normal derivative could also be defined at almost every point $\eta \in \gamma^o$ as the limit when $z \stackrel{\pm}{\to} \eta$ non-tangentially of the scalar product between $\nabla u(z)$ and $n_\gamma(\eta)$ [26, XVII, theorem 2.9]. In complex analysis, the normal derivative can be obtained as well as the limit when $z \stackrel{\pm}{\to} \eta$ along the normal of $u(z) - u^\pm(\eta)/|z - \eta|$. Whenever γ is Hölder smooth, these three definitions coincide everywhere on γ^o.

The *incomplete overdetermined* boundary data will typically be specified on a subset $K \subset \Gamma$ of positive Lebesgue measure, see for example [1, 7]. They consist of the additional knowledge of the temperature on K, given by a non-negative valued function $T \in C(K)$:

$$u = T \text{ on } K. \tag{3}$$

One then wants to know whether γ is empty or not and, in the latter case, what is its location. This can be expressed as the following inverse problem:

(IP_0) *Given* $\Phi \in L^2(\Gamma)$ *and* $T \in C(K)$, *find* $\gamma \subset D$ *such that the associated solution* u *of* (1) *satisfies* (3).

Whenever u solves for (1), we have that $u = \operatorname{Re} F$ for the function F analytic in D_γ defined by

$$F = u + \mathrm{i}\tilde{u} \tag{4}$$

where $\tilde{u}$ is the conjugate harmonic function to u in D_γ. In view of (1) and (2), $\tilde{u}$ is single-valued in D_γ, such that on Γ

$$\frac{\partial \tilde{u}}{\partial s} = \frac{\partial u}{\partial n_\Gamma}$$

and is unique up to an additive constant [5, 12]. It will follow from (12) and (13) later, together with Privalov's theorem [12, theorem 6], that F is bounded in D_γ and has a C^α extension to Γ given by

$$F(\xi) = u(\xi) + \mathrm{i} \int_a^\xi \frac{\partial u}{\partial n_\Gamma}(z)\, \mathrm{d}s(z) \tag{5}$$

for every $\xi \in \Gamma$ and some fixed $a \in \Gamma$, where the normalization $\tilde{u}(a) = 0$ has been imposed. The additional data (3) then provide us with the knowledge of F on K, since there

$$F(\xi) = T(\xi) + \mathrm{i} \int_a^\xi \Phi(z)\, \mathrm{d}s(z). \tag{6}$$

In a holomorphic framework, problem (IP_0) can be reformulated as:

(IP) *Given* $F \in C(K)$, *for* $K \subset \Gamma$ *of positive measure, find a smooth Jordan arc* $\gamma \subset D$ *such that* F *is analytic and bounded in* $D \setminus \gamma$ *and* $\operatorname{Im} F(\xi) = \int_a^\xi \Phi(z)\, \mathrm{d}s(z)$ *for every* $\xi \in \Gamma$.

The approach of bringing in the harmonic conjugate and rephrasing inverse problems for the 2D Laplacian into analytic continuation issues can be used to establish uniqueness and stability results of γ with respect to the measurements T and Φ, both for $\gamma \subset D$ (as in the present case) but with *complete* data ($K = \Gamma$), and for $\gamma \subset \Gamma$, with $K = \Gamma \setminus \gamma$, see [6, 20] and the references therein.

We wish to illustrate in this paper how meromorphic approximants to F on K provide information about the existence and the location of γ in D.

We first set up some notation in section 2. In section 3, we report on some results of [10] concerning the best uniform meromorphic approximation—also called AAK approximation†—that permit us to detect the presence of γ in D from the data (6), even under the realistic hypothesis that they are available only on a proper subset K of Γ; these results also allow us to constructively (and robustly) obtain some extension of F to the whole of Γ which is valuable already when initializing existing reconstruction algorithms [2, 8, 13]. In section 4, we establish a representation formula for F that exhibits its singular part in D as a Cauchy integral over γ. It is used in section 5 to give the asymptotic properties of poles

† Since solutions to these problems originate with the work of Adamjan, Arov *et al* [4].

of Padé approximants after a result of [25]. Unfortunately, Padé approximants are typically numerically unstable (their computations involve Fourier coefficients) and they possess slow convergence properties that are not conformally invariant. Therefore, it is natural to ask about the behaviour of the poles of optimal approximants in some conformally invariant metric, like AAK approximants. Here, we shall only investigate the very special case where D is the unit disk and γ is a line segment. In view of the striking result obtained in this case, we discuss possible extensions to more general situations in connection with the currently active area of research surrounding 'orthogonal' polynomials with respect to complex measures.

2. Notation and definitions

Let $H^\infty(D)$ be the Hardy space of bounded analytic functions in D and $H_N^\infty(D)$ the set of functions that are bounded on Γ and meromorphic with at most N poles in D ($H^\infty(D) = H_0^\infty(D)$ and $H_N^\infty(D) = H^\infty(D) + R_N(D)$, for $R_N(D)$ the set of rational functions with at most N poles in D, none on Γ); $H_N^\infty(D)_{|_K}$ is the set of traces on $K \subset \Gamma$ of $H_N^\infty(D)$ functions. Whenever $D = \mathbb{D}$, the unit disk, we simplify the above notation into $H_N^\infty(\mathbb{D}) = H_N^\infty$. We refer the reader to [16, 19] for properties of Hardy spaces. Introduce the Hardy space H^2 of the unit disk that consists of functions that are analytic in $\mathbb{D}$ and whose L^2 norms on circles of radius less than 1 are bounded. These functions possess L^2 traces on $\mathbb{T}$ and we let $\bar{H}_0^2$ be the orthogonal complement of H^2 into $L^2(\mathbb{T})$ (it also consists of functions analytic outside $\mathbb{D}$, vanishing at ∞, and whose L^2 norms on circles of radius greater than 1 are bounded). If $P_{\bar{H}_0^2}$ denotes the orthogonal projection from $L^2(\mathbb{T})$ onto $\bar{H}_0^2$, the Hankel operator $\mathcal{H}_f$ of symbol $f \in L^\infty(\mathbb{T})$ is defined on H^2 by [27]

$$\mathcal{H}_f g = P_{\bar{H}_0^2}(fg) \qquad \forall g \in H^2.$$

The operator $\mathcal{H}_f$ is compact as soon as f is continuous. The best meromorphic approximant with m poles, also called the AAK approximant of order m, to a function $f \in C(\mathbb{T})$ is defined as the unique solution to

$$\min_{g \in H_m^\infty} \|f - g\|_{L^\infty(\mathbb{T})} = \|f - g_m\|_{L^\infty(\mathbb{T})} \tag{7}$$

and is given by

$$g_m = f - \frac{\mathcal{H}_f v_m}{v_m} \tag{8}$$

for v_m the first element of some $(m + 1)$th Schmidt pair of $\mathcal{H}_f$, see [4, 18, 27].

3. Failure detection and data extension

To explain why the knowledge of (6) provides an economic way to find out whether γ is empty or not, observe that

$$\gamma = \emptyset \quad \text{if and only if} \quad F \in H^\infty(D)_{|_K}$$

for every function Φ that ensures the identifiability of γ; this holds whenever γ is not contained in a level line of the harmonic conjugate to the solution u to (1) (see [17] where the use of two such particular functions is shown to provide identifiability), a property which can be proved to be generic with respect to $\Phi \in L^2(\Gamma)$. We assume throughout that Φ is such a function.

Let us discuss how to detect γ and to extend the data to the whole boundary Γ using best bounded uniform meromorphic approximation to F on K. Since Φ and then $\operatorname{Im} F$ are known everywhere on Γ, the problem arises of computing the uniform distance on K from $F_{|_K}$ to the subset of $H^\infty(D)$ consisting of functions whose imaginary parts coincide up to some tolerance with $\mathrm{Im}F$ on $\Gamma \setminus K$. Unfortunately, we do not know yet how to solve this issue, nor even if it is well-posed. However, an approximate solution can be obtained by solving for a version of the bounded extremal problem (9) which we now describe.

Using the fact that any conformal map ϕ_0 from $\mathbb{D}$ into D preserves the corresponding H_N^∞ spaces [16, ch 10], we make use of the constructive results of [9, 10] that concern a family of bounded extremal problems in H_N^∞ from data on subsets of $\mathbb{T}$. They extend part of the Nehari–Adamjan–Arov–Krein theory related to problem (7). More precisely, for given $K_0 \subset \mathbb{T}$, $f \in L^\infty(K_0)$ and $\psi, M \in L^\infty(\mathbb{T} \setminus K_0)$, $\inf_{\mathbb{T}\setminus K_0} M > 0$, we handle in [10] the problem of constructively solving for

$$\min_{\substack{g\in H_N^\infty \\ |\psi-g|\leqslant M \text{ a.e. on } \mathbb{T}\setminus K_0}} \|f-g\|_{L^\infty(K_0)} = \|f-g_N\|_{L^\infty(K_0)} = \beta_N(M). \tag{9}$$

A solution g_N is shown to exist and to be unique at least when the concatenated function $f \vee \psi$, which is equal to f on K_0 and to ψ on $\mathbb{T} \setminus K_0$, belongs to $C(\mathbb{T})$; in this case, we also have that $|f - g_N| = \beta_N(M)$ a.e. on K_0 and $|\psi - g_N| = M$ a.e. on $\mathbb{T} \setminus K_0$, unless $f \in H_N^\infty$ which will never occur in the numerical practice.

For fixed M and continuous data f and ψ, the quantity $\beta_N(\alpha M)$ smoothly decreases to zero while $\alpha > 0$ increases with fixed N while, for fixed α, β_N decreases with N to zero. Moreover, $\beta_N(M)$ is continuous with respect to continuous data f and ψ and estimates of $\beta_N(M)$ that are uniformly robust with respect to measurement noise can be obtained by computing the $(N+1)$th singular value of some Hankel operator. Finally, g_N is computed from an associated Schmidt pair by an algorithm which is shown to be generically convergent in separable Hölder classes. It rests on a dichotomy procedure which is fully described in [10].

Our present interest here does not lie with problem (9) but rather with an analogous one where the minimization occurs over all $g \in H_N^\infty$ such that $\mathrm{Im}\,\psi = \mathrm{Im}\,g$ a.e. on $\mathbb{T} \setminus K_0$. However, by taking exponentials, a suboptimal solution to the problem of the best meromorphic approximation with prescribed imaginary part is furnished by solving for a version of (9) where $f = \exp(\mathrm{i}F)$, $\psi = 0$, and $M = \exp(-\mathrm{Im}F)$, although some adjustment is needed as $f \vee \psi$ is not continuous in this particular instance. This can be arranged upon premultiplying the data by some appropriate polynomial vanishing at the boundary points of K_0, at least when the latter is a finite union of intervals.

For our present purpose, let $K_0 \subset \mathbb{T}$ be such that $\phi_0(K_0) = K$, and let $F_0 = F \circ \phi_0$. Solving the above-mentioned problem with $N = 0$ provides us with an estimate of the distance of F_0 in $L^\infty(K_0)$ to traces of analytic functions having imaginary part $\mathrm{Im}F_0$ on $\mathbb{T}$. If this distance is smaller than the measurement error, we can conclude that D is free of cracks. Otherwise, $\gamma \neq \emptyset$ and we will turn to the question of its location. A first remark in this case is that we can let N grow larger than zero to obtain a meromorphic extension, say g_N, of F_0 to the whole of $\mathbb{T}$, which is convenient in order to initialize existing reconstruction algorithms [8, 17]. For this purpose, we pick N large enough so that the distance of F_0 in $L^\infty(K_0)$ to traces of meromorphic functions with N poles having imaginary part $\mathrm{Im}F_0$ on $\mathbb{T}$ is small enough. The trace on $\mathbb{T}$ of the associated solution g_N to (9) provides us with a completion of F_0 to the circle; now taking $G_N = g_N \circ \phi_0^{-1}$ thus gives an extension of F to the whole of Γ. It also furnishes an approximate solution to the direct problem since $u_N = \mathrm{Re}G_N$ approximates the Dirichlet data T on K, see (3). One may say, in some sense, that this solution was obtained upon approximating the data rather than discretizing the equation, and it provides us with a truly harmonic function, except at the poles when $N > 0$; these poles should then be regarded as a discretization of the singularities and it becomes most natural to ask whether they should converge in some appropriate sense to the crack.

Observe finally that such a scheme would still be useful in order to extend F to the whole Γ if nothing was known about its imaginary part (or the flux Φ) on $\Gamma \setminus K$. In this case, there is no estimate of the constraint M at first; however, one can find this by iteratively applying the above procedure: an appropriate M is a function that gives rise to a small value of $\beta_N(M)$, for some integer N, while the associated extension G_N behaves smoothly at the endpoints of K.

4. Representation by Cauchy integrals

We now establish the following result

Theorem 1. *Let u denote a solution to* (1). *Then the function $\sigma = u^+ - u^-$ is Hölder continuous on γ^o and $u = \operatorname{Re} f$ in D_γ, where*

$$f(z) = \frac{1}{2i\pi}\int_\gamma \frac{\sigma(\eta)}{z-\eta}\,d\eta + g(z) \qquad \forall z \in D_\gamma \tag{10}$$

for some function g belonging to $H^\infty(D)$.

In the thermal framework, σ represents the temperature jump across γ. Theorem 1 thus asserts that the function F defined by (4) is equal to f in D_γ for some $g \in H^\infty(D)$. This may be viewed as a double-layer potential representation for the solution of the interior Neumann problem (1), see [14, section 6.10].

Proof. Since D_γ has a $C^{1,\alpha}$ boundary $\gamma \cup \Gamma$, there exists a conformal map ϕ from an annulus Ω, delimited by inner and outer circles Γ_0 and Γ_1, into D_γ, such that ϕ' and $(\phi^{-1})'$ are continuous and non-vanishing up to $\Gamma_0 \cup \Gamma_1$ and $\gamma^o \cup \Gamma$, respectively. Although we do not provide the details here, we mention that this can be established by using [23, theorem 3.6] for the Hölder regularity of conformal maps onto simply connected smooth sets and then the McMillan twist theorem [23, 6.1] in order to work on smooth simply connected + or − regions in D_γ.

Assume that ϕ maps Γ_0 onto γ and Γ_1 onto Γ, call n the outer unit normal vectors to both Γ_0 and Γ_1, and consider the Neumann boundary value equation on Ω:

$$\begin{cases} \Delta v = 0 & \text{in } \Omega \\ \dfrac{\partial v}{\partial n} = \Phi \circ \phi & \text{on } \Gamma_1 \\ \dfrac{\partial v}{\partial n} = 0 & \text{on } \Gamma_0. \end{cases} \tag{11}$$

Since ϕ' is bounded from both sides on Γ_1, $\Phi \circ \phi \in L^2(\Gamma_1)$ and, in view of (2),

$$\int_{\Gamma_1} \Phi(\phi(z))\,ds(z) = 0.$$

Classical regularity results [14, section 6.14] imply that v belongs to the Sobolev space $\mathcal{W}^{3/2}(\Omega)$ and it is unique up to an additive constant. Moreover, for $i = 0, 1$, in view of trace theorems [14, ch 2],

$$v_{|\Gamma_i} \in C^\beta(\Gamma_i) \qquad \text{for } 0 < \beta \leqslant 1/2,\ \frac{\partial v}{\partial n}_{|\Gamma_i} \in L^2(\Gamma_i).$$

Hence,

$$u^\pm = v_{|\Gamma_0} \circ (\phi^\pm)^{-1} \in C^\beta(\gamma^o) \tag{12}$$

since $(\phi^{\pm})^{-1}$ has continuous derivative on γ^o, which proves the first assertion of the theorem (see also [17, lemma 2.1] or [15, section II.4.4, theorem 3]); moreover, at the endpoints, $u^+(\gamma_i) = u^-(\gamma_i)$ and

$$u_{|\Gamma} = v_{|\Gamma_1} \circ \phi^{-1} \in C^{\beta}(\Gamma). \tag{13}$$

Also, v admits a conjugate harmonic function $\tilde{v}$ which is constant on Γ_0, from the boundary condition in (11). Hence, the limits $\tilde{u}^{\pm}$ of $\tilde{u}$ coincide on γ:

$$\tilde{u}^+(\eta) - \tilde{u}^-(\eta) = \tilde{v} \circ (\phi^+)^{-1}(\eta) - \tilde{v} \circ (\phi^-)^{-1}(\eta) = 0 \qquad \forall \eta \in \gamma. \tag{14}$$

The remaining part of theorem 1 amounts to showing that, for F defined by (4), then $g = F - \mathcal{S} \in H^{\infty}(D)$, where

$$\mathcal{S}(z) = \frac{1}{2\mathrm{i}\pi} \int_{\gamma} \frac{\sigma(\eta)}{z - \eta} \mathrm{d}\eta \qquad \forall z \in D_{\gamma}.$$

We already know that g is analytic in D_{γ}; since $u^{\pm}$ is Hölder continuous on the $\pm$ regions of D, so is $\tilde{u}^{\pm}$, from the Privalov theorem [12, theorem 6], and thus $F^{\pm}$. In particular, g is then bounded on D_{γ}. To prove that g extends analytically across γ, observe first that the Plemelj–Sokhotski formula [12, theorem 3] leads to $\mathcal{S}^+(\eta) - \mathcal{S}^-(\eta) = \sigma(\eta)$ whenever $\eta \in \gamma^o$. Hence, it follows from (4), (14) and the definition of σ that $g^+(\eta) = g^-(\eta) = g(\eta)$ for $\eta \in \gamma^o$. Let C_{η} be a small circle centred at $\eta \in \gamma^o$ such that $C_{\eta} \cap \gamma \subset \gamma^o$ and consider the function g_{η} analytic in the interior domain D_{η} of C_{η}:

$$g_{\eta}(z) = \frac{1}{2\mathrm{i}\pi} \int_{C_{\eta}} \frac{g(\xi)}{\xi - z} \mathrm{d}\xi. \tag{15}$$

Whenever $z \in D_{\eta} \setminus \gamma$, then for closed contours $C^{\pm}_{\eta,\varepsilon}$ that consist of the $\pm$ part of C_{η} cut and closed by curves parallel to γ located at a distance $\varepsilon > 0$ from γ,

$$g_{\eta}(z) = \lim_{\varepsilon \to 0} \frac{1}{2\mathrm{i}\pi} \left[\int_{C^+_{\eta,\varepsilon}} \frac{g^+(\xi)}{\xi - z} \mathrm{d}\xi + \int_{C^-_{\eta,\varepsilon}} \frac{g^-(\xi)}{\xi - z} \mathrm{d}\xi \right] = g(z)$$

from the Cauchy formula [24, theorems 10.14, 11.21]. Indeed, $g^{\pm}$ are respectively analytic and bounded in the $\pm$ regions, and one of these two integrals is necessarily zero, while the other is equal to $g^{\pm}$. Thus, g_{η} extends g analytically across the portion of γ^o delimited by C_{η}, at each $\eta \in \gamma^o$.

Finally, g does not have a singularity at the endpoints γ_i of γ, because it is bounded on any neighbourhood of γ_i [24, theorem 10.20]; indeed, since σ vanishes at γ_i, the function $\mathcal{S}$ is bounded near γ_i, see [3, lemma 7.2.2]. □

Note in passing that the same proof would apply to providing a representation for F of the form (10) when γ is a smooth closed Jordan curve in D.

5. Meromorphic approximation and localization of curves γ

With the representation (10) for F we are ready to discuss the behaviour of poles of meromorphic approximants. Up to the conformal mapping ϕ_0, let us now work entirely over the unit disk for simplicity and assume that $D = \mathbb{D}$, $K = \mathbb{T}$. The section of the Fourier series of F corresponding to negative powers of $\mathrm{e}^{\mathrm{i}\theta}$ is equal to that of the Cauchy integral $\mathcal{S}$. If we assume that γ is a finite union of analytic arcs $\gamma^1, \ldots, \gamma^p$ linking points $y_1, \ldots, y_p$ and that σ extends analytically in a neighbourhood of these arcs which is large enough to contain the contour $\mathcal{M}$ of minimal capacity joining $y_1, \ldots, y_p$, a rather deep theorem [25, theorem 1.7] asserts that the poles of the diagonal Padé approximants to $\mathcal{S}$ at infinity converge in capacity to $\mathcal{M}$. More precisely, the probability measure on $\mathbb{C}$ having equal point masses at each of the poles of the Padé approximant converges weak-*, as the order of approximation becomes large, to the equilibrium distribution on $\mathcal{M}$ of the condenser obtained from $\mathcal{M}$ and the complement of the unit disk. This entails, in particular, that a full proportion of these poles converges to $\mathcal{M}$.

The assumption that γ is a succession of analytic arcs may perhaps be considered as a good approximation of a general γ. However, although it is true that σ extends analytically on a neighbourhood of $\gamma^1, \dots, \gamma^p$ (as follows from the reflection principle), we do not know *a priori* that the domain of analyticity will contain $\mathcal{M}$, as this will depend on Φ and the location of γ. Also, the very computation of Padé approximants runs into numerical difficulties because it will not be robust against any kind of error. Nevertheless, the result we just mentioned provides a strong incentive to explore the behaviour of the poles of AAK approximants. We shall be content here with analysing the simplest case where γ is a segment $[\gamma_0, \gamma_1] \subset (-1, 1)$ and where (10) can be written as

$$f(z) = \frac{1}{2i\pi} \int_{\gamma_0}^{\gamma_1} \frac{\sigma(t)}{z-t} \mathrm{d}t + g(z) \qquad \forall z \in \mathbb{D} \setminus [\gamma_0, \gamma_1] \tag{16}$$

involving a function g belonging to H^∞ and a Markov-type integral where σ may change sign on $[\gamma_0, \gamma_1]$. Note that, in this case, γ is already of minimal capacity and that, if σ does not vanish, a result in [21] improves [25, theorem 1.7] to the effect that *all* the poles of the diagonal Padé approximant to f converge to γ.

Since the function g in (16) has no singularity in $\mathbb{D}$, the poles of the AAK approximants to f are the same as those of the AAK approximants to

$$\mathcal{S}(z) = \frac{1}{2i\pi} \int_{\gamma_0}^{\gamma_1} \frac{\sigma(t)}{z-t} \mathrm{d}t.$$

Theorem 2. *Let f be defined by* (16)*. If the real-valued function σ changes sign exactly k times on (γ_0, γ_1), then at most k poles of the AAK approximant g_N of order N to f are located outside $[\gamma_0, \gamma_1]$.*

This phenomenon is illustrated by figure 2. Figures 2–4 show different locations of the crack γ in $D = \mathbb{D}$ together with the poles [x] and zeros [o] of the best meromorphic approximant associated with the corresponding data. These numerical results have been obtained using the Nag library to integrate (1) for some trigonometric polynomials Φ and then performing the AAK approximation of the solution using Matlab. In figure 2, $\gamma = [-1/2, 1/2]$ and $\Phi(\theta) = \cos\theta + \sin 3\theta$ provides $\sigma > 0$ on γ: the poles of g_{10} lie indeed *on* γ.

Proof. Since $g \in H^\infty$, we get $\mathcal{H}_f = \mathcal{H}_\mathcal{S}$ and we can assume $g = 0$. Let s_N denote the $(N+1)$th singular value of $\mathcal{H}_f$. As f is conjugate symmetric ($f(\bar{z}) = \overline{f(z)}$), the operator

$$v \to 1/z\overline{(\mathcal{H}_f v)(1/\bar{z})}$$

is self-adjoint on H^2 and one can choose v_N to be an eigenvector: $\pm s_N v_N(z) = 1/z(\mathcal{H}_f v_N)(1/z)$. Expressing $P_{\bar{H}_0^2}$ as a Cauchy integral in the definition of $\mathcal{H}_f$ and using successively Fubini's theorem and the Cauchy formula, we get for $z \in \mathbb{D}$

$$v_N(z) = \frac{1}{s_N} \int_{\gamma_0}^{\gamma_1} \frac{v_N(t)\sigma(t)\,\mathrm{d}t}{1 - zt}.$$

By AAK theory, v_N has at most $N_0 \leqslant N$ zeros in $\mathbb{D}$, say $z_1 \dots, z_{N_0}$, that are the poles of g_N from (8). Set $q(z) = \Pi_{j=1}^{N_0}(z - z_j)$ and let $\tilde{q}(z) = z^{N_0} q(1/z)$ be the reciprocal polynomial, noting

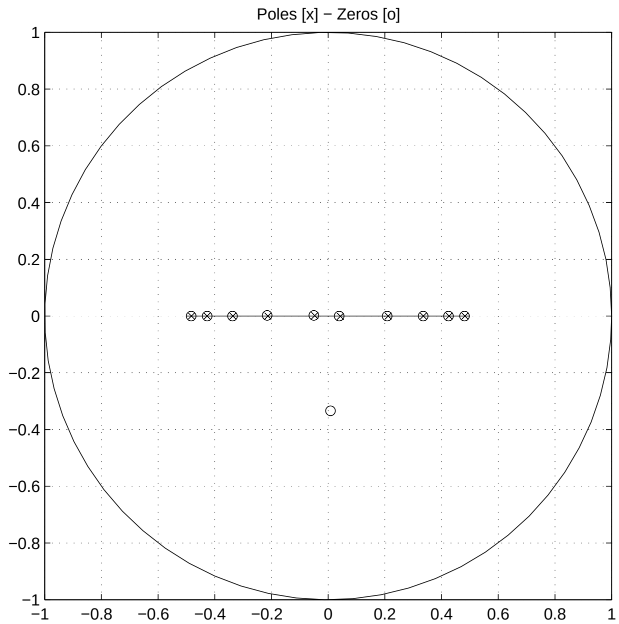

Figure 2. g_{10} for $\gamma = [-1/2, 1/2]$.

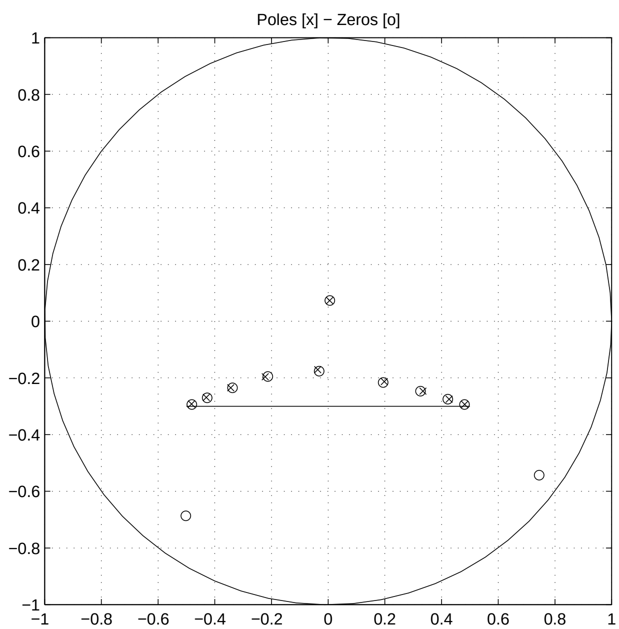

Figure 3. g_{10} for $\gamma = \{x - 0.3\mathrm{i},\ x \in [-1/2, 1/2]\}$.

that q has real coefficients since f and consequently g_N are conjugate symmetric. Evaluating the above formula at each of these zeros leads to the following orthogonality relation

$$\int_{\gamma_0}^{\gamma_1} \frac{v_N(t)\sigma(t)P_{N_0-1}(t)\,\mathrm{d}t}{\tilde{q}(t)} = 0$$

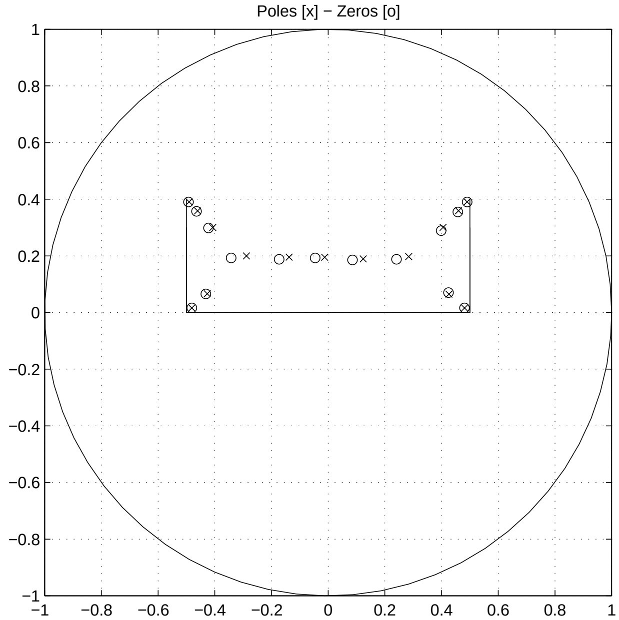

Figure 4. g_{15} for $\gamma = [-1/2, 1/2] \cup \{\pm 1/2 + \mathrm{i}y,\ y \in [0, 0.4]\}$.

for P_{N_0-1} an arbitrary polynomial of degree $N_0 - 1$. Now, $v_N = wq/\tilde{q}$ for some outer H^2 function w which consequently cannot vanish. The above orthogonality relation can then be rewritten as

$$\int_{\gamma_0}^{\gamma_1} \frac{q(t)w(t)\sigma(t)P_{N_0-1}(t)\,\mathrm{d}t}{\tilde{q}^2(t)} = 0. \tag{17}$$

If q has strictly less than $N_0 - k$ zeros on γ, then $qw\sigma$ is real valued and vanishes at most $N_0 - 1$ times on γ. One can then design P_{N_0-1} such that $qw\sigma P_{N_0-1} \geqslant 0$ on γ. The integrand in (17) is then non-negative and it should vanish identically, implying that $\sigma = 0$, a contradiction. □

Let us now briefly discuss the case of a crack γ that need not be (conformally equivalent to) a real segment in $\mathbb{D}$ but still an analytic curve. Then it has been proved [11] that all but a fixed number of poles of the best L^2 rational approximants to $\mathcal{S}$ asymptotically lie inside a given neighbourhood of the hyperbolic geodesic $\mathcal{C}$ joining the endpoints of γ, and that only $o(N)$ of these poles can lie outside this neighbourhood if the AAK approximation is considered; moreover, the asymptotic distribution of these poles is the equilibrium distribution of the condenser $(\mathcal{C}, \mathbb{T})$. This is illustrated by figure 3. Observe that the present situation is different from that in theorem 2, for the poles no longer accumulate on γ but rather on a special contour linking its endpoints. The procedure will therefore indicate the location of these endpoints, with a high density of poles due to the properties of the equilibrium measure, but to recover γ itself requires finding a curve between these two points along which $\mathcal{S}$ has a real jump. If the crack has small length, however, the hyperbolic geodesic arc $\mathcal{C}$ cannot differ too much from γ which is then expected to be recovered by a standard numerical search procedure from $\mathcal{C}$.

This leads us naturally to conjecture about the more realistic case where γ is not analytic. The non-analytic character of γ, if anything, should be helpful for crack detection. Indeed, one can then approximate γ by a chain of segments that may be chosen so small that σ can be analytically continued up to the contour $\mathcal{M}_G$ joining the endpoints and minimizing the capacity of the condenser $(\mathcal{M}_G, \mathbb{T})$ (note that when γ is analytic, $\mathcal{M}_G$ is just the hyperbolic geodesic $\mathcal{C}$). In doing so, the value of the Cauchy integral $\mathcal{S}$ is approximated in the Hölder norm on the circle and the AAK approximation is continuous with respect to this norm [22]. On the basis of [11] which treats the case of an analytic γ, and of numerical experiments with two, three, and four segments (see figure 4), the authors conjecture that the poles of the best approximants (in the AAK or the L^2 rational approximation) converge to $\mathcal{M}_G$. The proof of this conjecture is tantamount to showing that the zeros of q in the equation

$$\int_\gamma \frac{q(t)w(t)\sigma(t)P_{N_0-1}(t)\,\mathrm{d}t}{\tilde{q}^2(t)} = 0 \tag{18}$$

converge to $\mathcal{M}_G$. Note that the denominator of the Padé approximants is governed by an analogous equation, except for the important fact that there is no $\tilde{q}^2$ in the denominator of the integrand. The presence of this denominator is typical of the meromorphic approximation on $\mathbb{T}$ and will occur, for example, for L^p norms as well; it makes for a more difficult analysis, since the complex measure $w\sigma/\tilde{q}^2$ varies with N and depends on q itself, and should account for the fact that the logarithmic capacity is replaced by the condenser capacity.

6. Conclusion

In this paper, we have established a link between the location of the poles of the best uniform meromorphic approximants and the singularities of solutions to the Laplace equation over a slit domain of the complex plane that suggests an attractive approach to crack detection. It is tantamount to solving one of the most important issues facing the theory of orthogonal polynomials today, namely the asymptotic behaviour of their zeros, when orthogonality is taken with respect to a series of varying complex measures as in (18). The present paper contributes an incentive to study these issues, unexpectedly perhaps, from the point of view of non-destructive control.

As an aside, we have also illustrated the fact that a family of bounded extremal problems, in particular of best uniform meromorphic approximation, leads to approximate solutions for the 2D Laplacian: this provides an economic way to complete available partial data to the whole external boundary. Continuity properties of such approximants may also be used to establish stability results for the crack problem.

Complex approximation methods should allow us to solve for various other inverse problems (such as geometric issues or identification of unknown coefficients in the boundary conditions), although the isotropic and 2D restrictions remain necessary at this stage.

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