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Asymptotics of the Information Entropy for Jacobi and Laguerre Polynomials with Varying Weights*

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We study the asymptotic behavior as $n \to \infty$ of the entropy integral $S_n = -\int p_{n,n}^2(x) \ln p_{n,n}^2(x)\, w_n(x)\, dx$, where $p_{n,n}$ is the nth degree polynomial orthonormal with respect to a Jacobi or Laguerre weight function $w_n(x)$ whose parameters grow with n. For this purpose we use the weak-* convergence of the measures $p_{n,n}^2(x)\, w_n(x)$ to the Robin distribution of the support of the equilibrium measure in an external field, arising from the limit of the nth root of the sequence of weights. © 1999 Academic Press

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1. INTRODUCTION AND STATEMENT OF RESULTS

In the last few years much attention has been paid to the study of the information entropy

$$S_\rho = -\int \rho(x) \ln \rho(x)\, dx,$$

of the density $\rho(x) = |\Psi(x)|^2$ of a quantum mechanical system with the wave function $\Psi(x)$. This functional characterizes the localization of the density ρ. Based on this interpretation and using the work of Bialynicki-Birula and Mycielski [6], Gadre *et al.* [9] derived a new and stronger version of the quantum-mechanical Heisenberg uncertainty principle for finite physical systems which gives a quantitative expression to the property that a sharp position density $\rho(x)$ is associated to a diffuse momentum density $\gamma(\rho)$. In addition, both the position and momentum information entropies of a physical system are closely related to various fundamental and/or experimentally measurable quantities of the system (see [1]).

Very often, especially for single-particle systems in spherically symmetric quantum-mechanical potentials, the distribution function $\rho(x)$ can be expressed in terms of the classical orthogonal polynomials (Jacobi, Laguerre, etc.). Recently, it has been explicitly shown [4, 7, 23] that for the harmonic oscillator and Coulomb potentials the determination of the corresponding position and momentum information entropies reduces to the study of the functionals

$$S_n(w) = -\int_\Delta p_n^2(x) \ln p_n^2(x)\, w(x)\, dx, \tag{1}$$

where $p_n(x) = p_n(w; x)$ is the nth polynomial orthonormal with respect to a weight $w(x)$ on an interval $\Delta \subset \mathbb{R}$. These functionals are called, for obvious reasons, information entropies of the (orthonormal) polynomials $p_n(x)$.

The asymptotics of $S_n(w)$ has been thoroughly studied in [2, 3, 8, 22] for general orthogonal polynomials, when $w(x)$ belongs to a wide class of functions such as that of Bernstein-Szegő, Szegő, or Freud. However, this problem for polynomials orthogonal with respect to a classical weight $w(x)$ whose parameters grow with the degree n of the polynomial has till now not been investigated. Such a situation occurs in a great variety of physical systems [5, 23], and particularly for the hydrogen atom, where the asymptotic case corresponds to the so-called Rydberg region (i.e., the region of bound states characterized by a very high principal quantum number) where the transition from quantum to classical phenomena takes place.

Let $w_n(x)$ be a sequence of weights on an interval $\Delta \subset \mathbb{R}$ and $p_{n,m}(x) = l_{n,m} x^m + \cdots$ with $l_{n,m} > 0$ denotes the mth orthonormal polynomial with respect to w_n,

$$\int_{\Delta} p_{n,i}(x)\, p_{n,j}(x)\, w_n(x)\, dx = \delta_{ij}, \qquad i, j \in \mathbb{N}.$$

Then, the information entropy (1) becomes

$$S_n = -\int_{\Delta} p_{n,n}^2(x) \ln p_{n,n}^2(x)\, w_n(x)\, dx. \tag{2}$$

In this paper we study the asymptotic behavior of S_n in the following two cases:

- *Varying Jacobi weight*,

$$w_n(x) = (1-x)^{\alpha_n}(1+x)^{\beta_n}, \qquad \Delta = [-1, 1], \tag{3}$$

where $\alpha_n = \alpha n + o(n)$, $\beta_n = \beta n + o(n)$, with $\alpha, \beta > 0$;

- *Varying Laguerre weight*,

$$w_n(x) = x^{\alpha_n} \exp(-\beta_n x), \qquad \Delta = [0, +\infty), \tag{4}$$

where $\alpha_n = \alpha n + o(n)$, $\beta_n = \beta n + o(n)$, with $\alpha, \beta > 0$.

In [2] the entropy is computed for the fixed weight w as the limit value of the L^p norms when $p \to 1$, which in turn is studied using the well known asymptotic expressions for the polynomial inside and at the endpoints of the interval of orthogonality. Unfortunately, in the case of the varying weights the strong asymptotics of the orthogonal polynomials has not been well studied. For w_n given in (3) some asymptotic expressions for $p_{n,n}(x)$ were obtained in [11] using the steepest descent approach and in [13] by the Darboux's method. For the varying Laguerre weights analogous results have been obtained in [10]. In both cases the final formulas are not sufficient for the study of the functional (2).

Here, we shall use some weaker asymptotic properties of $\{p_{n,n}\}$. In fact, for general varying weights only the nth root asymptotics of such a sequence has been studied in detail by Gonchar, Rakhmanov, Mhaskar, and Saff (see [12, 14]). They proved that the main parameters describing the asymptotics are the support and the extremal constant of the equilibrium measure in an external field, arising from the limit of the nth root of the sequence of weights. We state these results in a weak form, sufficient for our purposes.

Let φ be continuous in the interior of Δ, such that uniformly on compact subsets inside Δ,

$$\lim_n \frac{1}{2n} \ln w_n(x) = -\varphi(x), \tag{5}$$

and if Δ is unbounded,

$$\lim_{x \to \infty,\, x \in \Delta} (\varphi(x) - \ln |x|) = +\infty. \tag{6}$$

Then there exists a unique probability measure $\lambda = \lambda(\varphi)$, with $K = \operatorname{supp} \lambda \subset \Delta$, such that for $x \in K$,

$$(V_\lambda + \varphi)(x) = \omega = \min_{x \in \Delta} (V_\lambda + \varphi)(x), \tag{7}$$

where

$$V_\lambda(z) = -\int \ln |z - x| \, d\lambda(x)$$

is the logarithmic potential of the *equilibrium measure* λ with the *external field* φ, and $\omega = \omega(\varphi)$ is the *extremal constant*. Furthermore,

$$\omega = \gamma_K + \int \varphi \, d\mu_K, \tag{8}$$

where γ_K and μ_K are the Robin constant and the Robin distribution of the compact set K, respectively; in particular, if $K = [a, b]$ then

$$\gamma_{[a, b]} = -\ln \left(\frac{b-a}{4} \right) \qquad \text{and} \qquad d\mu_{[a, b]}(x) = \frac{dx}{\pi \sqrt{(b-x)(x-a)}}.$$

From [12, 14] it follows that for certain weights $w_n(x)$ such as those in (3) and (4),

$$\lim_n \frac{1}{n} \ln |p_{n,n}(x)| = \omega - V_\lambda(x), \qquad \text{uniformly for } x \in \mathbb{R} \setminus K. \tag{9}$$

The main result of this paper can be formulated as the following theorem:

THEOREM. *Let w_n be as in* (3) *or* (4). *Then*

$$S_n = -2n(\omega(\varphi) - \gamma_K) + o(n) = -2n \int \varphi \, d\mu_K + o(n), \qquad n \to \infty. \tag{10}$$

From this theorem straightforward computations allow us to obtain explicit formulas for the entropies of the varying Jacobi and Laguerre weights:

COROLLARY 1. *For the varying Jacobi weight* $w_n(x) = (1-x)^{\alpha_n}(1+x)^{\beta_n}$ *given in* (3)

$$S_n = -2n\left\{\alpha \ln \frac{2}{\sqrt{1-a}+\sqrt{1-b}} + \beta \ln \frac{2}{\sqrt{1+a}+\sqrt{1+b}}\right\} + o(n),$$

where

$$a = \frac{\beta^2 - \alpha^2 - 4\sqrt{(1+\alpha)(1+\beta)(1+\alpha+\beta)}}{(\alpha+\beta+2)^2}, \tag{11}$$

$$b = \frac{\beta^2 - \alpha^2 + 4\sqrt{(1+\alpha)(1+\beta)(1+\alpha+\beta)}}{(\alpha+\beta+2)^2}. \tag{12}$$

In particular, if $\alpha = \beta$ (*the asymptotically Gegenbauer case*),

$$b = -a = \frac{\sqrt{1+2\alpha}}{1+\alpha},$$

and

$$S_n = -2\alpha n \ln\left(1 + \frac{1}{2\alpha+1}\right) + o(n).$$

COROLLARY 2. *For Laguerre weights* $w_n(x) = x^{\alpha_n}\exp(-\beta_n x)$ *given in* (4)

$$S_n = -n\left\{\alpha + 2 + \alpha \ln\left(\frac{\beta}{\alpha+1}\right)\right\} + o(n).$$

For applications it is necessary to study also the asymptotics for the Laguerre weights given in (4) with $\beta_n \equiv 1$. From the previous results it is immediate to obtain the formula:

COROLLARY 3. *For* $w_n(x) = x^{\alpha_n}\exp(-x)$ *with* $\alpha_n + o(n)$, $\alpha > 0$,

$$S_n(w_n) = \alpha_n \ln n - n\{\alpha + 2 - \alpha\ln(\alpha+1)\} + o(n).$$

Notice that although the dominant term in $S_n(w_n)$ in this case is $n \ln n$ and not n, this does not contradict general formula in (10) because condition (6) is not satisfied either.

The structure of the paper is as follows. In the next section we establish the weak-* asymptotics of the sequence $p^2_{n,n}(x)\,w_n(x)$, using the convergence of the coefficients of the recurrence relation involving the polynomials $p_{n,n-1}$, $p_{n,n}$, and $p_{n,n+1}$. Furthermore, we establish bounds that are used to estimate terms in the entropy integral. This enables us to prove in Section 3 the main result and its corollaries.

2. AUXILIARY RESULTS

The polynomials $p_{n,m}$ satisfy a three-term recurrence relation, namely

$$xp_{n,m}(x) = \sum_{j=-1}^{1} c_{m,m+j}(n)\, p_{n,m+j}(x), \tag{13}$$

where $c_{m,m}(n) \in \mathbb{R}$ and

$$c_{m,m+1}(n) = l_{n,m}/l_{n,m+1}, \qquad c_{m,m-1}(n) = l_{n,m-1}/l_{n,m}.$$

If for a fixed n the coefficients $c_{m,m+j}(n)$ converge as $m \to \infty$,

$$\lim_m c_{m,m}(n) = c_n, \qquad \lim_m c_{m,m+1}(n) = d_n > 0,$$

the weight $w_n(x)$ is said to belong to the Nevai-Blumenthal class $M(c_n, d_n)$. From [17, 18], a sufficient condition for this is $w_n(x) > 0$ a.e. on $[c_n - 2d_n, c_n + 2d_n]$. Then (see [15]), $p^2_{n,m}(x)\,w_n(x)\,dx$ weakly converges (as $m \to \infty$) to the Robin distribution on $[c_n - 2d_n, c_n + 2d_n]$. The fact that this assertion is true also in the case of varying n was first observed by Van Assche [21]; here we give for completeness a proof analogous to the one in the classical case.

Lemma 1. *If*

$$\lim_n c_{n,n+1}(n) = \lim_n c_{n,n-1}(n) = d > 0 \quad \textit{and} \quad \lim_n c_{n,n}(n) = c \in \mathbb{R}, \tag{14}$$

then for every polynomial q,

$$\int_\Delta q(x)\, p^2_{n,n}(x)\, w_n(x)\, dx \to \int q(x)\, d\mu_{[a,b]}(x), \tag{15}$$

where $a = c - 2d$ and $b = c + 2d$.

Proof. It is sufficient to show that all the moments of $p_{n,n}^2(x)\, w_n(x)\, dx$ converge to the corresponding moments of $\mu_{[a,b]}$. By Lemma 12 of [15, p. 45], for $j \in \mathbb{N}$,

$$x^j p_{n,m}^2(x)$$
$$= \sum_{1 \leqslant k_1, ..., k_j \leqslant 1} \prod_{i=0}^{j-1} c_{m+k_0+ \cdots +k_i,\, m+k_1+ \cdots +k_{i+1}}(n)\, p_{n,\, m+k_1+ \cdots +k_j}(x),$$

where $k_0 = 0$ and $k_j \in \mathbb{Z}$. Hence,

$$\mu_{n,m}^{(j)} := \int_\Delta x^j p_{n,m}^2(x)\, w_n(x)\, dx$$
$$= \sum{}' \prod_{i=0}^{j-1} c_{m+k_0+ \cdots +k_i,\, m+k_1+ \cdots +k_{i+1}}(n), \tag{16}$$

where the symbol $\sum'$ means that the indices k_j vary over the set $\{-1 \leqslant k_1, ..., k_j \leqslant 1,\ k_1 + \cdots + k_j = 0\}$. In what follows, we take $m = n$ and write $c_{n,n+j}$ instead of $c_{n,n+j}(n)$ whenever it cannot lead to confusion. Since by (14),

$$\lim_n c_{n,n+i} = (1 - |i|)\, c + |i|\, d, \qquad |i| \leqslant 1,$$

we have that

$$\mu^{(j)} := \lim_n \mu_{n,n}^{(j)} = \sum{}' \prod_{i=1}^{j} ((1 - |k_i|)\, c + |k_i|\, d)$$
$$= \sum{}' (d-c)^j \left(\frac{d}{d-c}\right)^{|k_1| + \cdots + |k_j|} \left(\frac{c}{d-c}\right)^{j - |k_1| - \cdots - |k_j|}$$
$$= \sum_{t=0}^{[j/2]} \binom{j}{t} \binom{j-t}{t} d^{2t} c^{j-2t}.$$

Now we compute the corresponding moments of $\mu_{[a,b]}$. Since

$$\frac{a+b}{2} = c \qquad \text{and} \qquad \frac{b-a}{2} = 2d,$$

an easy computation show that

$$\int_a^b x^j\, d\mu_{[a,b]}(x) = c^j \sum_{t=0}^{[j/2]} \binom{j}{2t} \left(\frac{2d}{c}\right)^{2t} \frac{1}{\pi} \int_{-1}^{1} \frac{x^{2t}\, dx}{\sqrt{1-x^2}} = \mu^{(j)}.$$

The assertion is established.

In order to prove results for Laguerre polynomials we need the following bounds:

LEMMA 2. *For the weights* (4), *there exists a constant* $b_1 > 0$ *such that for every* $m \in \mathbb{N}$,

$$\int_{b_1}^{+\infty} x^m p_{n,n}^2(x)\, w_n(x)\, dx \to 0, \qquad n \to \infty. \tag{17}$$

Proof. Let $L_n^{(\alpha)}(x)$ denote the Laguerre polynomial, orthonormal with respect to the weight $x^\alpha \exp(-x)$. Then

$$p_{n,n}(x) = (\beta_n)^{(\alpha_n+1)/2}\, L_n^{(\alpha_n)}(\beta_n x). \tag{18}$$

From the well known bounds on the largest zeros of Laguerre polynomials (see, e.g., inequality (6.31.7) in [20]) it follows that all the zeros of $p_{n,n}$ are uniformly bounded.

Furthermore, using the explicit expression (see (24) below) for the main coefficient of $L_n^{(\alpha)}(x)$ it is straightforward to work out the asymptotics for the leading coefficient $l_{n,n}$ of $p_{n,n}$,

$$l_{n,n}^2 = Ce^{\varkappa n}(1 + o(1)), \qquad n \to \infty,$$

where C and $\varkappa$ are constants which depend only on α and β (see (4)); their exact values play no role in what follows.

Now we can estimate the integral in the left hand side of (17). Take $b_1 > 1$ such that for every $n \in \mathbb{N}$ the zeros of $p_{n,n}$ lie on $[0, b_1]$. Then, on $[b_1, +\infty)$ the following bound is straightforward ($m \leqslant n$),

$$x^m p_{n,n}^2(x) \leqslant l_{n,n}^2 x^{3n}.$$

Using the Laplace method with $b_1 > (3+\alpha)/\beta$, we get

$$\int_{b_1}^{+\infty} x^m p_{n,n}^2(x)\, w_n(x)\, dx \leqslant C_1 l_{n,n}^2 e^{nf(b_1)},$$

where $f(x) = (3+\alpha)\ln x - \beta x$. Thus, it remains to take b_1 large enough to make $f(b_1) + \varkappa < 0$ and use the asymptotics for $l_{n,n}$. The lemma is proved.

Remark. The previous lemma also follows from Theorem 6.1 in [19, Sect. III.6].

In the next lemma we establish a weighted bound for the polynomial $p_{n,n}$ on Δ. More precise inequalities for the Jacobi weight can be found in [16].

LEMMA 3. *For the weights w_n given in* (3) *or* (4), *there exist positive constants C and q independent of n such that*

$$p_{n,n}^2(x)\, w_n(x) \leqslant Cn^q, \qquad \textit{for} \quad x \in \Delta. \tag{19}$$

Proof. We give a detailed proof for the varying Jacobi weight (3). Rewrite $w_n(x)$ in the form

$$w_n(x) = (1-x)^{2k_n + \hat{\alpha}_n}\,(1+x)^{2l_n + \hat{\beta}_n},$$

where $k_n, l_n \in \mathbb{N}$ and $\hat{\alpha}_n, \hat{\beta}_n \in [0, 2)$. Set

$$q_n(x) = p_{n,n}(x)(1-x)^{k_n}\,(1+x)^{l_n}, \qquad \hat{n} := \deg q_n = (1 + \alpha/2 + \beta/2)\, n + o(n).$$

Evidently,

$$\int_{-1}^{1} q_n^2(x)\, \hat{w}_n(x)\, dx = 1, \qquad \text{where} \quad \hat{w}_n(x) = (1-x)^{\hat{\alpha}_n}\,(1+x)^{\hat{\beta}_n}.$$

From the extremal property of the Christoffel kernel it follows that

$$q_n^2(x) \leqslant \sum_{m=0}^{\hat{n}} \hat{p}_m^2(x), \tag{20}$$

where $\hat{p}_m(x)$ are Jacobi polynomials orthonormal with respect to $\hat{w}_n(x)$. The following bounds are well known (see [20, Sect. 7.32]),

$$|\hat{p}_m(x)| \leqslant C_1 m^{5/2}, \qquad \text{for} \quad x \in [-1, 1],$$

with C_1 independent of n and m. Hence, from (20),

$$q_n^2(x) \leqslant \sum_{m=0}^{\hat{n}} C_1^2 m^5 \leqslant C_2 n^6.$$

Then,

$$p_{n,n}^2(x)\, w_n(x) = q_n^2(x)\, \hat{w}_n(x) \leqslant Cn^6.$$

The Laguerre case can be handled in a similar way. Representing the weight as above and making a change of variable, the problem is reduced to the estimation of the Christoffel kernel of degree $\hat{n} = (1 + \alpha/2)\, n + o(n)$ for the generalized Laguerre weights with bounded parameters $\hat{\alpha}_n$ and $\beta_n \equiv 1$. Lemma 3 then follows from the known estimates for Laguerre polynomials.

Finally, we apply Lemma 1 and Lemma 2 to the weights given in (3) and (4).

COROLLARY 4. *For the varying Jacobi weights* (3),

$$p_{n,n}^2(x)\, w_n(x)\, dx \to d\mu_{[a,b]}(x)$$

in the weak- topology, where a and b are given by* (11) *and* (12), *respectively.*

Proof. For the weight (3) the leading coefficient of orthonormal polynomial is well known (cf. formulas (4.21.6) and (4.3.4) in [20]),

$$l_{n,m} = \left(\frac{\alpha_n+\beta_n+2m+1}{m!\, 2^{\alpha_n+\beta_n+1}\Gamma(\alpha_n+\beta_n+m+1)\,\Gamma(\alpha_n+m+1)\,\Gamma(\beta_n+m+1)}\right)^{1/2} \times 2^{-m}\Gamma(\alpha_n+\beta_n+2m+1).$$

Hence, the coefficients in the recurrence relation (13) are

$$\frac{l_{n,n}^2}{l_{n,n+1}^2} = \frac{4(n+1)(\alpha_n+n+1)(\beta_n+n+1)(\alpha_n+\beta_n+n+1)}{(\alpha_n+\beta_n+2n+1)(\alpha_n+\beta_n+2n+2)^2\,(\alpha_n+\beta_n+2n+3)} \tag{21}$$

and

$$c_{n,n}(n) = \frac{\beta_n^2-\alpha_n^2}{(\alpha_n+\beta_n+2n+2)(\alpha_n+\beta_n+2n)}. \tag{22}$$

Thus, (14) holds with

$$c = \frac{\beta^2-\alpha^2}{(\alpha+\beta+2)^2}, \qquad \text{and} \qquad d = 2\,\frac{\sqrt{(1+\alpha)(1+\beta)(1+\alpha+\beta)}}{(\alpha+\beta+2)^2}. \tag{23}$$

Notice (cf. Example 1.17 in [19, Sect. IV.1]) that $[a, b]$ is the support K of the equilibrium measure μ corresponding to the external field

$$\varphi(x) = -\frac{\alpha}{2}\ln|1-x| - \frac{\beta}{2}\ln|1+x|.$$

Thus, the corollary is proved.

COROLLARY 5. *For the varying Laguerre weights* (4),

$$p_{n,n}^2(x)\, w_n(x)\, dx \to d\mu_{[a,b]}(x)$$

in the weak- topology, where a and b are given by*

$$a = \frac{\alpha+2-2\sqrt{\alpha+1}}{\beta}, \qquad \text{and} \qquad b = \frac{\alpha+2+2\sqrt{\alpha+1}}{\beta}.$$

Proof. By (5.1.1), (5.1.8) in [20] and (18) we have that

$$l_{n,m} = \beta^{((\alpha_n+1)/2)+m} / \sqrt{m!\,\Gamma(\alpha_n+m+1)}. \tag{24}$$

Thus, the coefficients of the recurrence relation (13) are

$$\frac{l_{n,n}}{l_{n,n+1}} = \frac{\sqrt{(n+1)(n+\alpha_n+1)}}{\beta_n}, \qquad \text{and} \qquad c_{n,n}(n) = \frac{2n+\alpha_n+1}{\beta_n}.$$

Weak asymptotics follows directly from (15) and Lemma 2. As above, (see Example 1.18 in [19, Sect. IV.1]), the equilibrium measure μ corresponding to the external field

$$\varphi(x) = -\frac{\alpha}{2}\ln x + \frac{\beta}{2}x$$

is supported on $K = [a, b]$.

3. ENTROPY ASYMPTOTICS FOR VARYING WEIGHTS

Denote $h_n(x) = p_{n,n}^2(x)\, w_n(x)$. Then the entropy (2) can be rewritten as

$$S_n = -\int_\Delta h_n(x) \ln h_n(x)\, dx + \int_\Delta h_n(x) \ln w_n(x)\, dx. \tag{25}$$

We estimate the first term in the right hand side of (25). By Lemma 3,

$$-\int_\Delta h_n(x) \ln h_n(x)\, dx \geqslant C - q \ln n \int_\Delta h_n(x)\, dx = C - q \ln n.$$

On the other hand, since $\psi(t) = t \ln t$ for $t > 0$ is convex, by Jensen's inequality,

$$-\int_\Delta \psi(h_n(x))\, dx \leqslant -\psi\left(\int_\Delta h_n(x)\, dx\right) = 0.$$

Thus, the first term in (25) is $O(\ln n)$. In order to find the main term of the asymptotic expansion of S_n it is sufficient to establish convergence and find the limit of

$$\frac{1}{n}\int_\Delta h_n(x) \ln w_n(x)\, dx.$$

Fix a bounded interval $\Delta' = [a', b']$ such that

$$K \subset \Delta' \subset \Delta,$$

where both inclusions are strict. Write

$$\int_{\Delta} h_n(x) \ln w_n(x)\, dx = \left(\int_{\Delta'} + \int_{\Delta \setminus \Delta'} \right) h_n(x) \ln w_n(x)\, dx. \tag{26}$$

We investigate first the Jacobi case. On the one hand, by (5),

$$\lim_n \int_{\Delta'} h_n(x) \left(\frac{1}{n} \ln w_n(x) + 2\varphi(x) \right) dx = 0. \tag{27}$$

Furthermore, by Corollary 4,

$$\lim_n \int_{\Delta'} \varphi(x)\, h_n(x)\, dx = \int \varphi(x)\, d\mu_{[a, b]}(x). \tag{28}$$

On the other hand, (9) implies that

$$\lim_n h_n(x) = 0, \tag{29}$$

uniformly on $\Delta \setminus \Delta'$, where the sequence $n^{-1} \ln w_n(x)$ is majorized by an L^1 function. Thus,

$$\lim_n \frac{1}{n} \int_{\Delta \setminus \Delta'} h_n(x) \ln w_n(x)\, dx = 0. \tag{30}$$

From (27), (28), and (30) we get

$$\lim_n \frac{1}{n} \int_{\Delta} h_n(x) \ln w_n(x)\, dx = \int \varphi(x)\, d\mu_{[a, b]}(x). \tag{31}$$

The Laguerre case is studied analogously with the added feature that Δ is unbounded. Splitting the integral like in (26), for the estimation of the first term the same arguments work. Moreover, they are valid also on $[0, a']$. Hence, it remains to consider

$$\frac{1}{n} \int_{b'}^{+\infty} h_n(x) \ln w_n(x)\, dx.$$

Since $n^{-1} \ln w_n(x) \leqslant cx$, $c > 0$, for $x \in [b', +\infty)$, we get

$$\frac{1}{n} \int_{b'}^{+\infty} h_n(x) \ln w_n(x)\, dx \leqslant c \int_{b'}^{+\infty} x h_n(x)\, dx.$$

It remains to take $b' > b_1$, where b_1 is as in Lemma 2, and use (17); thus the last integral vanishes as $n \to \infty$. This concludes the proof of (31) for the Laguerre weights (4) as well. Equation (8) is the last step to establish (10).

Corollaries 1 and 2 for the weights (3) and (4) are straightforward consequences of explicit formulas for the constants $\omega(\varphi)$ (see [19, Sect. IV.1]). Finally, for the weight $w_n(x) = x^{\alpha_n} \exp(-x)$ the change of variable $x = nt$ allows us to reduce the study of $S_n(w_n)$ to the situation of Corollary 2 with $\beta_n = n$.

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