ZERO ASYMPTOTIC BEHAVIOUR FOR ORTHOGONAL MATRIX POLYNOMIALS

By

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Abstract. Weak-star asymptotic results are obtained for the zeros of orthogonal matrix polynomials (i.e., the zeros of their determinants) on $\mathbb{R}$ from two different assumptions: first from the convergence of matrix coefficients occurring in the three-term recurrence for these polynomials; and, second, from conditions on the generating matrix measure. The matrix analogues of the Chebyshev polynomials of the first kind are also investigated.

1 Introduction

The study of zeros of orthogonal polynomials plays an important role in classical analysis because of its application to quadrature formulae, spectral analysis, digital filter design, etc. The asymptotic theory for such zeros has its origins in the works of Jentzsch [J] and Szegö [Sz] dealing with zeros of partial sums of power series. More recently, potential theoretic methods (see, e.g., [BSS], [SaT]) have been effectively utilized to obtain weak-star convergence of normalized zero counting distributions for asymptotically extremal sequences of polynomials (in particular, for orthogonal polynomials). In the matrix setting, the zeros of orthogonal polynomials again arise as nodes in quadrature formulae and as eigenvalues of block Jacobi matrices.

The purpose of this paper is to investigate zero asymptotics for orthogonal matrix polynomials.

We consider a $N \times N$ positive definite matrix of measures W for any Borel set $A \subset \mathbb{R}$, $W(A)$ is a positive semidefinite numerical matrix) having moments of every order, i.e., the matrix integral $\int_{\mathbb{R}} t^n dW(t)$ exists for any nonnegative integer n.

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Assuming that $\int P(t)dW(t)P^*(t)$ is nonsingular for all matrix polynomials P with nonsingular leading coefficient, the matrix inner product defined in the usual way by W in the space of matrix polynomials generates a sequence of orthonormal matrix polynomials $(P_n)_n$ satisfying

$$\int P_n(t)dW(t)P_m^*(t) = \delta_{n,m}I, \quad n,m \geq 0.$$

Here $P_n(t)$ is a matrix polynomial of degree n with a nonsingular leading coefficient and is defined up to a multiplication on the left by a unitary matrix.

As in the scalar case, the sequence of orthonormal matrix polynomials $(P_n)_n$ satisfies a three-term recurrence relation

$$P_n(t) = A_{n+1}P_{n+1}(t) + B_nP_n(t) + A_n^*P_{n-1}(t), \quad n \geq 0, \tag{1.1}$$

where ${}_{-1}(t) = \theta$, $P_0(t) \in \mathbb{C}^{N\times N} \setminus \{\theta\}$, A_n are nonsingular matrices and B_n are hermitian. (Here and in the rest of this paper, we write θ for the null matrix, the dimension of which can be determined from the context.) The polynomials $R_n(t) = U_nP_n(t)$ with $U_nU_n^* = I$ are also orthonormal with respect to the same positive definite matrix of measures with respect to which $(P_n)_n$ are orthonormal and satisfy a three-term recurrence relation like (1.1) with coefficients $U_{n-1}A_nU_n^*$ instead of A_n and $U_nB_nU_n^*$ instead of B_n. This three-term recurrence relation characterizes the orthonormality of a sequence of matrix polynomials with respect to a positive definite matrix of measures (see, for instance, [AN] or [DL]). In [D2], [D3] and [DV], a very close relationship between orthogonal matrix polynomials and scalar polynomials satisfying a higher order recurrence relation was established; this relationship has been used to show that matrix orthogonality is a useful tool for solving certain problems of scalar orthogonality (see [D2, Sect. 5]).

In [DL], it is proved that the zeros of the n-th orthonormal matrix polynomial P_n are real and have multiplicity at most N (by a zero of a matrix polynomial $P(t)$ we mean a zero of $\det(P_n(t))$). (We show in Section 4 that the zeros of $(P_n)_n$ are in the convex hull of the support of W.) We write $x_{n,k}$, $k = 1, \dots, m$ for the distinct zeros of the polynomial P_n, and l_k for the multiplicity of $x_{n,k}$. The aim of this paper is the study of the asymptotic behaviour of the discrete positive measures

$$\sigma_n = \frac{1}{nN}\sum_{k=1}^{m} l_k\delta_{x_{n,k}}, \quad n \geq 1. \tag{1.2}$$

The notion of convergence for measures (or matrices of measures) we use is the usual *weak convergence*: a sequence of measures $(\mu_n)_n$ (or matrices of measures) on a metric space X converges weakly to μ if

$$\lim_n \int f d\mu_n = \int f d\mu,$$

for any continuous and bounded function $f : X \to \mathbb{C}$ ($f : X \to C^{N\times N}$, respectively). For a treatment of this classical concept see [B].

We develop this study from two different perspectives: first, assuming that the sequences which appear in the three-term recurrence formula for $(P_n)_n$ converge; and second, assuming some conditions on the matrix of measures W.

In Section 2, we study the zero behaviour from the recurrence formula. To do that, we consider the matrix Nevai class, which was introduced in [D1]: given two matrices A and B, with B hermitian, we say that a sequence of orthonormal matrix polynomials $(P_n)_n$ satisfying (1.1) is in the matrix Nevai class $M(A, B)$ if $\lim_n A_n = A$, $\lim_n B_n = B$. Furthermore, we say that a positive definite matrix of measures W is in the matrix Nevai class $M(A, B)$ if some of its sequences of orthonormal polynomials are in $M(A, B)$. Let us notice that a positive matrix of measures W can belong to several Nevai classes, since the sequence of orthonormal polynomials with respect to W is not unique (recall that it is defined upto multiplication on the left by unitary matrices: see above).

When A is hermitian and nonsingular, we associate to the matrix Nevai class $M(A, B)$ the orthonormal matrix polynomials $(T_n^{A,B})_n$ defined by the recurrence formula

$$\text{(1.3)} \qquad \begin{cases} tT_0^{A,B}(t) = \sqrt{2}AT_1^{A,B}(t) + BT_0^{A,B}(t), \\ T_1^{A,B}(t) = AT_2^{A,B}(t) + BT_1^{A,B}(t) + \sqrt{2}AT_0^{A,B}(t), \\ T_n^{A,B}(t) = AT_{n+1}^{A,B}(t) + BT_n^{A,B}(t) + AT_{n-1}^{A,B}(t), \quad n \geq 2, \end{cases}$$

with initial conditions $T_0^{A,B}(t) = I$. This sequence is orthonormal with respect to a positive definite matrix of measures which we denote by $X_{A,B}$, and constitutes the matrix analogue of the orthonormal Chebyshev polynomials of the first kind. Let us notice that from $T_0^{A,B}(t) = I$ it follows that $\int dX_{A,B}(t) = I$. (Chebyshev matrix polynomials of the second kind were introduced in [D1].)

As the main result of this section, we state the existence of the zero asymptotic behaviour for the matrix Nevai class $M(A, B)$ (see Lemma 2.1), although we only find this asymptotic zero distribution explicitly when A is hermitian and nonsingular.

Theorem 1.1. *Let $(P_n)_n$ be orthonormal matrix polynomials satisfying the three-term recurrence relation (1.1). Assume that* $\lim_n A_n = A$, $\lim_n B_n = B$ *with A hermitian and nonsingular. Then the sequence of discrete positive measures defined by (1.2) has the asymptotic behaviour*

$$\lim_{n\to\infty} \sigma_n = \operatorname{tr}\left(\frac{1}{N}X_{A,B}\right),$$

where $X_{A,B}$ is the matrix weight for the Chebyshev matrix polynomials of the first kind defined by (1.3).

The key to establish this zero asymptotic behaviour is to consider the auxiliary sequence of discrete positive definite matrices of measures given by

$$\mu_n = \frac{1}{nN}\sum_{k=1}^{m}\left(\sum_{i=0}^{n-1} P_i(x_{n,k})\Gamma_{n,k}P_i^*(x_{n,k})\right)\delta_{x_{n,k}}, \quad n \geq 1, \tag{1.4}$$

where $\Gamma_{n,k}$ is the weight in the quadrature formula for $(P_n)_n$ associated to the zero $x_{n,k}$ (see [D4, Th. 3.1, p. 1186]). We then prove that $(\sigma_n)_n$ has the same asymptotic behaviour as $\left(\operatorname{tr}(\mu_n)\right)_n$ and that $\mu_n \to (1/N)X_{A,B}$, as n tends to ∞.

It is worth noting that in the scalar case ($N = 1$)

$$\Gamma_{n,k} = \frac{1}{\sum_{i=0}^{n-1} p_i^2(x_{n,k})},$$

and so $\mu_n = \sigma_n$; also,

$$\int \frac{d\sigma_n}{z-t} = \frac{p_n'(z)}{np_n(z)}.$$

Neither of these facts is true in the matrix case. Indeed, only when $x_{n,k}$ is a zero of the largest multiplicity ($l_k = N$) do we have

$$\Gamma_{n,k} = \left(\sum_{i=1}^{n-1} P_i^*(x_{n,k})P_i(x_{n,k})\right)^{-1}.$$

Otherwise, the matrix $\Gamma_{n,k}$ is singular, while the matrix $\sum_{i=1}^{n-1} P_i^*(x_{n,k})P_i(x_{n,k})$ is always nonsingular; moreover,

$$\Gamma_{n,k}\left(\sum_{i=1}^{n-1} P_i^*(x_{n,k})P_i(x_{n,k})\right)$$

could be non-hermitian.

For A hermitian and nonsingular, we have proved that the zero asymptotic behaviour is given by the matrix of measures $X_{A,B}$, with respect to which the Chebyshev matrix polynomials of the first kind are orthonormal. This sequence of matrix polynomials satisfies a three-term recurrence formula with constant coefficients $(A_n)_{n\geq 2}$, and $(B_n)_n$. This three-term recurrence formula is the same as the one for the Chebyshev matrix polynomials of the second kind, but with different initial conditions. In [D1] it is proved that the matrix weight for Chebyshev matrix polynomials of the second kind controls the ratio asymptotic behaviour of orthonormal matrix polynomials in the matrix Nevai class. Hence, for A hermitian and nonsingular, the constant sequences $A_n = A$, $n \geq 2$ and $B_n = B$, $n \geq 0$, control both the zero asymptotics and the ratio asymptotics in the matrix Nevai class $M(A, B)$. The difference between these two depends on the initial condition: we have $A_1 = \sqrt{2}A$ for zero asymptotics and $A_1 = A$ for ratio asymptotics. We give an example showing that this is not the case when A is non-hermitian and nonsingular.

In Section 3, using Markov's theorem for orthogonal matrix polynomials established in [D4], we give, when A is positive definite, the following explicit expression for the weight $X_{A,B}$ with respect to which the Chebyshev matrix polynomials $(T_n^{A,B})_n$ are orthonormal. Consider the matrix polynomial

$$K_{A,B}(z) = \frac{A^{-\frac{1}{2}}(B - zI)A^{-\frac{1}{2}}}{2},$$

where $A^{-\frac{1}{2}}$ is the inverse of the unique positive definite square root of A. Since for x real $K_{A,B}(x)$ is hermitian, we can diagonalize it in the form $K_{A,B}(t) = U(t)D(t)U^*(t)$, where $D(t)$ is a diagonal matrix with entries $d_{i,i}(t) \in \mathbb{R}$, $i = 1, \dots, N$ and $U(x)U(x)^* = I$. Then

$$dX_{A,B}(x) = \frac{1}{\pi}A^{-\frac{1}{2}}U(x)U^*(x)A^{-\frac{1}{2}}\,dx, \tag{1.5}$$

where $T(x)$ is the diagonal matrix with entries

$$t_{i,i}(x) = \begin{cases} \dfrac{1}{\sqrt{1 - d_{ii}^2(x)}}, & \text{if } d_{ii}(x) \in (-1,1), \\ 0, & \text{if } d_{ii}(x) \notin (-1,1). \end{cases}$$

The support of $X_{A,B}$ is then the set of real numbers

$$\operatorname{supp}(X_{A,B}) = \{x \in \mathbb{R} : A^{-\frac{1}{2}}(xI - B)A^{-\frac{1}{2}} \quad \text{has an eigenvalue in } [-2,2]\},$$

which consists of a finite union of at most N disjoint bounded nondegenerate intervals, whose endpoints are roots of the scalar polynomial

$$\det\Big(K_{A,B}^2(z) - I\Big).$$

We illustrate this with an example: for the particular case $A = \frac{1}{2}I$, we have

$$\lim_n \sigma_n = \frac{1}{\pi N}\sum_{k=1}^{N} \frac{\chi_{[b_k-1,b_k+1]}(t)dt}{\sqrt{1-(b_k-t)^2}},$$

where $b_1, \dots, b_N$ are the eigenvalues of the matrix B.

In Section 4, we deduce the zero asymptotics for the orthogonal matrix polynomials from the matrix of measures W. To do that, we consider the matrix W' formed with the Radon–Nikodym derivatives of the entries of the matrix of measures W with respect to the trace measure of W. Under the assumption that each of the scalar measures $(\Delta_k/\Delta_{k-1})d(\mathrm{tr}(W))$, $k = 1, \dots, N$(Δ_k denotes the k-th principal determinant of W') is finite and has the Erdös–Turán property on $[-1, 1]$ (i.e., they are supported in $[-1, 1]$ and have strictly positive Radon–Nikodym derivatives with respect to Lebesgue measure a.e. on $[-1, 1]$), we prove (see Theorem 4.5) that the normalized zero counting measures σ_n in (1.2) for the orthonormal polynomials P_n with respect to W satisfy

$$\lim_n \sigma_n = \frac{dt}{\pi\sqrt{1-t^2}}, \quad t \in [-1, 1].$$

2 Zero asymptotics from the recurrence relation

In this section, we prove Theorem 1.1. To do that we use the so-called *method of moments*. Suppose that μ_n and μ are probability measures on $\mathbb{R}$ with moments of every order and that μ has compact support. If $\lim_n \int t^k d\mu_n(t) = \int t^k d\mu(t)$ for $k = 0, 1, \dots$, then $\mu_n \to \mu$ (see [F]). This method can easily be extended to positive definite matrices of measures.

We recall that both the set of zeros of a sequence of matrix polynomials and the support of a matrix of measures in the matrix Nevai class are bounded. Hence the matrices of measures defined by (1.4) and any of their weak limit points have compact support.

We now prove a lemma which establishes the existence of zero asymptotic behavior for orthogonal matrix polynomials in the matrix Nevai class $M(A, B)$ and the relationship between these asymptotics and the asymptotics of the sequence of matrices of measures $(\mu_n)_n$ (see (1.4)). We emphasize that we prove this lemma without any restriction on the matrix A.

Lemma 2.1. *Let $(P_n)_n$ be orthonormal matrix polynomials satisfying the three-term recurrence relation (1.1). Assume that $\lim_n A_n = A$ and $\lim_n B_n = B$. Then there exists a positive definite matrix of measures μ such that the sequence of discrete positive measures defined by (1.2) and the sequence of discrete positive definite matrices of measures defined by (1.4) have the following asymptotic behaviour:*

$$\lim_n \mu_n = \mu \quad \textit{and} \quad \lim_n \sigma_n = \mathrm{tr}(\mu).$$

Proof. The quadrature formula for $(P_n)_n$ (see (3.2) in Th. 3.1 of [D4]) gives $\int d\mu_n(t) = 1/N$; then, since μ_n is a positive definite matrix of measures for $n \geq 0$ we conclude (by using the Banach–Alaoglu theorem, for instance) that there exists at least one positive definite matrix of measures μ which is a limit point of $(\mu_n)_n$. From the method of moments, it will be enough to prove that

$$\lim_n \int t^l d\sigma_n(t) = \operatorname{tr}\left(\int t^l d\mu(t)\right), \quad l = 0, 1, 2, \dots,$$
$$\lim_n \int t^l d\mu_n(t) = \int t^l d\mu(t), \quad l = 0, 1, 2, \dots.$$

We now consider the N-Jacobi matrix associated to the sequence $(P_n)_n$, that is, the $(4N-1)$-banded infinite hermitian matrix defined by

$$J = \begin{pmatrix} B_0 & A_1 & \theta & \theta & \dots \\ A_1^* & B_1 & A_2 & \theta & \dots \\ \theta & A_2^* & B_2 & A_3 & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix},$$

and the matrix

$$J_{A,B} = \begin{pmatrix} B & A & \theta & \theta & \dots \\ A^* & B & A & \theta & \dots \\ \theta & A^* & B & A & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{2.1}$$

Notation

Throughout this paper we denote

- $J_{(n,m)}$ = entry (n, m) of the matrix J.
- $J_{[n,m]}$ = block entry (n, m) of the block matrix J.
- $(J)_{|n}$ = truncation of J of size $n \times n$.

First of all, we compute $\lim_{n\to\infty} \int t^l d\sigma_n(t)$ for $l \geq 0$. From the definition of σ_n we have

$$\int t^l d\sigma_n(t) = \frac{1}{nN} \sum_{k=1}^{m} l_k x_{n,k}^l.$$

In Lemma 2.1, p. 101 of [DL], it is proved that the zeros of P_n are the eigenvalues of $(J)_{|nN}$. Hence, $x_{n,k}^l$ are the eigenvalues of $[(J)_{|nN}]^l$; and we get

$$\int t^l d\sigma_n(t) = \frac{1}{nN} \operatorname{tr}\left[(J)_{|nN}\right]^l. \tag{2.2}$$

But the matrices $[(J)_{|nN}]^l$ and $(J^l)_{|nN}$ are equal except for the last lN rows and lN columns; hence, taking into account that A_n and B_n converge, we see that

$$\lim_n \frac{1}{nN}\text{tr}\Big[(J)_{|nN}\Big]^l = \lim_n \frac{1}{nN}\text{tr}(J^l)_{|nN}.$$

If we consider the matrix $(J^l)_{|nN}$ as a block matrix formed by blocks of size $N \times N$, we can then write

$$\lim_n \frac{1}{nN}\text{tr}\Big[(J)_{|nN}\Big]^l = \lim_{n\to\infty} \frac{1}{n}\Big[\sum_{i=1}^{n} \frac{1}{N}\text{tr}\Big((J^l)_{[i,i]}\Big)\Big]. \tag{2.3}$$

It is clear that $(J^l_{A,B})_{[n,n]} = (J^l_{A,B})_{[l,l]}$ for $n \geq l$. Since A_n and B_n converge to A and B respectively, $(J^l)_{[i,i]}$ (which only contains recurrence coefficients A_{n+j} and B_{n+j} with j bounded) converges to $(J^l_{A,B})_{[l,l]}$ as i tends to ∞. Hence, from (2.2) and (2.3), we deduce that

$$\lim_{n\to\infty} \int t^l d\sigma_n(t) = \frac{1}{N}\text{tr}\Big((J^l_{A,B})_{[l,l]}\Big).$$

Now the lemma will be proved if we show that

$$\lim_{n\to\infty} \int t^l d\mu_n = \frac{1}{N}(J^l_{A,B})_{[l,l]}. \tag{2.4}$$

From the definition of μ_n,

$$\begin{aligned}\int t^l d\mu_n =& \frac{1}{nN}\sum_{k=1}^{m}\sum_{i=0}^{n-1} x^l_{n,k} P_i(x_{n,k})\Gamma_{n,k}P_i^*(x_{n,k})\\ =& \frac{1}{nN}\sum_{k=1}^{m}\sum_{i=0}^{n-l-1} x^l_{n,k} P_i(x_{n,k})\Gamma_{n,k}P_i^*(x_{n,k})\\ &+ \frac{1}{nN}\sum_{k=1}^{m}\sum_{i=n-l}^{n-1} x^l_{n,k} P_i(x_{n,k})\Gamma_{n,k}P_i^*(x_{n,k}).\end{aligned}$$

Write

$$\nu_{n,l} = \frac{1}{nN}\sum_{k=1}^{m}\sum_{i=n-l}^{n-1} P_i(x_{n,k})\Gamma_{n,k}P_i^*(x_{n,k})\delta_{x_{n,k}}, \quad n \geq l. \tag{2.5}$$

Using the quadrature formula for the polynomials $(P_n)_n$ (taking into account that $l + 2i \leq 2n - 1$, for $0 \leq i \leq n - l - 1$ and $l \geq 0$), we have

$$\int t^l d\mu_n = \frac{1}{nN}\sum_{i=0}^{n-l-1} \int t^l P_i(t)dW(t)P_i^*(t) + \int t^l d\nu_{n,l}(t). \tag{2.6}$$

We first prove that $\lim_{n\to\infty}\int t^l d\nu_{n,l}(t)=\theta$. Moreover, we prove that the sequence $(\nu_{n,l})_n$ tends to θ as n tends to ∞. Indeed, $(\nu_{n,l})$ is a sequence of positive definite matrices of measures and

$$\begin{aligned}\int d\nu_{n,l} =& \frac{1}{nN}\sum_{k=1}^{m}\sum_{i=n-l}^{n-1} P_i(x_{n,k})\Gamma_{n,k}P_i^*(x_{n,k})\\ =& \frac{1}{nN}\sum_{i=n-l+1}^{n-1}\int P_i(t)dW(t)P_i^*(t)=\frac{lI}{nN}\le I.\end{aligned}$$

The sequence of positive definite matrices of measures $(\nu_{n,l})_n$ must have some limit point. Since $\lim_{n\to\infty}\int d\nu_{n,l}(t)=\theta$, we conclude that all the limit points ν of $(\nu_{n,l})_n$ satisfy $\int d\nu=\theta$, that is, $\nu=\theta$. This proves that the sequence $(\nu_{n,l})_n$ tends weakly to θ.

Finally we prove that

$$\lim_{n\to\infty}\frac{1}{nN}\sum_{i=0}^{n-l-1}\int t^l P_i(t)dW(t)P_i^*(t)=\frac{1}{N}(J^l_{A,B})_{[l,l]}. \tag{2.7}$$

Indeed, writing

$$t^l P_i(t)=\sum_{k=-i}^{l}\Delta_{l,i,k}P_{i+k},$$

we have

$$\frac{1}{nN}\sum_{i=0}^{n-l-1}\int t^l P_i(t)dW(t)P_i^*(t)=\frac{1}{nN}\sum_{i=0}^{n-l}\Delta_{l,i,0}. \tag{2.8}$$

Consider again the matrix J^l as a block matrix formed by blocks of size $N\times N$. Then the three-term recurrence relation for the polynomials $(P_n)_n$ clearly shows that, for $i\ge l$, $\Delta_{l,i,0}=(J^l)_{[i,i]}$. Since A_n and B_n converge to A and B respectively, $(J^l)_{[i,i]}$ (which only contains recurrence coefficients A_{n+j} and B_{n+j} with j bounded) converges to $(J^l_{A,B})_{[l,l]}$ as i tends to ∞; hence, from (2.8) we deduce (2.7). □

In the next lemma, we identify the matrix of measures μ which appears in the previous lemma. In this case, we need to assume that A is hermitian and nonsingular. Theorem 1.1 will then be a consequence of Lemmas 2.1 and 2.2.

Lemma 2.2. *Let $(P_n)_n$ be orthonormal matrix polynomials satisfying the three-term recurrence relation (1.1). Assume that* $\lim_n A_n=A$ *and* $\lim_n B_n=B$, *with A hermitian and nonsingular. Then*

$$\lim_n \mu_n=\frac{1}{N}X_{A,B},$$

where $X_{A,B}$ is the matrix weight for the Chebyshev matrix polynomials of the first kind defined by (1.3).

Proof. From the definition, we have

$$\int T_l^{A,B}(t)dX_{A,B}(t) = \begin{cases} I, & \text{for } l = 0, \\ \theta, & \text{for } l \neq 0. \end{cases}$$

Since μ (the matrix of measures which appears in Lemma 2.1) and $X_{A,B}$ both have compact support, and $(T_l^{A,B})_l$ forms a basis of matrix polynomials, the lemma will follow if we prove that

$$\lim_{n\to\infty} \int T_l^{A,B}(t)d\mu_n(t) = \begin{cases} \frac{1}{N}I, & \text{for } l = 0, \\ \theta, & \text{for } l \neq 0. \end{cases}$$

Proceeding as before, when we compute the moments of μ_n, we have

$$\int T_l^{A,B}(t)d\mu_n(t) = \frac{1}{nN}\sum_{i=0}^{n-l-1}\int T_l^{A,B}(t)P_i(t)dW(t)P_i^*(t) + \int T_l^{A,B}(t)d\nu_n(t),$$

where ν_n is given by (2.5). Since $(\nu_n)_n$ tends to θ, we have

$$\lim_{n\to\infty}\int T_l^{A,B}(t)d\mu_n(t) = \lim_{n\to\infty}\frac{1}{nN}\sum_{i=0}^{n-l-1}\int T_l^{A,B}(t)P_i(t)dW(t)P_i^*(t).$$

The theorem will follow if we prove that

$$\lim_{n\to\infty}\int T_l^{A,B}(t)P_n(t)dW(t)P_n^*(t) = \begin{cases} I, & \text{for } l = 0, \\ \theta, & \text{for } l \neq 0. \end{cases} \tag{2.9}$$

To do this, write

$$T_l^{A,B}(t)P_n(t) = \sum_{k=n-l}^{n+l}\Delta_{k,l,n}P_k(t), \tag{2.10}$$

where $\Delta_{k,l,n}$, $k = n-l, \dots, n+l$, are numerical matrices. Then

$$\int T_l^{A,B}(t)P_n(t)dW(t)P_n^*(t) = \Delta_{n,l,n},$$

and (2.9) will follow if we prove that

$$\lim_{n\to\infty}\Delta_{k,l,n} = \begin{cases} I, & \text{for } k = n,\ l = 0, \\ \frac{1}{\sqrt{2}}I, & \text{for } l \geq 1 \text{ and } k = n-l, n+l, \\ \theta, & \text{for } l \geq 1 \text{ and } n-l+1 \leq k \leq n+l-1. \end{cases} \tag{2.11}$$

We use induction on l. When $l = 0$, the result is immediate.

For $l = 1$, taking into account that $T_1^{A,B}(t) = (\sqrt{2}A)^{-1}(tI - B)$ and the three-term recurrence formula for $(P_n)_n$ we have

$$T_1^{A,B}(t)P_n(t) = (\sqrt{2}A)^{-1}A_{n+1}P_{n+1}(t) + \Big((\sqrt{2}A)^{-1}B_n - (\sqrt{2}A)^{-1}B\Big)P_n(t) + (\sqrt{2}A)^{-1}A_n^*P_{n-1}(t).$$

Since A_n and B_n converge to A and B respectively, we have (2.11) for $l = 1$.

For $l = 2$, proceeding in a similar way, we find

$$\begin{aligned} T_2^{A,B}(t)P_n(t) = A^{-1}\Big[&\Delta_{n+1,1,n}A_{n+2}P_{n+2}(t) \\ &+ \big(\Delta_{n,1,n}A_{n+1} + \Delta_{n+1,1,n}B_{n+1}B\Delta_{n+1,1,n}\big)P_{n+1}(t) \\ &+ \big(\Delta_{n-1,1,n}A_n + \Delta_{n,1,n}B_n + \Delta_{n+1,1,n}A_{n+1}^* \\ &\qquad - B\Delta_{n,1,n} - \sqrt{2}A\big)P_n(t) \\ &+ \big(\Delta_{n-1,1,n}B_{n-1} + \Delta_{n,1,n}A_n^* - \Delta_{n-1,1,n}\big)P_{n-1}(t) \\ &+ \Delta_{n-1,1,n}A_{n-1}^*P_{n-2}(t)\Big], \end{aligned}$$

which shows that (2.11) holds for $l = 2$ as $n \to \infty$.

Now suppose the result valid up to $l \geq 2$. The three-term recurrence formula for the matrix polynomials $(T_n^{A,B})_n$ gives

$$T_{l+1}^{A,B}(t)P_n(t) = \Big(A^{*-1}tT_l^{A,B}(t) - A^{*-1}BT_l^{A,B}(t) - A^{*-1}AT_{l-1}^{A,B}(t)\Big)\,P_n(t).$$

Formula (2.10) and the three-term recurrence formula for the matrix polynomials $(P_n)_n$ give

$$\begin{aligned} \Delta_{k,l+1,n} = A^{*-1}&\left(\Delta_{k-1,l,n}A_k + \Delta_{k,l,n}B_k + \Delta_{k+1,l,n}A_{k+1}^*\right) \\ &- A^{*-1}B\Delta_{k,l,n} - A^{*-1}A\Delta_{k,l-1,n}. \end{aligned}$$

For $k \geq n-l+2$ or $k \leq n+l-2$, the induction hypothesis shows that $\lim_{n\to\infty}\Delta_{k,l+1,n} = \theta$. For $k = n-l-1, n-l, n-l+1, n+l-1, n+l$ and $n+l+1$, from the induction hypothesis and taking into account that $\lim_n A_n = A$ and $\lim_n B_n = B$, it follows respectively that

$$\lim_{n\to\infty} \Delta_{n-l-1,l+1,n} = A^{*-1}\frac{1}{\sqrt{2}}A = \frac{1}{\sqrt{2}}I,$$
$$\lim_{n\to\infty} \Delta_{n-l,l+1,n} = A^{*-1}\frac{1}{\sqrt{2}}B - A^{*-1}\frac{1}{\sqrt{2}}B = \theta,$$
$$\lim_{n\to\infty} \Delta_{n-l+1,l+1,n} = A^{*-1}\frac{1}{\sqrt{2}}A - A^{*-1}\frac{1}{\sqrt{2}}A = \theta,$$
$$\lim_{n\to\infty} \Delta_{n+l-1,l+1,n} = A^{*-1}\frac{1}{\sqrt{2}}A^* - A^{*-1}\frac{1}{\sqrt{2}}A = \theta,$$
$$\lim_{n\to\infty} \Delta_{n+l,l+1,n} = A^{*-1}\frac{1}{\sqrt{2}}B - A^{*-1}\frac{1}{\sqrt{2}}B = \theta,$$
$$\lim_{n\to\infty} \Delta_{n+l+1,l+1,n} = A^{*-1}\frac{1}{\sqrt{2}}A = \frac{1}{\sqrt{2}}I.$$

Thus (2.11) is proved. □

In Lemma 2.1, the existence of the zero asymptotic behaviour for orthogonal matrix polynomials in the matrix Nevai class $M(A, B)$ is proved independently of the hermitian character of A. Moreover, the moments of the matrix of measures of the zero asymptotics are explicitly given. When A is hermitian, we have proved in Lemma 2.2 that the orthonormal matrix polynomials with respect to this matrix of measures have constant coefficients $(A_n)_{n\geq 2}$, $(B_n)_n$. In [D1], it is proved that the matrix weight for Chebyshev matrix polynomials of the second kind controls the ratio asymptotic behaviour of orthonormal matrix polynomials in the matrix Nevai class; these Chebyshev polynomials of the second kind have a recurrence formula with constant coefficients $A_n = A^*, n \geq 1$, $B_n = B, n \geq 0$. Hence, for A hermitian and nonsingular, the constant sequences $A_n = A$, $n \geq 2$ and $B_n = B$, $n \geq 0$, control both the zero asymptotics and the ratio asymptotics in the matrix Nevai class $M(A, B)$. The difference between these two depends on the initial condition: we have $A_1 = \sqrt{2}A$ for zero asymptotics and $A_1 = A$ for ratio asymptotics. We give an example showing that this is not the case when A is non-hermitian and nonsingular.

Suppose the three-term recurrence relation for the orthonormal matrix polynomials with respect to the matrix of measures μ which appears in Lemma 2.1 is

$$T_n(t) = \alpha_{n+1}T_{n+1}(t) + \beta_n T_n(t) + \alpha_n^* T_{n-1}(t), \quad n \geq 0,$$

with $\alpha_{-1} = \theta$.

For $n = 0$, we get $T_1(t) = \alpha_1^{-1}(tI - \beta_0)$; and, using the three-term recurrence relation for P_n, we get

$$T_1(t)P_n(t) = \alpha_1^{-1}A_{n+1}P_{n+1}(t) + \alpha_1^{-1}(B_n - \beta_0)P_n(t) + \alpha_1^{-1}A_n^*P_{n-1}(t). \tag{2.12}$$

Proceeding as in the proof of Lemma 2.2, we get

$$\Delta_{n+1,1,n} = \alpha_1^{-1} A_{n+1},$$
$$\Delta_{n,1,n} = \alpha_1^{-1}(B_n - \beta_0),$$
$$\Delta_{n-1,1,n} = \alpha_1^{-1} A_n^*.$$

Thus, since $\lim_{n\to\infty} \Delta_{n,1,n} = \theta$, we get $\beta_0 = B$. For $n = 1$ in (2.12), using the three-term recurrence relation for P_n, we find

$$\lim_{n\to\infty} \Delta_{n,2,n} = \alpha_2^{-1}\{\alpha_1^{-1}A^*A + \alpha_1^{-1}AA^* - \alpha_1^*\} = \theta,$$

which gives $\alpha_1\alpha_1^* = AA^* + A^*A$. Thus the possible solutions α_1 are RC, where R is a fixed square root of $(AA^* + A^*A)$ and C is any unitary matrix.

For $n = 2$, we find in the same way that

$$\lim_{n\to\infty} \Delta_{n,3,n} = \alpha_3^{-1}\alpha_2^{-1}\{\alpha_1^{-1}(A^*BA + ABA^*) - \beta_1\alpha_1^*\}\theta,$$

which gives $\beta_1 = \alpha_1^{-1}(A^*BA + ABA^*)\alpha_1^{*-1}$.

For $n = 3$, we find

$$\lim_{n\to\infty} \Delta_{n,4,n} = \alpha_4^{-1}\alpha_3^{-1}\{\alpha_2^{-1}\{\alpha_1^{-1}(A^*A^*AA + A^*BBA + ABBA^* + AAA^*A^*) - \beta_1^2\alpha_1^*\} - \alpha_1\alpha_1^*\} = \theta$$

which gives

$$\alpha_2 = \{\alpha_1^{-1}A^*A^*AA + A^*BBA + ABBA^* + AAA^*A^*) - \beta_1^2\alpha_1^*\}(\alpha_1\alpha_1^*)^{-1}. \tag{2.13}$$

We now show that for the example

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$\beta_1 \neq B$ and $\alpha_2 \neq A^*$.

We choose the square root R of $AA^* + A^*A$ to be

$$R = \frac{1}{2}\begin{pmatrix} 1+\sqrt{5} & -1+\sqrt{5} \\ -1+\sqrt{5} & 1+\sqrt{5} \end{pmatrix};$$

then from $\beta_1 = \alpha_1^{-1}(A^*BA + ABA^*)\alpha_1^{*-1}$ it follows that if $\beta_1 = B$ we would have $B = C^*R^{-1}(A^*BA + ABA^*)R^{-1}C$, that is,

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2\sqrt{5}}\begin{pmatrix} 1+\sqrt{5} & 1-\sqrt{5} \\ 1-\sqrt{5} & 1+\sqrt{5} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{2\sqrt{5}}\begin{pmatrix} 1+\sqrt{5} & 1-\sqrt{5} \\ 1-\sqrt{5} & 1+\sqrt{5} \end{pmatrix} C.$$

Taking the determinant of this expression, we get $-1 = -\frac{1}{5}$, which is not possible; hence $\beta_1 \neq B$.

We now show that $\alpha_2 \neq A^*$, for any choice of the unitary matrix C. From (2.13), taking into account that

$$(A^*A^*AA + A^*BBA + ABBA^* + AAA^*A^*) = \begin{pmatrix} 9 & 6 \\ 6 & 9 \end{pmatrix},$$

we get

$$\begin{aligned}
\alpha_2 =& \alpha_1^{-1} \begin{pmatrix} 9 & 6 \\ 6 & 9 \end{pmatrix} (\alpha_1\alpha_1^*)^{-1} - \alpha_1^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \alpha_1^{*-1}\alpha_1^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \alpha_1^{*-1}\alpha_1^*(\alpha_1\alpha_1^*)^{-1} \\
=& \alpha_1^{-1} \begin{pmatrix} 9 & 6 \\ 6 & 9 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix} \\
&- \alpha_1^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix} \\
=& \frac{14}{15} C^* R^{-1}.
\end{aligned}$$

If $\alpha_2 = A^*$, then $A^* = \frac{14}{15}C^*R^{-1}$. Taking the determinant, we obtain $1 = \frac{14}{15}\frac{1}{\sqrt{5}}$, which is false; hence $\alpha_2 \neq A^*$.

3 Chebyshev matrix polynomials of the first kind

In this section, we give an explicit expression for the matrix of measures $X_{A,B}$ when A is positive definite.

We prove that the matrix of measures $X_{A,B}(x)$, $x \in \mathbb{R}$, has the form given in the introduction (see (1.5)).

First of all, observe that the definition of $X_{A,B}$ does not depend on the choice of the diagonal form of $K_{A,B}$, as can easily be proved (see also [HJ2, pp. 407–408]). From the definition, it is clear that $X_{A,B}$ is positive definite matrix of measures with support in the set of real numbers

(3.1) $\Delta = \{z \in \mathbb{C} : A^{-\frac{1}{2}}(zI - B)A^{-\frac{1}{2}}/2$ has at least one eigenvalue in $[-1, 1]\}$.

Theorem 3.1. *If A is positive definite and B is hermitian, the matrix weight $X_{A,B}$ for the Chebyshev matrix polynomials of the first kind defined by (1.3) is the matrix of measures given by (1.5). $X_{A,B}$ lives in a finite union of at most N disjoint bounded nondegenerate intervals whose endpoints are roots of the scalar polynomial*

$$\det\left(A^{-\frac{1}{2}}(B - zI)A^{-1}(B - zI)A^{-\frac{1}{2}} - 4I\right);$$

$X_{A,B}$ is absolutely continuous with respect to the Lebesgue measure multiplied by the identity matrix, with continuous matrix Radon–Nikodym derivative except at the endpoints of the intervals which form the support of $X_{A,B}$, where this derivative is unbounded.

Proof. We start by finding an expression for the Hilbert transform of $X_{A,B}$:

$$G_{A,B}(z) = \int \frac{dX_{A,B}}{z-t}, \quad z \in \mathbb{C} \setminus \operatorname{supp}(X_{A,B});$$

then we use the inversion formula to get the expression of $X_{A,B}$.

The key to find the expression for $G_{A,B}$ is to use Markov's Theorem for orthogonal matrix polynomials. Indeed, from Theorem 1.1 of [D4], we have

$$\lim_{n\to\infty} \left(T_n^{A,B}\right)^{-1}(z)Q_n^{A,B}(z) = G_{A,B}(z), \tag{3.2}$$

where $(Q_n^{A,B}(z))$ is the sequence of polynomials of the second kind defined by

$$Q_n^{A,B}(z) = \int \frac{T_n^{A,B}(z) - T_n^{A,B}(t)}{z-t} dX_{A,B}(t).$$

These associated polynomials (or polynomials of the second kind) satisfy the recurrence formula (recall that A is positive definite, and hence hermitian)

$$tQ_n^{A,B}(t) = AQ_{n+1}^{A,B}(t) + BQ_n^{A,B}(t) + AQ_{n-1}^{A,B}(t), \quad n \geq 1,$$

with initial conditions $Q_0^{A,B}(t) = \theta$, $Q_1^{A,B}(t) = (\sqrt{2}A)^{-1}$.

Hence it follows from (1.1) of [D1] that

$$Q_n^{A,B}(t) = \frac{1}{\sqrt{2}} U_{n-1}^{A,B}(t)A^{-1}, \quad n \geq 1, \tag{3.3}$$

where $(U_n^{A,B}(t))_n$ is the sequence of Chebyshev matrix polynomials of the second kind introduced in [D1]. Since A is positive definite, these matrix polynomials have the form

$$U_n^{A,B}(t) = A^{-\frac{1}{2}} u_n \left(\frac{A^{-\frac{1}{2}}(tI - B)A^{-\frac{1}{2}}}{2} \right) A^{\frac{1}{2}}, \tag{3.4}$$

where $(u_n(t))_n$ is the sequence of scalar Chebyshev polynomials of the second kind, which are orthonormal with respect to the positive measure $\frac{2}{\pi}\sqrt{1-t^2}\,\chi_{[-1,1]}(t)dt$ (see [D1, Sect. 3]).

It can be proved analogously that the matrix polynomials $(T_n^{A,B}(t))_n$ have the form

$$T_n^{A,B}(t) = A^{-\frac{1}{2}} t_n \left(\frac{A^{-\frac{1}{2}}(tI - B)A^{-\frac{1}{2}}}{2} \right) A^{\frac{1}{2}}, \tag{3.5}$$

where $(t_n(t))_n$ is the sequence of scalar Chebyshev polynomials of the first kind, which are orthonormal with respect to the positive measure $\chi_{[-1,1]}(t)dt/\pi\sqrt{1-t^2}$.

From the decomposition

$$\frac{A^{-\frac{1}{2}}(tI-B)A^{-\frac{1}{2}}}{2} = U(t)D(t)U^*(t),$$

(3.4) and (3.5) can be written as follows:

$$U_n^{A,B}(t) = A^{-\frac{1}{2}}U(t)\begin{pmatrix} u_n(d_{1,1}(x)) & 0 & 0 & \cdots & 0 \\ 0 & u_n(d_{2,2}(x)) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ & 0 & 0 & \cdots & u_n(d_{N,N}(x)) \end{pmatrix} U^*(t)A^{\frac{1}{2}},$$

$$T_n^{A,B}(t) = A^{-\frac{1}{2}}U(t)\begin{pmatrix} t_n(d_{1,1}(x)) & 0 & 0 & \cdots & 0 \\ 0 & t_n(d_{2,2}(x)) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ & 0 & 0 & \cdots & t_n(d_{N,N}(x)) \end{pmatrix} U^*(t)A^{\frac{1}{2}}.$$

So (3.2) gives

$$G_{A,B}(z) = \lim_{n\to\infty} A^{-\frac{1}{2}}U(t)\begin{pmatrix} \frac{u_n(d_{1,1}(x))}{\sqrt{2}t_n(d_{1,1}(x))} & 0 & 0 & \cdots & 0 \\ 0 & \frac{u_n(d_{2,2}(x))}{\sqrt{2}t_n(d_{2,2}(x))} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ & 0 & 0 & \cdots & \frac{u_n(d_{N,N}(x))}{\sqrt{2}t_n(d_{N,N}(x))} \end{pmatrix} U^*(t)A^{-\frac{1}{2}}.$$

We now use Markov's theorem for the scalar Chebyshev polynomials:

$$\lim_{n\to\infty} \frac{u_n(z)}{\sqrt{2}t_n(z)} = \frac{1}{\sqrt{z^2-1}}, \quad z \in \mathbb{C}\setminus[-1,1],$$

where we take the square root so that $|z-\sqrt{z^2-1}|<1$, for $z \in \mathbb{C}\setminus[-1,1]$. Then we have

$$G_{A,B}(z) = A^{-\frac{1}{2}}U(t)\begin{pmatrix} \frac{1}{\sqrt{d_{1,1}(z)^2-1}} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{\sqrt{d_{2,2}(z)^2-1}} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{\sqrt{d_{N,N}(z)^2-1}} \end{pmatrix} U^*(t)A^{-\frac{1}{2}}$$

for $z \in \Delta$, where

$$\Delta = \{z \in \mathbb{C} : A^{-\frac{1}{2}}(zI - B)A^{-\frac{1}{2}}/2 \text{ has at least one eigenvalue in } [-1,1]\}.$$

The inversion formula says that

$$dX_{A,B}(x) = -\frac{1}{2\pi i}\lim_{y\to 0}\Big(G_{A,B}(x+iy) - G_{A,B}(x-iy)\Big), \quad x \in \mathbb{R}.$$

This limit can now be computed as in Theorem 3.1 of [D1]; hence, we conclude that $X_{A,B}$ is given by (1.5).

We now see that the supports of $X_{A,B}$ and $W_{A,B}$ coincide, where $W_{A,B}$ is the matrix weight for the Chebyshev polynomials of the second kind (see (3.1) of [D1]). Consequently the rest of the theorem follows from Theorem 3.1 of [D1]. □

4 Zero asymptotics from the matrix of measures

Deducing the zero asymptotics for the matrix orthogonal polynomials $(P_n)_n$ directly from properties of the matrix of measures W, without knowledge of the recurrence coefficients, appears to be a rather challenging problem. In this section, we take a first step in this direction by generalizing the following theorem well-known in the scalar case (cf. [ET], [StT, Section 4.1]).

Theorem 4.1. *Let ν be a finite positive scalar measure with support $S(\nu) = [-1,1]$, and let ν' denote the Radon–Nikodym derivative of ν with respect to Lebesgue measure. If $\nu' > 0$ a.e. on $[-1,1]$ (with respect to Lebesgue measure), then for the polynomials $(p_n)_n$ orthonormal with respect to $d\nu$, the normalized zero counting measures*

$$\sigma_n = \frac{1}{n}\sum_{p_n(x)=0}\delta_x$$

satisfy

$$\lim_n \sigma_n = \frac{dt}{\pi\sqrt{1-t^2}}, \quad t \in [-1,1]. \tag{4.1}$$

A scalar measure ν satisfying the hypothesis of Theorem 4.1 is said to have the *Erdös–Turán property* on $[-1,1]$. The limit measure in (4.1), i.e., the arcsine distribution, is the equilibrium distribution for the interval $[-1,1]$ in the sense that it minimizes the logarithmic energy over all probability measures supported on $[-1,1]$ (cf. [SaT]). In the scalar case, Theorem 4.1 has been generalized in several directions. Namely, for regular measures ν whose support $S(\nu) \subset \mathbb{R}$ is compact set having positive logarithmic capacity, the zero counting measures σ_n converge to the equilibrium measure for $S(\nu)$ (cf. [StT, Theorem 3.14]). The main theorem of this section concerning matrix orthogonal polynomials (Theorem 4.5) likewise extends to this more general setting.

The Erdös–Turán property for the measure ν on $[-1,1]$ (or the more general property of regularity on $S(\nu)=[-1,1]$) implies that the orthonormal polynomials $(p_n)_n$ satisfy

$$\lim_n \|p_n\|_{[-1,1]}^{1/n} = 1, \tag{4.2}$$

where $\|\cdot\|_{[-1,1]}$ is the sup norm on $[-1,1]$ (cf. [StT, Theorem 3.2.3]); that is, in an n-th root sense, the $L^2(d\nu)$-norms and the L^∞-norms are asymptotically the same. On expanding an arbitrary polynomial q_n with $\deg(q_n) \le n$ in terms of $p_0,\dots,p_n$, we deduce from (4.2) that

$$\limsup_n \|q_n\|_{[-1,1]}^{1/n} \le \limsup_n \|q_n\|_{L^2(\nu)} \tag{4.3}$$

for any sequence of scalar polynomials $(q_n)_n$ with $\deg(q_n) \le n,\ n=0,1,2,\dots$.

For our purposes, it is more convenient to work with monic orthogonal polynomials. If $p_n(x)=\kappa_n x^n+\cdots$, we set $\mathbf{p}_n(x)=p_n(x)/\kappa_n = x^n+\cdots$. Then on comparing $\mathbf{p}_n$ with the monic scalar Chebyshev polynomials $\tilde{t}_n(x) = (\cos(n\theta))/2^{n-1}$, $x=\cos\theta$, $n\ge 1$, we obtain

$$\frac{1}{2} = \liminf_n \|\tilde{t}_n\|_{[-1,1]}^{1/n} \le \liminf_n \|\mathbf{p}_n\|_{[-1,1]}^{1/n}.$$

On the other hand, we deduce from (4.3) that

$$\limsup_n \|\mathbf{p}_n\|_{[-1,1]}^{1/n} = \limsup_n \|\mathbf{p}_n\|_{L^2(\nu)}^{1/n} \le \limsup_n \|\tilde{t}_n\|_{L^2(\nu)}^{1/n} \le \frac{1}{2},$$

where we have used the minimality property of $\mathbf{p}_n$ in the $L^2(\nu)$-norm. Hence

$$\lim_n \|\mathbf{p}_n\|_{[-1,1]}^{1/n} = \frac{1}{2}. \tag{4.4}$$

(We remark that the constant $1/2$ is the logarithmic capacity of the set $[-1,1]$.) The limit (4.4) shows that the monic orthogonal polynomials $(\mathbf{p}_n)_n$ have, in the

n-th root sense, asymptotically minimal L_∞-norm on $[-1,1]$. It is known (cf. [BSS]) that any sequence of monic polynomials having this asymptotic minimality property necessarily has the arcsine distribution as its limiting zero distribution.

Theorem 4.2 ***([BSS]).*** *Let $(q_n)_n$ be a sequence of monic polynomials with $\deg(q_n) = n$, $n = 0, 1, \ldots$. If*

$$\limsup_n \|q_n\|_{[-1,1]}^{1/n} \leq \frac{1}{2},$$

then

$$\sigma(q_n) = \frac{1}{n} \sum_{q_n(x)=0} \delta_x \to \frac{dt}{\pi\sqrt{1-t^2}}, \quad \text{as } n \to \infty.$$

We now turn to the matrix case. With the notation of Section 1, we write $P_n(t) = t^n K_n + \cdots$ for the orthonormal polynomials with respect to the nondegenerate positive definite matrix of measure W and set

$$\mathcal{P}_n(t) = K_n^{-1} P_n(t) = t^n I + \cdots, \tag{4.5}$$

so that now $\mathcal{P}_n$ is a monic matrix orthogonal polynomial. In the proof of our main result we shall appeal to the following L_2 minimality property of the $\mathcal{P}_n$'s, which is of independent interest.

Lemma 4.3. *With the above notation, given any $v = (v_1, \ldots, v_N) \in \mathbb{C}^N$, we have*

$$\int_{\mathbb{R}} (v\mathcal{P}_n) dW (v\mathcal{P}_n)^* \leq \int_{\mathbb{R}} (v\mathcal{Q}) dW (v\mathcal{Q})^* \tag{4.6}$$

for any monic $\mathcal{Q} = t^n I + \cdots \in \mathbb{C}_n^{N\times N}[t]$, with equality if and only if $v\mathcal{Q} = v\mathcal{P}_n$.

Proof. Since $\mathcal{Q} - \mathcal{P}_n$ has degree at most $n-1$, we can write

$$\mathcal{Q} - \mathcal{P}_n = \sum_{k=0}^{n-1} C_k P_k$$

for suitable $C_k \in \mathbb{C}^{N\times N}$. Then

$$v\mathcal{Q} = v(\mathcal{P}_n + (\mathcal{Q} - \mathcal{P}_n)) = v(\mathcal{P}_n + \sum_{k=0}^{n-1} C_k P_k),$$

and we obtain from the orthogonality of the P_n's

$$\int_{\mathbb{R}} (v\mathcal{Q}) dW (v\mathcal{Q})^* = \int_{\mathbb{R}} (v\mathcal{P}_n) dW (v\mathcal{P}_n)^* + \sum_{k=0}^{n-1} (vC_k)(vC_k)^* \geq \int_{\mathbb{R}} (v\mathcal{P}_n) dW (v\mathcal{P}_n)^*,$$

with equality if and only if $vC_k = \theta$, $0 \le k \le n-1$, that is, if and only if $v\mathcal{P}_n = vQ$. □

As a corollary, we deduce the following location property for the zeros of $(P_n)_n$.

Corollary 4.4. *The zeros of $(P_n)_n$ are in the convex hull of the support of W.*

Proof. If there is a zero a outside the convex hull of the support of W, then one of the numbers inf supp(W) or sup supp(W) is finite; write α for the one closest to a. We have $\det P_n(a) = 0$, so there exists $v \neq \theta$ with $vP_n(a) = \theta$; and thus we can divide $vP_n(x)$ by $x - a$.

Since $|x-\alpha|/|x-a| < 1$, for all $x \in \operatorname{supp}(W)$ and $v \neq \theta$, we have

$$vP_n(x)\frac{(x-\alpha)}{(x-a)} = vQ_n(x),$$

$Q_n(x)$ being a monic polynomial in $\mathbb{C}_n^{N\times N}[x]$ not equal to $P_n(x)$, and

$$\int vQ_n(x)dW(x)(vQ_n(x))^* < \int vP_n(x)dW(x)(vP_n(x))^*,$$

which contradicts the previous lemma. □

We remark that a slight modification of the above argument provides another proof that all the zeros of $P_n(t)$ are real.

To state the main result in this section, we begin by writing the matrix of measures $W = [w_{i,j}]$ in the form

$$dW = \begin{pmatrix} w'_{11} & \cdots & w'_{1N} \\ \vdots & \ddots & \vdots \\ w'_{N1} & \cdots & w'_{NN} \end{pmatrix} d\mathrm{tr}(W), \tag{4.7}$$

where $w'_{ij} = dw_{ij}/d\mathrm{tr}(W)$ is the Radon–Nikodym derivative. Let $\Delta_k = \Delta_k(t)$ denote the k-th principal determinant of the matrix $[w'_{ij}]$, i.e.,

$$\Delta_1 = w'_{11}, \quad \Delta_2 = \begin{vmatrix} w'_{11} & w'_{12} \\ w'_{21} & w'_{22} \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} w'_{11} & w'_{12} & w'_{13} \\ w'_{21} & w'_{22} & w'_{23} \\ w'_{31} & w'_{32} & w'_{33} \end{vmatrix}, \quad \text{etc.}$$

Further, set $\Delta_0 = 1$, and note that since W is positive definite, we have $\Delta_k \ge 0$ for $t \in [-1,1]$, $k = 0, \ldots, N$. We now state the main result of this section.

Theorem 4.5. *Suppose that the nondegenerate, positive definite matrix of measures W has support $S(W) = S(\mathrm{tr}(W)) = [-1,1]$. If each of the scalar measures $(\Delta_k/\Delta_{k-1})d(\mathrm{tr}(W))$, $k = 1, \ldots, N$, is finite and has the Erdös–Turán* *property on* $[-1,1]$, *then the normalized zero counting measures* σ_n *in (1.2) for the orthonormal polynomials* P_n *with respect to* dW *satisfy*

$$\lim_n \sigma_n = \frac{dt}{\pi\sqrt{1-t^2}}, \quad t \in [-1,1]. \tag{4.8}$$

Proof. Let $\mathcal{P}_n$ denote the corresponding monic orthogonal polynomials of (4.5). Then, from Theorem 4.2 and the fact that $\det\mathcal{P}_n$ is a scalar monic polynomial of degree nN, it suffices to prove that

$$\limsup_n \|\det\mathcal{P}_n\|_{[-1,1]}^{1/nN} \le \frac{1}{2}. \tag{4.9}$$

Write $\mathcal{P}_n = [\mathbf{p}_{i,j,n}]_{i,j=1}^N$, where the scalar polynomials $\mathbf{p}_{i,j,n}$ satisfy $\deg \mathbf{p}_{i,j,n} \le n-1$ for $i \ne j$, and $\mathbf{p}_{i,i,n} = t^n + \cdots$ is monic for $i = 1, \ldots, N$. Hadamard's inequality gives

$$\|\det\mathcal{P}_n\|_{[-1,1]} \le \prod_{i=1}^N \left\| (\sum_{j=1}^N \mathbf{p}_{i,j,n}^2)^{1/2} \right\|_{[-1,1]},$$

and so (4.9) will follow if we show that

$$\limsup_n \|\mathbf{p}_{i,j,n}\|_{[-1,1]}^{1/n} \le \frac{1}{2} \tag{4.10}$$

for all $i,j = 1,2,\ldots,N$.

For simplicity, we shall establish (4.10) only for the case $i=1$ (the general case being similar). Set

$$\mathbf{p}_{j,n} = \mathbf{p}_{1,j,n}, \quad j = 1,\ldots,N.$$

By the extremality property (4.6) with $v = (1,0,\ldots,0)$ and $\mathbf{q} = I\tilde{t}_n$, where $\tilde{t}_n$ is the monic scalar Chebyshev polynomial of degree n, we obtain

$$\int_{-1}^1 (v\mathcal{P}_n)dW(v\mathcal{P}_n)^* \le \int_{-1}^1 \tilde{t}_n^2 \Delta_1 d(\mathrm{tr}\,(W)) \le \frac{M}{2^{2n}},$$

for some constant M. Thus

$$\int_{-1}^1 \left(\sum_{j,k=1}^N w'_{j,k}\mathbf{p}_{j,n}\mathbf{p}_{k,n} \right) d(\mathrm{tr}(W)) \le \frac{M}{2^{2n}}. \tag{4.11}$$

From this last estimate, we shall deduce (4.10) for $i=1, j=1,\ldots,N$, by rewriting the quadratic form on the left-hand side as the sum of squares. By hypothesis, the determinants Δ_k, $k = 0,1,\ldots,N$, are positive for almost all t on $[-1,1]$ (with respect to Lebesgue measure). Hence for such t, the Jacobi formula (cf. [G, pp. 304–308]) for the quadratic form gives

$$\sum_{j,k=1}^{N} w'_{j,k}\mathbf{p}_{j,n}\mathbf{p}_{k,n} = \sum_{l=1}^{N} Y_l^2 \frac{\Delta_l}{\Delta_{l-1}}, \tag{4.12}$$

where

$$Y_l = \mathbf{p}_{l,n} + \frac{1}{\Delta_l}\sum_{j=l+1}^{N} M_{l,j}\mathbf{p}_{j,n}, \tag{4.13}$$

and

$$M_{l,j} = \begin{vmatrix} w'_{11} & \cdots & w'_{1,l-1} & w'_{1j} \\ w'_{21} & \cdots & w'_{2,l-1} & w'_{2j} \\ \vdots & \ddots & \vdots & \vdots \\ w'_{l,1} & \cdots & w'_{l,l-1} & w'_{lj} \end{vmatrix}, \quad l \geq 2, \quad j \geq l+1,$$

$$M_{1,j} = w'_{1,j}, \quad \text{for } l = 1, j \geq 2,$$

$$M_{l,j} = 0, \quad \text{for } j < l,$$

and the empty sum $\sum_{j=N+1}^{N}$ is given the value zero.

Without loss of generality, we assume that $\mathrm{tr}(W)$ is absolutely continuous with respect to Lebesgue measure (otherwise, we can work with its absolutely continuous part).

It then follows from (4.11) that, for all n,

$$\sum_{l=1}^{N} \int_{-1}^{1} \left(\mathbf{p}_{l,n} + \frac{1}{\Delta_l}\sum_{j=l+1}^{N} M_{l,j}\mathbf{p}_{j,n}\right)^2 \frac{\Delta_l}{\Delta_{l-1}} d(\mathrm{tr}(W)) \leq \frac{M}{2^{2n}}. \tag{4.14}$$

We note also that from the Jacobi formula (4.12) we see (by taking $\mathbf{p}_{s,n} \equiv 1$, $\mathbf{p}_{j,n} \equiv 0$ for $j \neq s$) that the inequality

$$\left(\frac{1}{\Delta_l} M_{l,s}\right)^2 \frac{\Delta_l}{\Delta_{l-1}} \leq w'_{s,s}, \quad s \geq l+1, \tag{4.15}$$

holds almost everywhere for each $l = 1, \ldots, N$. Consequently, since W is a finite matrix of measures, each of the functions on the left in (4.15) is integrable with respect to $d(\mathrm{tr}(W))$. From this observation and (4.14), we show, by induction, that

$$\limsup_n \|\mathbf{p}_{j,n}\|_{[-1,1]}^{1/n} \leq \frac{1}{2}, \quad \text{for } j = 1, \ldots, N. \tag{4.16}$$

Indeed, for $j = N$, we deduce from (4.14) that

$$\int_{-1}^{1} \mathbf{p}_{N,n}^2 \frac{\Delta_N}{\Delta_{N-1}} d(\operatorname{tr}(W)) \leq \frac{M}{2^{2n}};$$

and since $(\Delta_N/\Delta_{N-1})d(\operatorname{tr}(W))$ satisfies the Erdös–Turán property, we get from (4.3) that (4.16) holds for the $\mathbf{p}_{N,n}$. Assuming now that (4.16) is valid for $j = N, N-1, \ldots, k+1$, we see that it holds for $j = k$ because, from (4.14), we have

$$\int_{-1}^{1} \left(\mathbf{p}_{kn} + \frac{1}{\Delta_k} \sum_{j=k+1}^{N} M_{k,j} \mathbf{p}_{k,j} \right)^2 \frac{\Delta_k}{\Delta_{k-1}} d(\operatorname{tr}(W)) \leq \frac{M}{2^{2n}}.$$

Then from (4.15) and the induction hypothesis, it follows by the triangle inequality that

$$\limsup_n \left(\int_{-1}^{1} \mathbf{p}_{k,n}^2 \frac{\Delta_k}{\Delta_{k-1}} d(\operatorname{tr}(W)) \right)^{1/2n} \leq \frac{1}{2},$$

which implies by the Erdös–Turán property that $\limsup_n \|\mathbf{p}_{kn}\|_{[-1,1]}^{1/n} \leq 1/2$. This completes the proof of the theorem. □

As remarked at the beginning of this section, the same argument used to prove Theorem 4.5 yields the following more general result.

Theorem 4.6. *Suppose that the nondegenerate, positive definite matrix of measures W has compact support $S(W) = S(\operatorname{tr}(W)) \subseteq \mathbb{R}$ of positive logarithmic capacity and that each of the measures $(\Delta_k/\Delta_{k-1})d\operatorname{tr}(W)$, $k = 1, \ldots, N$ is finite and regular on $S(W)$. Then the normalized zero counting measures σ_n in (1.2) for the matrix orthogonal polynomials with respect to dW converge weak-star to the equilibrium measure (Robin measure) for $S(W)$.*

An interesting open problem is to obtain a generalization of Theorem 4.3 that applies when the determinants Δ_k are positive only on a portion of the support of W.

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