On Meromorphic Approximation

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Dedicated to Richard S. Varga on the occasion of his seventieth birthday, with much appreciation for his mentoring and friendship.

Abstract

Let G be a bounded $N-$connected domain, the boundary Γ of which consists of closed analytic Jordan curves. We assume that $0 \in G$. For any nonnegative integers n and m, denote by $\mathcal{M}_{n,m}$ the class of all meromorphic functions on G that can be represented in the form $h = p/qz^m$, where p belongs to the Smirnov class $E_\infty(G)$, q is a polynomial of degree at most n, $q \not\equiv 0$. Theorems giving necessary and sufficient conditions for a function belonging to the class $\mathcal{M}_{n,m}$ to be an element of best approximation to a continuous function f on Γ in the space $L_\infty(\Gamma)$ by functions in the class $\mathcal{M}_{n,m}$ are proved. Some questions concerning orthogonal polynomials and the theory of Hankel operators are also considered.

1 Introduction

Let G be a bounded domain with boundary Γ consisting of N disjoint closed analytic Jordan curves, $0 \in G$. In the present paper we investigate the problem describing properties of an element of best approximation to a continuous complex–valued function f on Γ in the space $L_\infty(\Gamma)$ by functions in the class $\mathcal{M}_{n,m}$. For any nonnegative integers n and m, the class $\mathcal{M}_{n,m}$ consists of all meromorphic functions on G that can be represented in the form $h = p/qz^m$, where $p \in E_\infty(G)$, q is a polynomial of degree at most

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n, $q \not\equiv 0$. We prove Theorems 1 and 2 giving necessary and sufficient conditions for a function belonging to the class $\mathcal{M}_{n,m}$ to be an element of best approximation.

The methods of this paper are closely related to the Carathéodory–Fejér method (CF–method). The CF–method relates to the situation when the function f being approximated is defined on the boundary of the unit disk. In this regard we single out the works of Carathéodory and Fejér [4], Schur [20], Takagi [21], Trefethen and Gutknecht [23]–[24]. We also mention the paper of Hayashi, Trefethen and Gutknecht [8], in which the authors give several characterizations of an element of best approximation. We remark that in [8] the extension of the corresponding results to Faber–CF approximation on a Jordan region with rectifiable boundary is also considered. In the present paper for the case when G is a multiply connected domain we obtain the characterization of best approximation in terms different from those in [8].

Let $L_p(\Gamma)$, $1 \leq p < \infty$, be the Lebesgue space of functions with integrable pth power. The norm is given by

$$\|g\|_p = \left(\int_\Gamma |g(\xi)|^p |d\xi| \right)^{1/p}.$$

$L_\infty(\Gamma)$ is the Lebesgue space of essentially bounded functions, with the norm

$$\|g\|_\infty = \operatorname*{ess\,sup}_{\xi \in \Gamma} |g(\xi)|.$$

It will be assumed that the boundary Γ is positively oriented with respect to G.

Denote by $E_p(G)$, $1 \leq p \leq \infty$, the Smirnov class of analytic functions on G. Each function in $E_p(G)$ has limit values at almost all points $\xi \in \Gamma$ along paths nontangential to Γ. The function determined on Γ by these limit values belongs to the space $L_p(\Gamma)$. The condition

$$\int_\Gamma \frac{g(\xi)\, d\xi}{\xi - z} = 0 \qquad \text{for all} \;\; z \in \overline{\mathbf{C}} \setminus \overline{G} \tag{1.1}$$

is necessary and sufficient for a function g, belonging to $L_1(\Gamma)$, to be the boundary value of a function in the Smirnov class $E_1(G)$ (see [14] and [25] for more details about the classes $E_p(G)$).

The outline of this paper is as follows. In Section 2 we state Theorems 1 and 2. In Sections 3 and 4 proofs of these theorems are given. Sections 5, 6 and 7 are devoted to some problems of the theory of Hankel operators. In Section 8 questions relating to the orthogonal polynomials are considered.

2 Meromorphic Approximation

Let f be a continuous function on Γ. Fix any nonnegative integers n and m. The *deviation* of f in $L_\infty(\Gamma)$ from the class $\mathcal{M}_{n,m}$ is denoted by $\Delta_{n,m}$:

$$\Delta_{n,m} = \Delta_{n,m}(f; G) = \inf_{h \in \mathcal{M}_{n,m}} \|f - h\|_\infty. \tag{2.1}$$

For each function $h \in \mathcal{M}_{n,m}$ we define the defect $\operatorname{def}(h, \mathcal{M}_{n,m})$ of h in the class $\mathcal{M}_{n,m}$ as the greatest nonnegative integer d such that $h \in \mathcal{M}_{n-d,m}$.

It is not difficult to prove that there exists a function $h_{n,m}$ in the class $\mathcal{M}_{n,m}$ for which the infimum in (2.1) is attained (for the case when $m = 0$ see [15]). Let $d = \operatorname{def}(h_{n,m}, \mathcal{M}_{n,m})$. We can represent $h_{n,m}$ in the form $h_{n,m} = P/Q\xi^m$, where $P \in E_\infty(G)$, Q is a monic polynomial with zeros in G, $\deg Q = n - d$, $P \neq 0$ at zeros of Q. We set $Q \equiv 1$ if $h_{n,m} \equiv 0$. Let ω be an arbitrary monic polynomial of degree d with zeros in G (for $d = 0$ we can take $\omega \equiv 1$).

It follows from the definition of the quantity $\Delta_{n,m}$ that

$$\Delta_{n,m} = \inf_{u \in E_\infty(G)} \|f - u/(Q\omega)(\xi)\xi^m\|_\infty.$$

In view of the results of Tumarkin and Khavinson (see [26]–[28]; for the case when G is the unit disk see [6] and [9]) obtained on the basis of duality relations, the last equality implies that there exists a function $\psi \in E_1(G)$, $\psi \not\equiv 0$, such that

$$Q(\xi)\xi^m\omega(\xi)\psi(\xi)(f - h_{n,m})d\xi = \Delta_{n,m}|Q(\xi)\xi^m\omega(\xi)\psi(\xi)|\,|d\xi| \tag{2.2}$$

almost everywhere on Γ.

The key element in Theorem 1 (necessary conditions for a function belonging to the $\mathcal{M}_{n,m}$ to be an element of best approximation) is a constant 2 in the power of Q (see (2.3)). The presence of the constant 2 is due to the fact that the extremal problem (2.1) is a problem of approximation by meromorphic functions with partially free poles. We note that Q is determined by free poles of meromorphic approximants (compare with the factors in (2.3) related to "fixed part" of the element of best approximation).

Equation (2.3) can be applied for investigating questions of meromorphic and rational approximation. In fact, using (2.3) in Section 7 we prove a generalization of the well-known Adamyan–Arov–Kreĭn theorem (see [1], [2]). The theorem proved allows one to investigate the rate of decrease of ray sequences from the Walsh table of best rational approximations of analytic functions (see [10], [18]). Moreover, it follows from equation (2.3) that Q is the n-th nonhermitian orthogonal polynomial for a weight function that varies with n (see Section 8). In this regard we also mention the paper [3].

Theorem 1 *Let f be a continuous function on Γ, and let $h_{n,m}$ be an element of best approximation to f in the class $\mathcal{M}_{n,m}$, $d = \operatorname{def}(h_{n,m}, \mathcal{M}_{n,m})$. Then there exists a function $\varphi \in E_1(G)$, $\varphi \not\equiv 0$, with at least d zeros in G, such that*

$$Q^2(\xi)\xi^m\varphi(\xi)(f - h_{n,m})(\xi)d\xi = \Delta_{n,m}|Q^2(\xi)\xi^m\varphi(\xi)||d\xi| \qquad (2.3)$$

almost everywhere on Γ.

Note, that it follows directly from (2.2) and (2.3) that $|(f - h_{n,m})(\xi)| = \Delta_{n,m}$ almost everywhere on Γ.

Suppose that a continuous function $g : \Gamma \to \mathbf{C}$ is different from zero on Γ. We define the index $\operatorname{ind}_\Gamma g$ of the function g with respect to Γ with the help of the formula

$$\operatorname{ind}_\Gamma g = \frac{1}{2\pi}\Delta_\Gamma \arg g,$$

where $\Delta_\Gamma \arg g$ is the increment of the argument of the function g upon traversing Γ.

It is not hard to see that the function $Q^2\xi^m\varphi$ can be represented in the form $Q^2\xi^m\varphi = \varphi_1\varphi_2$, where $\varphi_1 \in E_1(G)$ and φ_2 is a polynomial with zeros in G, $\operatorname{ind}_\Gamma \varphi_2 = 2n - d + m$.

We now formulate Theorem 2 giving a sufficient condition for a function $h \in \mathcal{M}_{n,m}$ to be an element of best approximation to the function f in the space $L_\infty(\Gamma)$.

Theorem 2 *Let f be a continuous function on Γ, and let h be a function in the class $\mathcal{M}_{n,m}$ such that $\Delta = |(f - h)(\xi)|$, $\Delta \geq 0$, almost everywhere on Γ, $d = \operatorname{def}(h, \mathcal{M}_{n,m})$. Suppose that there exists a function $\Phi = \varphi_1\varphi_2$, $\Phi \not\equiv 0$, where $\varphi_1 \in E_1(G)$, φ_2 is a rational function with zeros and poles in G, $\operatorname{ind}_\Gamma \varphi_2 = 2n - d + m + N - 1$, such that*

$$\Phi(\xi)(f - h)(\xi)d\xi = \Delta|\Phi(\xi)||d\xi| \qquad (2.4)$$

almost everywhere on Γ, where N is the connectivity index of the domain G.

Then $\Delta = \Delta_{n,m}$ and h is the unique element of best approximation to the function f in the class $\mathcal{M}_{n,m}$.

3 Proof of Theorem 1

This section is devoted to the proof of Theorem 1 giving necessary conditions for an element of $\mathcal{M}_{n,m}$ to be a best approximation to the function f in the class $\mathcal{M}_{n,m}$.

Let ω be an arbitrary polynomial of degree d with zeros in G (for $d = 0$ we can take $\omega \equiv 1$).

We first establish

Lemma 1 *For any function $u \in E_\infty(G)$, there holds*

$$\operatorname*{ess\,sup}_{\xi\in\Gamma} \mathrm{Re}\left(-\frac{\overline{(f-h_{n,m})}(\xi)u(\xi)}{(Q^2\omega)(\xi)\xi^m}\right) \geq 0. \tag{3.1}$$

Proof. Fix an arbitrary $u \in E_\infty(G)$. It is not hard to see that we can represent u in the form $u = \alpha Q - \beta P$, where $\alpha \in E_\infty(G)$, β is a polynomial, and $\deg \beta \leq n$.

For an arbitrary $\varepsilon > 0$ denote by $h_{n,m,\varepsilon}$ the following function belonging to the class $\mathcal{M}_{n,m}$:

$$h_{n,m,\varepsilon} = \frac{P\omega + \varepsilon\alpha}{(Q\omega + \varepsilon\beta)\xi^m}. \tag{3.2}$$

Choose ε sufficiently small so that $Q\omega + \varepsilon\beta \neq 0$ on Γ. Then we have almost everywhere on Γ

$$\begin{aligned} |(f-h_{n,m,\varepsilon})(\xi)|^2 &= |(f-h_{n,m})(\xi)|^2 - 2\,\mathrm{Re}(\overline{(f-h_{n,m})}(h_{n,m,\varepsilon}-h_{n,m}))(\xi) \\ &\quad + |(h_{n,m,\varepsilon}-h_{n,m})(\xi)|^2 \\ &= \Delta_{n,m}^2 - 2\varepsilon\,\mathrm{Re}\left(\overline{(f-h_{n,m})}(\xi)\frac{u(\xi)}{(Q\omega+\varepsilon\beta)(\xi)Q(\xi)\xi^m}\right) \\ &\quad + |(h_{n,m,\varepsilon}-h_{n,m})(\xi)|^2. \end{aligned} \tag{3.3}$$

Letting now ε tend to 0 and using the definition of $\Delta_{n,m}$ and the fact that $h_{n,m,\varepsilon}$ belongs to the class $\mathcal{M}_{n,m}$, we get (3.1).

$\square$

The following lemma shows that $\Delta_{n,m}$ is also the solution of a related extremal problem.

Lemma 2 *The best approximant to f by functions $u/Q^2\omega\xi^m$, $u \in E_\infty(G)$, in the space $L_\infty(\Gamma)$ is $h_{n,m}$, that is*

$$\Delta_{n,m} = \inf_{u\in E_\infty(G)} \|f - u/Q^2\omega\xi^m\|_\infty. \tag{3.4}$$

Proof. Fix an arbitrary $u \in E_\infty(G)$. Let us estimate $|(f - u/Q^2\omega\xi^m)(\xi)|^2$ on Γ. We have

$$\begin{aligned}|(f - u/Q^2\omega\xi^m)(\xi)|^2 = \Delta_{n,m}^2 \quad &- \quad 2\,\mathrm{Re}(\overline{(f - h_{n,m})}(\xi)(u/Q^2\omega\xi^m - h_{n,m})(\xi)) \\ &+ \quad |(u/Q^2\omega\xi^m - h_{n,m})(\xi)|^2\end{aligned}$$

almost everywhere on Γ.

From this, the formula $u/Q^2\omega\xi^m - h_{n,m} = v/Q^2\omega\xi^m$, where $v = u - PQ\omega \in E_\infty(G)$, and Lemma 1 we get

$$\|f - u/Q^2\omega\xi^m\|_\infty \geq \Delta_{n,m}.$$

Taking into account now that u is an arbitrary function in $E_\infty(G)$ and the fact that $h_{n,m}$ can be represented in the form $h_{n,m} = u/Q^2\omega\xi^m$, where $u = PQ\omega$, we obtain (3.4).

□

Using Lemma 2, we get with the help of the duality relations (for more details, see [26]–[28]) that

$$\Delta_{n,m} = \sup_v |\int_\Gamma (vQ^2\omega\xi^m f)(\xi)d\xi|, \tag{3.5}$$

where the supremum is taken over all $v \in E_1(G)$ such that $\|vQ^2\omega\xi^m\|_1 = 1$. Moreover, there exists a function $q \in E_1(G)$, $\|qQ^2\omega\xi^m\|_1 = 1$, for which the supremum is attained in (3.5) and

$$(Q^2\omega)(\xi)\xi^m q(\xi)(f - h_{n,m})(\xi)d\xi = \Delta_{n,m}|(Q^2\omega)(\xi)\xi^m q(\xi)||d\xi|$$

almost everywhere on Γ. Letting $\varphi = \omega q$, we get (2.3). Theorem 1 is proved.

4 Sufficient Conditions for a Best Approximant

Proof of Theorem 2. It will be assumed that $\Delta > 0$. For $\Delta = 0$ the corresponding assertion is obvious.

Assume that there exists a function $h_1 \in \mathcal{M}_{n,m},\ h_1 \not\equiv h$, such that the inequality holds

$$|(f-h_1)(\xi)| \le \Delta$$

almost everywhere on Γ. From this, by the equality

$$(h_1-h)/(f-h) = 1-(f-h_1)/(f-h),$$

and the fact that $|(f-h)(\xi)| = \Delta$ almost everywhere on Γ, we get

$$\operatorname{Re}(h_1-h)(\xi)/(f-h)(\xi) \ge 0, \tag{4.1}$$

almost everywhere on Γ; moreover, since $h_1 \not\equiv h$, there exists a set of the positive measure on the boundary Γ on which $\operatorname{Re}(h_1-h)(\xi)/(f-h)(\xi) > 0$. We represent the function $(h_1-h)/(f-h)$ in the form

$$(h_1-h)/(f-h) = u+iv,$$

where, according to (4.1), $u \ge 0$ almost everywhere on Γ. It follows from the formula (2.4) that

$$\Phi(\xi)(h_1-h)(\xi)d\xi = \Delta(u+iv)(\xi)|\Phi(\xi)||d\xi| \tag{4.2}$$

almost everywhere on Γ.

We note that the function h_1-h can be represented in the form

$$h_1-h = p/\omega\xi^m, \tag{4.3}$$

where $p \in E_\infty(G),\ p \not\equiv 0$, and ω is a polynomial with the zeros in G, $\deg\omega \le 2n-d$.

We now use the representation of the function Φ in the form $\Phi = \varphi_1\varphi_2$, where $\varphi_1 \in E_1(G)$ and φ_2 is a rational function with zeros and poles in G, $\operatorname{ind}_\Gamma \varphi_2 = 2n-d+m+N-1$. By the formula (4.3), we can rewrite the relation (4.2) in the form

$$(\varphi_1 p)(\xi)d\xi = \Delta\frac{\omega(\xi)\xi^m}{\varphi_2(\xi)}(u(\xi)+iv(\xi))|\Phi(\xi)||d\xi|. \tag{4.4}$$

Since $\operatorname{ind}_\Gamma(\omega\xi^m/\varphi_2) \le 1-N$, it follows from the theory of boundary value problems for analytic functions (see, for example, [5]) that there exists a function $g,\ g \ne 0$ on Γ, holomorphic on G and continuous on $\bar{G}$, such that

$$(g\omega\xi^m/\varphi_2)(\xi) = |(g\omega\xi^m/\varphi_2)(\xi)|, \quad \xi \in \Gamma.$$

Using now the last relation and (4.4), we get

$$(g\varphi_1 p)(\xi)d\xi = \Delta(u(\xi) + iv(\xi))|(g\omega\xi^m\varphi_1)(\xi)||d\xi|$$

almost everywhere on Γ. ¿From this, by the Cauchy theorem,

$$\int_\Gamma (g\varphi_1 p)(\xi)d\xi = \Delta \int_\Gamma u(\xi)|(g\omega\varphi_1)(\xi)||d\xi| = 0. \tag{4.5}$$

On the other hand, since there exists a set of positive measure on Γ on which $u(\xi)|(g\omega\varphi_1)(\xi)| > 0$, we obtain the inequality

$$\int_\Gamma u(\xi)|(g\omega\varphi_1)(\xi)||d\xi| > 0,$$

which contradicts the relation (4.5). Therefore, for an arbitrary function $h_1 \in \mathcal{M}_{n,m}, h_1 \not\equiv h$,

$$\|f - h_1\|_\infty > \Delta.$$

This inequality implies the equality $\Delta_{n,m} = \Delta$ and uniqueness of an element of best approximation.

□

5 A Generalization of the Adamyan-Arov-Kreĭn Theorem

In recent years the theory of Hankel operators has been widely used in studying the degree of rational approximation of analytic functions. The methods of the theory of Hankel operators allow one to investigate the convergence of best rational approximations, and to obtain estimates of the degree of rational approximation for different classes of analytic functions. In this regard we single out the papers [11], [16], [17] (see also [13], [12]). The methods used are based on the Adamyan–Arov–Kreĭn theorem [1], [2] and a generalization of the Adamyan–Arov–Kreĭn theorem for the case when a function f from which the Hankel operator A_f is constructed is given on the boundary of a multiply connected domain (see [15]). It is essential to underscore that the results obtained relate to the situation when analytic functions are approximated by rational functions with free poles.

In connection with problems of rational approximation related to investigating the behavior of the ray sequences $\{\rho_{n,m}\}$, $m/n \to \theta$ as $m+n \to \infty$, of the best uniform rational approximation of analytic functions it is important to develop the methods of the theory of Hankel operators such that the corresponding methods can be applied to investigate the degree of approximation of analytic functions by rational functions with partially fixed poles. In Section 7 we prove Theorem 3, which is a generalization of the Adamyan-Arov-Kreĭn theorem for the case when f is given on the boundary of a multiply connected domain and approximated by meromorphic functions with partially fixed poles.

Fix a nonnegative integer m. Let $L_{2,m}(\Gamma)$ be the Lebesgue space of functions g measurable on Γ with the norm

$$\|g\|_{2,m} = \left(\int_\Gamma |g(\xi)|^2 |\xi|^m |d\xi|\right)^{1/2} < \infty.$$

The inner product in the Hilbert space $L_{2,m}(\Gamma)$ is denoted by

$$(g,\psi) = \int_\Gamma (g\overline{\psi})(\xi)\,|\xi|^m\,|d\xi|, \qquad g,\psi \in L_{2,m}(\Gamma).$$

Denote by $H = H_m$ the class of all functions q representable in the form $q = g/\xi^m$, where $g \in E_2(G)$. Here and in what follows we will consider H and $E_2(G)$ as the subspaces of $L_{2,m}(\Gamma)$.

Let f be a continuous function on Γ. We define the Hankel operator A_f as the composition of the operator of multiplication by the function f, and the orthogonal projection $\mathbf{P}_-$ of $L_{2,m}(\Gamma)$ onto $H^\perp$, where $H^\perp$ is the orthogonal complement of H in $L_{2,m}(\Gamma)$. For any function $q \in E_2(G)$ we have $A_f q = \mathbf{P}_-(qf)$ by definition.

Let $\{s_k\}$, $s_k = s_k(f;G)$, $k = 0,1,2\ldots$, be the sequence of singular numbers of the operator A_f (the sequence of eigenvalues of operator $(A_f^* A_f)^{1/2}$, where $A_f^* : H^\perp \to E_2(G)$ is the adjoint of A_f). We assume that the sequence $\{s_n\}$ is nonincreasing. The following formula is valid

$$s_k = \inf_K \|A_f - K\|, \qquad k = 0,1,2,\ldots, \tag{5.1}$$

where the infimum is over the collection of all linear operators $K : E_2(G) \to H^\perp$ of rank at most k, and $\|\cdot\|$ is the norm of the corresponding linear operator (see, for example, [7]). It is not hard to see that the operator A_f is a compact operator.

We formulate a theorem establishing a connection between the singular numbers s_n of operator A_f and the best meromorphic approximations $\Delta_{n,m}$ of f.

Theorem 3 *Let G be a bounded domain with boundary Γ consisting of N disjoint closed analytic Jordan curves and let f be a continuous function on Γ. Then for all integers $n \geq N-1$,*

$$\Delta_{n+N-1,m} \leq s_n \leq \Delta_{n,m}.$$

The Adamyan–Arov–Kreĭn theorem relates to the case when $G = \{z : |z| < 1\}$, $m = 0$. We then have $s_n = \Delta_{n,0}$, $n = 0, 1, 2, \ldots$. In [15] a generalization of the Adamyan–Arov–Kreĭn theorem was proved for the case when G is an N–connected domain and $m = 0$.

We remark that in this paper we extend the definition of the Hankel operator. Namely, the Hankel operator A_f is defined as an operator which is operating from $E_2(G)$ into $H^\perp$. In the case when $m = 0$ (see, for example, [15]), we have $A_f : E_2(G) \to E_2^\perp(G)$. It is not hard to see that the corresponding definitions coincide for $m = 0$.

6 Some Formulas

In this section we present the needed auxiliary assertions. Since for any function $q \in E_2(G)$, we have

$$A_f q = \mathbf{P}_-(qf),$$

where $\mathbf{P}_-$ is the orthogonal projection of $L_{2,m}(\Gamma)$ onto $H^\perp$, the equality

$$A_f q = qf - p$$

holds, where the function $p \in H$ is determined by the relation

$$\|qf - p\|_{2,m} = \inf_{\tilde{p} \in H} \|qf - \tilde{p}\|_{2,m}. \tag{6.1}$$

It follows directly from (6.1) that

$$\|A_f\| \leq \|f\|_\infty, \tag{6.2}$$

where $\|A_f\|$ is the operator norm of A_f.

Lemma 3 *The following formulas hold for the singular numbers $\{s_n\}_{n=0}^\infty$:*

$$s_0 = \sup_{q,v} \left| \int_\Gamma (qv\xi^m f)(\xi) d\xi \right| \tag{6.3}$$

and

$$s_n = \inf_{\psi_1,\dots,\psi_n \in E_2(G)} \{\sup_{q,v} |\int_\Gamma (qv\xi^m f)(\xi) d\xi|\}, \quad n = 1, 2, \dots, \tag{6.4}$$

where the suprema in (6.3) *and* (6.4) *are taken over all functions* $q, v \in E_2(G)$*,* $\|q\|_{2,m} = 1$*, and* $||v||_{2,m} = 1$*, with the functions* q *in* (6.7) *satisfying the conditions* $(q, \psi_i) = 0$*,* $i = 1, \dots, n$*.*

Before proceeding to the proof of Lemma 3, we state some remarks.

First, for any function $u \in H^\perp$ there exists a function $v \in E_2(G)$ such that

$$\overline{u}(\xi)|\xi|^m \, |d\xi| = v(\xi) \, \xi^m \, d\xi \tag{6.5}$$

almost everywhere on Γ. Indeed, for $q(\xi) = 1/\xi^m(\xi - z)$ (z is an arbitrary point in $\overline{\mathbf{C}} \setminus \overline{G}$) we have

$$(q, u) = \int_\Gamma \frac{1}{(\xi - z)\xi^m} \, \overline{u}(\xi) \, |\xi|^m \, |d\xi| = 0.$$

This implies (see (1.1)) that there exists a function $v \in E_2(G)$ such that (6.5) holds.

Second, for any function $v \in E_2(G)$ the function u belongs to $H^\perp$ if

$$\overline{u}(\xi)|\xi|^m|d\xi| = v(\xi)\xi^m d\xi$$

almost everywhere on Γ.

Third, it is a well-known fact that the following formulas hold for singular numbers:

$$s_0 = \sup_{q,u} |(A_f q, u)| \tag{6.6}$$

and

$$s_n = \inf_{\psi_1,\dots,\psi_n \in E_2(G)} (\sup_{q,u} |(A_f q, u)|), \quad n = 1, 2, \dots, \tag{6.7}$$

where the suprema in (6.6) and (6.7) are taken over all functions $q \in E_2(G)$, $u \in H^\perp$, $\|q\|_{2,m} = 1$, $||u||_{2,m} = 1$, with the functions q in (6.7) satisfying the conditions $(q, \psi_i) = 0$, $i = 1, \dots, n$ (see [7] for more details about the singular numbers).

Proof of Lemma 3. Let $q \in E_2(G)$ and $u \in H^\perp$. Since the equality holds

$$A_f q = qf - p,$$

where $p \in H$, and $p = g/\xi^m, g \in E_2(G)$, by (6.5), we have

$$\begin{aligned}(A_f q, u) &= \int_\Gamma ((qf - p)\overline{u})(\xi)|\xi|^m|d\xi| = \int_\Gamma ((qf-p)v\xi^m)(\xi)d\xi \\ &= \int_\Gamma (qv\xi^m f)(\xi)d\xi.\end{aligned}$$

From the last formula, on account of (6.6) and (6.7), we obtain the required equalities.

□

7 Proof of Theorem 3

7.1 Lower Bound for $\Delta_{n,m}$

Fix a nonnegative integer n. Let h be any function in the class $\mathcal{M}_{n,m}$. We set $Kq = \mathbf{P}_-(qh)$, $q \in E_2(G)$, where $\mathbf{P}_-$ is the orthogonal projection of $L_{2,m}(\Gamma)$ onto $H^\perp$. It is easy to see that $K : E_2(G) \to H^\perp$ is a linear operator with rank at most n.

We have the following estimate of the norm of the operator $A_f - K$:

$$\|A_f - K\| \leq \|f - h\|_\infty.$$

The inequalities

$$s_n \leq \Delta_{n,m}, \qquad n = 0, 1, 2, \dots, \tag{7.1}$$

follow immediately from the definition of $\Delta_{n,m}$ and formula (5.1) for singular numbers.

7.2 Estimates for $\Delta_{n,m}$ from above

In this subsection we prove the inequalities

$$\Delta_{n+N-1,m} \leq s_n, \qquad n = N-1, N, \dots. \tag{7.2}$$

To prove (7.2) it is sufficient to show that for all integers $n \geq 2N - 2$

$$\Delta_{n,m} \leq s_{n-N+1}. \tag{7.3}$$

Fix an integer $n \geq 2N - 2$. We assume without loss of generality that $n - N + 1 \geq 1$.

We can assume that f is holomorphic on Γ. Indeed, the general case can be obtained directly from this case with help of the fact that there exists a sequence $\{r_k\}$, $k = 0, 1, 2, \ldots$, of rational functions with poles off Γ such that $\|f - r_k\|_\infty \to \infty$ as $k \to \infty$, and the inequalities

$$|s_n(f;G) - s_n(r_k;G)| \leq \|A_f - A_{r_k}\| \leq ||f - r_k||_\infty$$

and

$$|\Delta_{n+N-1,m}(f;G) - \Delta_{n+N-1,m}(r_k;G)| \leq \|f - r_k\|_\infty \tag{7.4}$$

which follows from general properties of singular numbers (see [7]), (6.2) and the definition of $\Delta_{n+N-1,m}$.

We remark that in the case when f is holomorphic on Γ, the functions P and φ can be continued analytically across Γ, and (2.3) holds for all $\xi \in \Gamma$ (see [26]–[28] for more details).

It follows from (2.3) that

$$\begin{aligned} Q(\xi)\xi^m(f - h_{n,m})(\xi)d\xi &= (Q(\xi)f(\xi) - P(\xi)/\xi^m)\xi^m d\xi \\ &= \Delta_{n,m}(\overline{Q(\xi)\varphi(\xi)})|\xi^m|/|\varphi(\xi)||d\xi| \end{aligned} \tag{7.5}$$

on Γ. Let

$$\lambda(\xi) = (Q\varphi)(\xi)/|\varphi(\xi)|, \qquad \xi \in \Gamma.$$

We choose arbitrary functions $\psi_1, \ldots, \psi_{n-N+1} \in E_2(G)$. Since $\mathrm{ind}_\Gamma\, \overline{\lambda} \leq -n$ (the function $Q\varphi$ has at least n zeros in G), there exist functions u, v holomorphic on G and continuous on $\overline{G}$, $u, v \not\equiv 0$ (see [29]), such that

$$(\overline{\lambda}u)(\xi) = \overline{v}(\xi), \qquad \xi \in \Gamma. \tag{7.6}$$

Moreover, since there are $n - N + 2$ linearly independent solutions of the equation (7.6),

$$(q, \psi_i) = 0, \qquad i = 1, \ldots, n - N + 1, \tag{7.7}$$

where $q = uQ$. Let $p = uP/\xi^m$. By (7.5),

$$(qf - p)(\xi)\xi^m\, d\xi = \Delta_{n,m}\overline{v}(\xi)\, |\xi^m|\, |d\xi|, \qquad \xi \in \Gamma.$$

¿From this we deduce that

$$\int_\Gamma (qv\xi^m f)(\xi)d\xi = \Delta_{n,m} \int_\Gamma |v(\xi)|^2 \, |\xi^m| \, |d\xi|.$$

Taking into account now that $|v| = |q|$ on Γ, we get

$$\sup_{\tilde{q},\tilde{v}} | \int_\Gamma (\tilde{q}\tilde{v}\xi^m f)(\xi)d\xi| \geq \Delta_{n,m},$$

where the supremum is taken over all $\tilde{q}, \tilde{v} \in E_2(G)$, such that $\|\tilde{q}\|_2 = 1$, $||\tilde{v}||_2 = 1$, and $(\tilde{q}, \psi_i) = 0$, $i = 1, \ldots, n - N + 1$. Minimizing over all choices of $\psi_1, \ldots, \psi_{n-N+1}$, we obtain (see (6.4)) the desired estimates for $\Delta_{n,m}$:

$$s_{n-N+1} \geq \Delta_{n,m}.$$

□

8 A Connection with the Orthogonal Polynomials

8.1 The Case when G is the Unit Disk

In this section we prove that the polynomial Q constructed from the (nonzero) poles of $h_{n,m}$ is the n-th orthogonal polynomial with respect to a weight that varies with n.

Here and in what follows we assume that f is a holomorphic on $\overline{\mathbf{C}} \setminus E$, where E is a compact set, $E \subset G$. Let D be an arbitrary domain lying with its closure in G and containing the compact set E. We assume that D is bounded by finitely many disjoint contours and the boundary γ of D is positively oriented with respect to D.

Let G be the unit disk with the center at 0. We assume without loss of generality that Q is monic and $\deg Q = n-d \geq 1$. Denote by $\alpha_{1,n}, \ldots, \alpha_{n-d,n}$ the zeros of Q_n (counting multiplicity) so that

$$Q(z) = \prod_{k=1}^{n-d} (z - \alpha_{k,n}).$$

Let B be the corresponding Blaschke product

$$B(z) = \prod_{k=1}^{n-d} \frac{z - \alpha_{k,n}}{1 - \overline{\alpha}_{k,n} z}$$

and

$$w_n(z) = \prod_{k=1}^{n-d}(1-\overline{\alpha}_{k,n}z).$$

Let us consider equation (2.3), which, mentioned in Section 7, holds for all $\xi \in \Gamma$. Assume that $\varphi \neq 0$ on Γ. We represent the function $Q^2\varphi$ in the form

$$(Q^2\varphi)(z) = B^2(z)B_1(z)\psi^2(z),$$

where $B_1(z)$ is the Blaschke product constructed from the zeros of φ in G. Note that since $\varphi \neq 0$ on Γ, we have $\psi \neq 0$ on $\overline{G}$. Denote by l_n, $l_n \geq d$, the number of zeros of φ in G. In the case when $l_n \geq 1$, we set

$$\omega_n(z) = \prod_{k=1}^{l_n}(1-\overline{\beta}_{k,n}z),$$

where $\beta_{1,n}, \ldots, \beta_{l_n,n}$ are the zeros of φ in G. For $l_n = 0$ let $\omega_n(z) \equiv 1$. In what follows we assume without loss of generality that $l_n \geq 1$.

We can rewrite the relation (2.3) in the form

$$B^2(\xi)B_1(\xi)\xi^m\psi^2(\xi)(f-h_{n,m})(\xi)\,d\xi = \Delta_{n,m}|\psi^2(\xi)|\,|d\xi|,\ \xi \in \Gamma. \tag{8.1}$$

Denote by E^{-1} the reflection of E in the unit circle. Let $\psi^*(z) = \overline{\psi(\overline{z})},\ z \in G$.

We have the following assertion.

Theorem 4 *With the assumption that the function φ of* (2.3) *does not vanish on Γ, the function $B\xi^m\psi(f-h_{n,m})$ can be extended analytically to $\overline{\mathbf{C}}\setminus E$ and has zeros at the points $1/\overline{\alpha}_{k,n}$, $k = 1,\ldots,n-d$, and $1/\overline{\beta}_{k,n}$, $k = 1,\ldots,l_n$, and*

$$(B^2B_1\xi^m\psi)(\xi)(f-h_{n,m})(z) = \frac{1}{2\pi i}\int_\gamma \frac{(B^2B_1\xi^m\psi)(\xi)\,d\xi}{z-\xi}, \quad z \in \overline{\mathbf{C}}\setminus\overline{D}. \tag{8.2}$$

The function ψ^* can be extended analytically to $\overline{\mathbf{C}}\setminus E^{-1}$, and satisfies the equation

$$\frac{1}{2\pi}\int_\gamma \frac{(B^2B_1\xi^m\psi f)(\xi)d\xi}{z-\xi} = \Delta_{n,m}\psi^*\left(\frac{1}{z}\right)\frac{1}{z}, \quad z \in \overline{\mathbf{C}}\setminus\overline{D}. \tag{8.3}$$

The following orthogonality relations are valid:

$$\int_\gamma \xi^\nu Q(\xi)\frac{\xi^m\psi(\xi)f(\xi)d\xi}{w_n^2(\xi)\omega_n(\xi)} = 0 \quad \textit{for} \quad \nu = 0,1,\ldots,n-1. \tag{8.4}$$

Proof. It follows from the relation (8.1) that on Γ

$$\begin{aligned} B(\xi)\xi^m\psi(\xi)(f-h_{n,m})(\xi) &= \Delta_{n,m}\frac{1}{B(\xi)B_1(\xi)}\overline{\psi(\xi)}\frac{1}{i\xi} \\ &= \Delta_{n,m}\frac{1}{B(\xi)B_1(\xi)}\psi^*\left(\frac{1}{\xi}\right)\frac{1}{i\xi}. \end{aligned} \tag{8.5}$$

Using now the last formula we can conclude that $B\xi^m\psi(f-h_{n,m})$ can be extended analytically to $\overline{\mathbf{C}}\setminus E$ and has zeros at the points $1/\overline{\alpha}_{k,n},\ k=1,\dots,n-d$, and $1/\overline{\beta}_{k,n},\ k=1,\dots,l_n$, and a zero at infinity. From (8.5) we also obtain that $\psi^*(\xi)$ can be extended analytically to $\overline{\mathbf{C}}\setminus E^{-1}$.

Fix $z\in\overline{\mathbf{C}}\setminus\overline{D}$. Using now the fact that $B^2B_1\xi^m\psi(f-h_n)$ is holomorphic in the region $\overline{\mathbf{C}}\setminus E$, we get, by the Cauchy formula,

$$(B^2B_1\xi^m\psi)(z)(f-h_{n,m})(z)=\frac{1}{2\pi i}\int_\gamma\frac{(B^2B_1\xi^m\psi f)(\xi)d\xi}{z-\xi},\qquad z\in\overline{\mathbf{C}}\setminus\overline{D}. \tag{8.6}$$

¿From this and from (8.5), we obtain

$$\frac{1}{2\pi}\int_\gamma\frac{(B^2B_1\xi^m\psi f)(\xi)d\xi}{z-\xi}=\Delta_{n,m}\psi^*\left(\frac{1}{z}\right)\frac{1}{z},\quad z\in\overline{\mathbf{C}}\setminus\overline{D}.$$

The function

$$\frac{B\xi^m\psi(f-h_n)}{w_n\omega_n}$$

is holomorphic on $\overline{\mathbf{C}}\setminus E$, and in a neighborhood of infinity

$$\frac{B\xi^m\psi(f-h_n)}{w_n\omega_n}(z)=\frac{A}{z^{n+1}}+\dots, \tag{8.7}$$

where the right-hand side is a series in increasing powers of $1/z$.

By (8.7), we get

$$\int_\gamma\xi^\nu Q(\xi)\frac{\xi^m\psi(\xi)f(\xi)d\xi}{w_n^2(\xi)\omega_n(\xi)}=0\quad\text{for}\quad\nu=0,1,\dots,n-1.$$

The last relations imply that the polynomial $Q(z)$ is the nth orthogonal polynomial with respect to the complex-valued measure $\xi^m\psi f d\xi/w_n^2\omega_n$.

□

Remark 1. Concerning the assumption that φ does not vanish on Γ we remark that a verification of this property in specific cases requires a more detailed analysis (see e.g. [3]). In case φ does vanish at points on Γ, the orthogonality equation (8.4) must be modified accordingly.

Remark 2. In connection with Theorem 4 we single out the paper [3], where the authors (together with L. Baratchart) investigate the rate of decrease of $\Delta_{n,m}$, convergence of the best meromorphic approximants, and the limiting distribution of poles of the best approximants in the case when f is the Markov function

$$f(z) = \frac{1}{2\pi i} \int_a^b \frac{d\mu(x)}{z - x}$$

with the measure μ, $\operatorname{supp} \mu = [a, b] \subset (-1, 1)$, satisfying the Szegő condition

$$\int_a^b \frac{\log(d\mu/dx)}{\sqrt{(x-a)(b-x)}}\, dx > -\infty.$$

The methods used are based on the orthogonality relations (8.4), the equation (8.3) and the results of Totik [22] and Stahl [19] concerning Szegő type asymptotics for orthogonal polynomials with a weight that varies with n.

8.2 G is a Multiply Connected Domain

In this subsection we consider the case when G is N-connected domain. We assume without loss of generality that $N \geq 2$. Let $\Gamma = \bigcup_{k=0}^{N-1} \Gamma_k$, where $\Gamma_k,\ k = 0, \ldots, N-1$, are disjoint closed analytic Jordan curves. It will be assumed that G lies interior to Γ_0. For $k = 1, 2, \ldots, N-1$ let a_k be an arbitrary point lying inside of Γ_k.

We assume that $\varphi \neq 0$ on Γ. Let $d\xi = S(\xi)|d\xi|,\ \xi \in \Gamma$, and let $\chi = \operatorname{ind}_\Gamma(\overline{(\varphi Q)(\xi)}|\xi^m|/(|\varphi(\xi)|S(\xi))$. It is not hard to see that $\chi \leq N - 2 - n$. We assume that $n \geq N$. In this case we have $\chi \leq -2$.

Since $\chi \leq -2$, we obtain, by virtue of the results of the theory of boundary value problems for analytic functions (see, for example, [5]), that there exists a function $g,\ g \neq 0$ on Γ, holomorphic on G and continuous on $\overline{G}$, such that (see (7.5)) the function $(gQ\xi^m)(\xi)(f - h_{n,m})(\xi)$ can be extended analytically to $\overline{\mathbf{C}} \setminus G$,

$$(gQ\xi^m)(\xi)(f - h_{n,m})(\xi) = \Delta_{n,m} g(\xi)\overline{Q(\xi)\varphi(\xi)}|\xi^m|/(|\varphi(\xi)|S(\xi)) \text{ on } \Gamma, \quad (8.8)$$

and in a neighborhood of infinity

$$(gQz^m)(z)(f-h_{n,m})(z)=\frac{A}{z^{|\chi|}}+\dots, \tag{8.9}$$

where the right-hand side is a series in increasing powers of $1/z$.

From (8.9) it follows immediately that Q satisfies the orthogonality relations

$$\int_\gamma \xi^\nu (gQ\xi^m f)(\xi)d\xi=0, \qquad \nu=0,\dots,|\chi|-2.$$

Let T be arbitrary polynomial T of degree at most $|\chi|-1$. By the Cauchy formula, we get

$$(TgQz^m)(z)(f-h_{n,m})(z)=\frac{1}{2\pi i}\int_\gamma \frac{(TgQ\xi^m f)(\xi)d\xi}{z-\xi}, \quad z\in\overline{\mathbf{C}}\setminus\overline{D}.$$

Moreover, we have the following formulas (see (8.8))

$$\frac{1}{2\pi i}\int_\gamma \frac{(TgQ\xi^m f)(\xi)d\xi}{z-\xi}=\Delta_{n,m}T(z)g(z)\overline{Q(z)\varphi(z)}|z^m|/(|\varphi(z)|S(z)),\ \ z\in\Gamma,$$

and (see [5])

$$g(z)=\prod_{k=1}^{N-1}(z-a_k)\exp(-\Psi(z)), \quad z\in G,$$

where

$$\Psi(z)=\frac{1}{2\pi i}\int_\Gamma \frac{\log\left(\xi^{-\chi}\prod_{k=1}^{N-1}(\xi-a_k)\overline{Q(\xi)\varphi(\xi)}|\xi^m|/(|\varphi(\xi)|S(\xi))\right)}{\xi-z}\,d\xi.$$

References

[1] V.M. Adamyan, D.Z. Arov, and M.G. Kreĭn, Infinite Hankel matrices and generalized Caratheodory-Fejer and Riesz problem, Functional Anal. Appl. 2 (1968), 1–18.

[2] V.M. Adamyan, D.Z. Arov, and M.G. Kreĭn, Analytic properties of Schmidt pairs, Hankel operators, and the generalized Schur-Takagi problem, Mat. Sb. 86(128) (1971), 34–75; English transl. in Math. USSR Sb. 15 (1971).

[3] L. Baratchart, V.A. Prokhorov, E.B. Saff, On meromorphic approximation of Markov functions (to appear).

[4] C. Carathéodory and L. Fejér, Űber den Zusammenhang der Extremen von harmonsichen Funktionen mit ihren Koeffizienten und űber den Picard-Landauschen Satz., Rend. Circ. Mat. Palermo 32, (1917), 218–339.

[5] F.D. Gakhov, *Boundary value problems*, Pergamon Press, Oxford, 1966.

[6] John B. Garnett, *Bounded analytic functions*, Academic Press, New York, 1981.

[7] I.Ts. Gokhberg [Israel Gohberg] and M.G. Kreĭn, *Introduction to the theory of linear nonselfadjoint operators in Hilbert space*, "Nauka", Moscow, 1965; English transl., Amer. Math. Soc., Providence, RI,

[8] E. Hayashi, L.N. Trefethen, and M.H. Gutknecht, The CF table, Constr. Approx. 6 (1990), 195–223.

[9] Paul Koosis, *Introduction to H^p-spaces*, Cambridge Univ. Press, Cambridge, 1980.

[10] A. Krot, V.A Prokhorov, and E.B. Saff, Ray sequences of the rational approximants to entire functions, in: *Approximation Theory IX, Volume 1*: *Theoretical Aspects*, eds. Charles K. Chui and Larry L. Schumaker, 1998, Vanderbilt Univ. Press, pp. 177–202.

[11] O.G. Parfenov, Estimates of the singular numbers of a Carleson operator, Mat. Sb. 131(173) (1986), 501–518; English transl. in Math. USSR Sb. 59 (1988).

[12] V.V. Peller, A description of Hankel operators of class σ_P for $p > 0$, an investigation of the rate of rational approximation, and other applications, Mat. Sb. 122(164) (1983), 481–510; English transl. in Math. USSR Sb. 50 (1985).

[13] V.V. Peller and S.V. Khrushchev, Hankel operators, best approximations, and stationary Gaussian processes, Uspekhi Mat. Nauk 37 (1982), no. 1 (223), 53–124; English transl. in Russian Math. Surveys 37 (1982).

[14] I.I. Privalov, *Boundary properties of analytic functions*, 2nd ed., GITTL, Moscow, 1950; German transl., VEB Deutscher Verlag Wiss., Berlin, 1956.

[15] V.A. Prokhorov, On a theorem of Adamyan, Arov, and Kreĭn, Mat. Sb. 184 (1993), 89-104; English transl. in Russian Acad. Sci. Sb. Math. 78 (1994).

[16] V.A. Prokhorov, Rational approximation of analytic function, Mat. Sb. 184 (1993), 3–32; English transl. in Russian Acad. Sci. Sb. Math. 78 (1994).

[17] V.A. Prokhorov, On the degree of rational approximation of meromorphic functions, Mat. Sb. 185 (1994), 3–26; English transl. in Russian Acad. Sci. Sb. Math. 81 (1995).

[18] V.A. Prokhorov and E.B. Saff, Rates best uniform rational approximation of analytic functions by ray sequences of rational functions, Constr. Approx. 15 (1999), 155–173.

[19] H. Stahl, Strong asymptotics for orthogonal polynomials with varying weights (to appear).

[20] I. Schur, Űber Potenzreihen, die im Innern des Einheitskreises beschrănkt sind, J. Reine Angew. Math. 148 (1918), 122–145.

[21] T. Takagi, On an algebraic problem related to an analytic theorem of Carathéodory and Fejér, Japan J. Math. 1 (1924), 83–93; 2 (1925), 13–17.

[22] V. Totik, *Weighted Approximation with Varying Weights*, volume 1300 of *Lecture Notes in Math.* Springer–Verlag, Berlin–Heidelberg–New York, 1994.

[23] L.N. Trefethen, Rational Chebyshev approximation on the unit disk, Numer. Math. 37 (1981), 297–320.

[24] L.N. Trefethen, and M.H. Gutknecht, Padé, stable Padé, and Chebyshev–Padé approximation, in: *Algorithms for Approximation*, eds. J. C. Mason and M. G. Cox, Clarendon Press, Oxford, 1986, pp. 227–264.

[25] G.Ts. Tumarkin and S.Ya. Khavinson, On the definition of analytic functions of class E_p in multiply connected domains, Uspekhi Mat. Nauk 13 (1958), no. 1 (79), 201–206 (Russian).

[26] G.Ts. Tumarkin and C.Ya. Khavinson, Extremal problems for certain classes of analytic functions on finitely connected domains, Mat. Sb. 36(78) (1955), 445–478 (Russian).

[27] G.Ts. Tumarkin and C.Ya. Khavinson, Investigation of the properties of extremal functions with the help of duality relations in extremal problems for classes of analytic functions on multiply connected domains, Mat. Sb. 46(88) (1958), 195–228 (Russian).

[28] G.Ts. Tumarkin and C.Ya. Khavinson, Qualitative properties of solutions of extremal problems of certain kinds, in: *Studies of Current Problems in the Theory of Functions of a Complex Variable*, Fizmatgiz, Moscow, 1960, pp. 77–94; French transl., Fonctions d'une Variable Complexe. Problèmes Contemporains, Gauthiers-Villars, Paris, 1962, pp. 73–92.

[29] E.I. Zverovich, Boundary value problems in the theory of analytic functions in Hölder classes on Riemann surfaces, Uspekhi Mat. Nauk 26 (1971), no. 1 (157), 113–179; English transl. in Russian Math. Surveys 26 (1971).

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