Fast Decreasing Rational Functions

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Abstract

Necessary and sufficient conditions are obtained for the existence of sequences of rational functions of the form $r_n(x) = p_n(x)/p_n(-x)$, with p_n a polynomial of degree n, that decrease geometrically on (0, 1] in accordance with a specified rate function. The technique of proof involves minimum energy problems for Green potentials in the presence of an external field. Applications are given for the construction of rational approximations of $| \ x \ |$ and sgn (x) on [-1, 1] having geometric rates of convergence for $x \neq 0$.

1 Introduction

Rational functions of the form

$$r_n(x) = P_n(x)/P_n(-x) \tag{1.1}$$

(P_n stands for a polynomial of degree $\leq n$) play an important role in rational approximation on $[-1,1]$. Once we have constructed r_n of the form (1.1) that is fast decreasing on $[0,1]$ (note that $r_n(0) = 1$), we automatically get good rational approximants for $|x|, \operatorname{sgn} x$, and related functions (cf. [11], [4], [19], [8]). On the other hand, Newman's inequality (cf. [11], Lemma 3.1)

$$\int_x^1 \log\left|\frac{\zeta - t}{\zeta + t}\right| \frac{dt}{t} \ \geq \ -\frac{\pi^2}{2}, \quad 0 < x < 1, \quad \zeta \in \mathrm{C}, \tag{1.2}$$

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implies that any r_n of the form (1.1) must satisfy

$$\max_{[x,1]} |r_n| \ \geq \ \exp\left\{-\pi^2 n/2\log\frac{1}{x}\right\}, \quad 0 < x < 1, \tag{1.3}$$

so that such r_n cannot decrease "too fast" (see also [3]). Let us formulate a general problem.

Given a function φ that is continuous on $[0,1]$ and satisfies $\varphi(0) = 0$, we wish to find a sequence $r_n, n \geq 1$, of functions of the form (1.1), with the property:

$$|r_n(x)| \ \leq \ Ce^{-n\varphi(x)}, \quad 0 \leq x \leq 1, \quad n \geq 1. \tag{1.4}$$

Under what conditions on φ do there exist such r_n ? We also consider a weaker property, namely

$$|r_n(x)| \ \leq \ e^{o(n)-n\varphi(x)}, \quad 0 \leq x \leq 1, \quad n \geq 1, \tag{1.5}$$

where $o(n)$ is uniform in x. Still weaker, is the property:

$$|r_n(x)| \ \leq \ Ce^{-cn\varphi(x)}, \quad 0 \leq x \leq 1, \quad n \geq 1 \tag{1.6}$$

for some positive constants C, c.

The existence of r_n satisfying (1.6) is settled by

Theorem 1.1 *Let $\varphi \in C[0,1], \varphi(0) = 0$. Then there exist r_n with property* (1.6) *if and only if*

$$\int_0^1 \frac{\varphi(x)}{x} dx \quad < \quad \infty. \tag{1.7}$$

This result was proved by V. Maimeskul and the authors in [8]. (Although not stated explicitly, it follows from Lemma 3.1 and Theorem 4.2 in [8]). The proof also provides an estimate for a constant c in the exponential term of (1.6). For example, if φ is increasing on $[0,1]$, one can set

$$c = \left\{\sum_{k=0}^{\infty} \varphi(2^{-k})\right\}^{-1}.$$

However, the method of [8] is not suitable for producing sharp estimates.

In this paper we utilize a potential theoretic approach that yields best possible estimates. The method is similar to one used by V. Totik in his paper [16] on fast decreasing polynomials, and it is based on a study of a certain equilibrium problem. In our case, the problem is the following one.

Given a measure $\mu \geq 0$ in $D := \{z \in \mathbf{C} : \mathrm{Re}\, z > 0\}$, let

$$U_g^\mu(z) := \int \log \left| \frac{z+\bar{t}}{z-t} \right| d\mu(t), \quad z \in D \tag{1.8}$$

denote the Green potential of μ. Find μ of total mass $||\mu|| = 1$, with support $S(\mu) \subseteq [0,1]$, that satisfies, with some constant c_μ:

$$U_g^\mu - \varphi = c_\mu \quad on \quad S(\mu) \tag{1.9}$$

$$U_g^\mu - \varphi \geq c_\mu \quad on \quad [0,1]. \tag{1.10}$$

Note that this is a non-standard "singular" problem, since $[0,1]$ touches the boundary of D. Another complication is that the condition (1.7) allows φ of the form

$$\varphi(x) = \left(\log \frac{c}{x}\right)^{-\alpha}, \quad \alpha > 1, \quad c > 1.$$

Such a φ has an integrable derivative but φ' is not in $L_p[0,1]$, for any $p > 1$. Therefore, known results concerning the density of μ (cf. [9], [16]) are not applicable.

The relevance of the above equilibrium problem is obvious: given a polynomial

$$P_n(z) = \prod_{j=1}^{n} (z - \zeta_j),$$

we may write

$$\left| P_n(z)/P_n(-\overline{z}) \right| = \exp\left(-nU_g^{\nu_n}(z)\right),$$

where

$$\nu_n := \frac{1}{n}\sum_{j=1}^{n}\delta_{\zeta_j}, \qquad ||\nu_n|| = 1,$$

and δ_{ζ_j} denotes the unit point mass at ζ_j. Then the property (1.5) can be formulated as

$$U_g^{\nu_n}(x) - \varphi(x) \geq o(1), \quad x \in [0,1] \tag{1.11}$$

while (1.4) becomes

$$U_g^{\nu_n}(x) - \varphi(x) \geq O(n^{-1}), \quad x \in [0,1]. \tag{1.12}$$

Thus, our goal is to describe φ for which c_μ of (1.9) is zero; for then, on discretizing μ, we may hope to get r_n of the form (1.1) that satisfy (1.4) or (1.5).

The paper is organized as follows. In the next section we collect some auxiliary results. In particular, a generalization of Mhaskar-Saff's F-*functional* is given. In Section 3 we show that the problem (1.9), (1.10) has a unique solution, and then we prove

Theorem 1.2 *Given* $\varphi \in C[0,1], \varphi(0) = 0$, *let* U_g^μ, c_μ *be as in* (1.9). *Then the following statements are equivalent:*

(i) *There exist* $r_n, n \geq 1$, *satisfying* (1.5);

(ii) *There holds*

$$\frac{2}{\pi^2}\int_0^1 \left(U_g^\mu(x) - c_\mu\right) \frac{dx}{x\sqrt{1-x^2}} \ \leq \ 1; \tag{1.13}$$

(iii) c_μ=0.

Of course, it is desirable to have a condition in terms of φ. The necessary condition is immediate - if we have (1.5), then $c_\mu = 0$ by (iii), so that $\varphi \leq U_g^\mu$

by (1.10). Then (1.13) implies that

$$\frac{2}{\pi^2}\int_0^1 \varphi(x)\,\frac{dx}{x\sqrt{1-x^2}} \ \leq\ 1. \tag{1.14}$$

In many important cases this condition is also sufficient. In Section 4 we prove (compare with Theorem 3.3 in [16]) the following.

Theorem 1.3 *Let $\varphi \in C[0,1]$, $\varphi(0) = 0$, and assume additionally, that φ is increasing and concave on $[0,1]$. Then there exist $r_n, n \geq 1$, with property* (1.5) *if and only if* (1.14) *holds.*

In Section 5 we consider the stronger property (1.4). Here we need to analyze the density of the equilibrium measure μ. In doing so we impose extra conditions on φ that ensure a regular behavior of μ' near 0. Assume that for x small enough,

(a) $x\varphi'(x)$ is increasing;

(b) for some $\delta > 0$, $x\varphi'(x) \leq (1-\delta)\varphi(x)$.

Then we have

Theorem 1.4 *Let φ be as in Theorem* 1.3*, and assume that φ satisfies* (a), (b) *for x small enough. Then there exist $r_n, n \geq 1$, with property* (1.4) *provided strict inequality holds in* (1.14)*, i.e.*

$$\frac{2}{\pi^2}\int_0^1 \varphi(x)\,\frac{dx}{x\sqrt{1-x^2}} \ <\ 1. \tag{1.14'}$$

Remark. Many important functions, like $x^\alpha, \alpha < 1.$ or $(\log c/x)^{-\alpha}, \alpha > 1$, satisfy (a) and (b) above. Moreover, applying a finer discretization technique (cf. [7], [17]), one can replace (1.14') by (1.14), at least for the case $\varphi(x) = cx^\alpha, \alpha < 1$.

Finally, we apply our results to rational approximation of $|x|$. It is well-known that the error in best approximation of $|x|$ by rational functions of order n behaves like $\exp(-\pi\sqrt{n})$ (cf. [19],[14]). However, the best approximants do not converge to $|x|$ geometrically fast on any subinterval of $[-1,1]$ (cf. [12]). In [8] near best rational approximants R_n were constructed with the property

$$(1.15) \quad \big|\,|x| - R_n(x)\,\big| \;\le\; C\exp\{-c_1\sqrt{n} - c_2 n\varphi(|x|)\}, \quad x \in [-1,1].$$

The same method shows that c_1 can be chosen arbitrarily close to π, but no sharp estimate of c_2 was given. Here we prove a more precise result.

Theorem 1.5 *Let φ be continuous increasing function on $[0,1]$ with $\varphi(0) = 0$. Let $0 \le \varepsilon \le 1$, and assume that a sequence $\{R_n\}, n \ge 1$, exists such that for $x \in [-1,1]$*

$$(1.16) \qquad \Big|\,|x| - R_n(x)\,\Big| \;\le\; C\exp\left\{-\pi\sqrt{n(1-\varepsilon)} - n\varphi(|x|)\right\}.$$

Then

$$(1.17) \qquad \frac{2}{\pi^2}\int_0^1 \varphi(x)\,\frac{dx}{x\sqrt{1-x^2}} \;\le\; \varepsilon.$$

In particular, if $\varepsilon = 0$, then $\varphi \equiv 0$ so that geometric convergence is not possible.

Conversely, if strict inequality holds in (1.17) *and φ is as in Theorem* 1.4*, then there exist $\{R_n\}$ satisfying* (1.16).

2 Auxiliary Results

We start with a simple observation. Let E be a compact set in the open right half-plane D, and let $\widetilde{E} := \{-\overline{z} : z \in E\}$ be its reflection about the

y-axis. The pair $(E, \widetilde{E})$ is called a (symmetric) *condenser.* Given a positive measure μ on E, define $\widetilde{\mu}$ on $\widetilde{E}$, by symmetry. Then the Green potential of μ can be written as an ordinary logarithmic potential of the signed measure $\mu - \widetilde{\mu}$:

$$U_g^{\mu}(z) = \int \log\left|\frac{z+\bar{t}}{z-t}\right| d\mu(t) \;=\; \int \log \frac{1}{|z-t|}\, d(\mu - \widetilde{\mu})(t).$$

With this observation in mind, the classical result of T. Bagby [2] can be stated as follows (see also [20] or [13, Section II.5]).

There exists a unique positive measure ω_E on E, of total mass $||\omega_E|| = 1$, such that

$$U_g^{\omega_E} = \frac{1}{\mathrm{cap}_g E} := \frac{1}{2\mathrm{cap}(E, \widetilde{E})}, \quad \text{q.e. on } E \tag{2.1}$$

and

$$U_g^{\omega_E} \le \frac{1}{\mathrm{cap}_g E}, \quad \text{in } \overline{\mathbf{C}}. \tag{2.2}$$

(As usual, q.e. means except for a set of zero logarithmic capacity.) The measure ω_E is called the *equilibrium measure* (or distribution) on E relative to D, and the constant $\mathrm{cap}_g E$ is called the *Green capacity* of E relative to D (it also coincides with the capacity of the condenser formed by E and the y-axis).

In particular, for $E = [\varepsilon, 1], 0 < \varepsilon < 1$, we have (cf. [1])

$$\frac{1}{\mathrm{cap}_g E} = \frac{\pi K(\varepsilon)}{K'(\varepsilon)}, \tag{2.3}$$

where $K(\varepsilon), K'(\varepsilon)$ denote the complete elliptic integrals for moduli $\varepsilon, \sqrt{1-\varepsilon^2}$, respectively. The equilibrium measure $\omega_{[\varepsilon,1]} =: \omega_E$ is given by

$$d\omega_E(t) = \frac{1}{K'(\varepsilon)} \left\{(1-t^2)(t^2-\varepsilon^2)\right\}^{-1/2} dt, \quad t \in [\varepsilon, 1]. \tag{2.4}$$

Next, we shall need a maximum principle for Green potentials (cf. [13, Section II.5]). We state it for the special case $S(\mu) \subseteq [0,1]$, since this will suffice for our purposes, and the proof in this case is elementary. Indeed, it is easy to see that

$$\left|\frac{z+t}{z-t}\right| < \left|\frac{\operatorname{Re} z+t}{\operatorname{Re} z-t}\right|, \quad \operatorname{Re} z > 0, \quad \operatorname{Im} z \neq 0, \quad t > 0, \tag{2.5}$$

and

$$\left|\frac{z+t}{z-t}\right| < \frac{1+t}{1-t}\ , \quad \operatorname{Re} z > 1, \quad 0 < t < 1. \tag{2.6}$$

Therefore, U_g^μ attains its maximum in D only on $[0,1]$. Since U_g^μ is convex on any subinterval of $[0,+\infty) \setminus S(\mu)$, we obtain that

$$U_g^\mu(z) \leq \sup_{S(\mu)} U_g^\mu, \quad z \in D,\ S(\mu) \subseteq [0,1]. \tag{2.7}$$

Moreover, the maximum principle for harmonic functions shows that strict inequality holds for $z \notin S(\mu)$, unless $S(\mu) = \{0\}$, in which case $U_g^\mu \equiv 0$.

We are now in a position to prove an important result that can be viewed as a strengthened version of Newman's inequality (1.2).

Lemma 2.1 *Let σ be the measure on $(0,1)$ given by*

$$d\sigma(t) = \frac{2}{\pi^2}\frac{dt}{t\sqrt{1-t^2}}, \quad t \in (0,1) \tag{2.8}$$

(note that $\|\sigma\| = \infty$). Then

(i)

$$U_g^\sigma(z) = 1, \quad z \in (0,1] \tag{2.9}$$

$$U_g^\sigma(z) < 1, \quad z \in D \setminus (0,1]. \tag{2.10}$$

Furthermore,

(ii) *for any positive finite measure* μ *in* D *with* $S(\mu) \subset \overline{D}$, *we have*

$$\int U_g^\mu d\sigma \le \|\mu\| \tag{2.11}$$

and

$$\int U_g^\mu d\sigma = \|\mu\| - \mu\{0\}, \quad if\ S(\mu) \subseteq [0,1]. \tag{2.12}$$

Proof (i) For $0 \le \varepsilon < 1$, set

$$I_\varepsilon(x) := \int_\varepsilon^1 \log\left|\frac{x+t}{x-t}\right| \Big\{(1-t^2)(t^2 - \varepsilon^2)\Big\}^{-1/2} dt.$$

Then (see (2.1), (2.3), (2.4)) for $\varepsilon > 0$, we have

$$I_\varepsilon(x) = \pi K(\varepsilon), \quad \varepsilon \le x \le 1.$$

Since $K(\varepsilon) \to \pi/2$ as $\varepsilon \to 0$ (cf. [1]), equation (2.9) will follow provided we can show that I_ε converges to I_0. Fix $0 < x \le 1$ and write, for $\varepsilon < x/2$,

$$I_\varepsilon = \int_\varepsilon^{x/2} + \int_{x/2}^1 =: I_{\varepsilon 1} + I_{\varepsilon 2}.$$

The integrand in $I_{\varepsilon 2}$ is decreasing as $\varepsilon \to 0$, so the monotone convergence theorem applies. In $I_{\varepsilon 1}$, we make a substitution $t^2 = x^2 s/4 + \varepsilon^2(1-s)$. Then

$$I_{\varepsilon 1} = \sqrt{(x^2/4) - \varepsilon^2} \int_0^1 \psi(t) \frac{ds}{\sqrt{s}},$$

where

$$\psi(t) := \frac{1}{2t} \log\left|\frac{x+t}{x-t}\right| \frac{1}{\sqrt{1-t^2}}, \qquad t^2 = \frac{x^2}{4}s + \varepsilon^2(1-s).$$

Since $\psi(t)$ is bounded on $(0, x/2]$, we may appeal to the bounded convergence theorem. Consequently, (2.9) holds and then (2.10) follows by the maximum principle.

(ii) By the Fubini-Tonelli theorem,

$$\int U_g^{\mu} d\sigma = \int U_g^{\sigma} d\mu,$$

so that (2.11), (2.12) follow from (2.9) and (2.10).

■

We turn now to the equilibrium problem in the presence of an external field. Given a continuous real-valued function φ on a compact set $E \subset D$, with cap $E > 0$, then it is known (cf. [13, Section II.5]) that there exists a unique measure μ on E, having total mass 1 and such that, with some constant c_μ,

$$U_g^{\mu} - \varphi \le c_\mu \quad \text{on } S(\mu), \tag{2.13}$$

$$U_g^{\mu} - \varphi \ge c_\mu \quad \text{q.e on } E. \tag{2.14}$$

Now, let K be a compact subset of E (of positive capacity). Integrating (2.14) against the equilibrium measure ω_K, we obtain (recall (2.1)):

$$c_\mu \le \int U_g^{\mu} d\omega_K - \int \varphi \, d\omega_K \;=\; \int U_g^{\omega_K} d\mu - \int \varphi \, d\omega_K \;\le\; \frac{1}{\mathrm{cap}_g K} - \int \varphi \, d\omega_K.$$

For $K = S(\mu)$ the opposite inequality holds (integrate (2.13) against ω_K and apply (2.1), recalling that sets of zero capacity have zero ω_K-measure). We have thus proved the following.

Theorem 2.2 *Define, for any compact* $K \subset E$,

$$F(K) := -\frac{1}{\mathrm{cap}_g K} + \int \varphi \, d\omega_K. \tag{2.15}$$

Then

$$\max_K F(K) = F(S(\mu)) = -c_\mu.$$

Remark. The F-functional

$$F(K) = \log(\operatorname{cap} K) - \int Q \, d\omega_K$$

(note: cap not cap_g) was introduced and studied by Mhasker and Saff [10] in connection with the equilibrium problem for logarithmic potentials of positive measures. Theorem 2.2 is an extension of their result. (Our φ is $-Q$ in the notation of [10]). Note that this can be extended further to the non-symmetric case.

We conclude with two observations. Let E be a regular compact set (e.g., a segment) in D, let ν be any measure in D, and assume that, for some continuous φ on E, the relation $U_g^\nu - \varphi \geq c$ holds q.e. on E. Then it must hold everywhere on E (cf. [16, p.138]).

Next, since

$$U_g^\nu(z) = \int \log \frac{1}{|z-t|} d\nu(t) + u(z),$$

where u is harmonic in D, the principle of descent and the lower envelope theorem (cf. [15, Appendix], [6], or [18]) can be applied to U_g^ν.

3 The equilibrium problem

Let $E_\varepsilon := [\varepsilon, 1]$. We have seen in Section 2 that there exists unique μ_ε on $[\varepsilon, 1]$, $\|\mu_\varepsilon\| = 1$, such that

$$U_g^{\mu_\varepsilon} - \varphi = c_\varepsilon, \quad \text{on } S(\mu_\varepsilon), \tag{3.1}$$

$$U_g^{\mu_\varepsilon} - \varphi \geq c_\varepsilon, \quad \text{on } [\varepsilon, 1], \tag{3.2}$$

where c_ε is some constant. We now show that, as $\varepsilon \to 0$, μ_ε approaches the desired equlibrium measure μ, that satisfies (1.9), (1.10). Integrating (3.2) against dt/t we obtain

$$\left\{c_\varepsilon + \min_{[\varepsilon,\delta]}\varphi\right\}\log\frac{\delta}{\varepsilon} \;\leq\; \int_\varepsilon^\delta U_g^{\mu_\varepsilon}(t)\frac{dt}{t} \;=\; \int\left\{\int_\varepsilon^\delta \log\left|\frac{x+t}{x-t}\right|\frac{dt}{t}\right\}d\mu_\varepsilon(x) \;\leq\; \frac{\pi^2}{2}.$$

In the last step we used Newman's inequality (1.2) and the fact that $\|\mu_\varepsilon\| = 1$. Since φ is continuous and $\varphi(0) = 0$, we obtain, on letting first $\varepsilon \to 0$ and then $\delta \to 0$, that

$$\limsup_{\varepsilon\to 0} c_\varepsilon \;\leq\; 0.$$

On the other hand, as $U_g^{\mu_\varepsilon} \geq 0$, inequality (3.1) shows that c_ε is bounded from below, by $-\max_{[0,1]}\varphi$.

Now, let μ be any weak-star limit of $\{\mu_\varepsilon\}$. Passing to a subsequence we may assume that $c_\varepsilon \to c_\mu \leq 0$.

Applying (in D) the principle of descent to (3.1) and the lower envelope theorem to (3.2), we obtain that

$$U_g^\mu - \varphi \leq c_\mu \quad \text{on } S(\mu)\setminus\{0\}$$

$$U_g^\mu - \varphi \geq c_\mu \quad \text{q.e. on } [0,1].$$

The second relation holds (see the end of Section 2) everywhere on $(0,1]$, and as $c_\mu \leq 0$ it also holds at 0. The first relation holds at 0 if 0 is an isolated point of $S(\mu)$, since then U_g^μ must be continuous at 0. In the opposite case we have

$$c_\mu \geq \liminf_{\substack{x\to 0\\ x\in S(\mu)}}(U_g^\mu - \varphi) \;\geq\; 0$$

Thus, $c_\mu = 0$ and again $U_g^\mu - \varphi = c_\mu$ at $0 \in S(\mu)$. Therefore we finally obtain

$$U_g^\mu - \varphi = c_\mu, \quad \text{on } S(\mu) \tag{3.3}$$

$$U_g^\mu - \varphi \geq c_\mu, \quad \text{on } [0,1]. \tag{3.4}$$

Being a weak-star limit of unit measures, μ is also unit, and the existence of the solution of problem (1.9), (1.10) is established.

Remarks.

(a) It may happen above that μ has a mass at 0. For example, if $\varphi \equiv 0$, then c_μ must be 0 (since $U_g^\mu \geq 0,\ c_\mu \leq 0$), and then (3.3) yields $U_g^\mu = 0$ on $S(\mu)$. By the maximum principle, $U_g^\mu \leq 0$ in D, so that U_g^μ is identically 0. Therefore, the corresponding μ is a unit mass placed at the origin.

(b) By (3.3), U_g^μ is continuous on $S(\mu)$, hence ([18, Theorem III.2) it is continuous in $\mathbf{C}$, except perhaps at 0. Now, if 0 is not a limit point of $S(\mu)$, this potential is obviously continuous at 0. In the opposite case, $c_\mu = 0$ and (3.3) shows that $\lim_{x\to 0+} U_g^\mu(x) = 0$. Then we have $\lim_{z\to 0,\ z\in D} U_g^\mu(z) = 0$ by the same argument that was used in the proof of the maximum principle (2.7). Finally, as $U_g^\mu(-z) = U_g^\mu(z)$, we obtain that U_g^μ is continuous in $\mathbf{C}$.

Next, we prove the uniqueness of above μ. This follows from a more general result that shows that $U_g^\mu - c_\mu$ is the least superharmonic majorant for φ, in the class of all majorants of the form $U_g^\nu + \text{const},\ \|\nu\| \leq 1$.

Lemma 3.1 *Let μ satisfy* (3.3), (3.4), $\|\mu\| = 1$. *Assume that $U_g^\nu - a \geq \varphi$ on* $[0,1]$, *where $\|\nu\| \leq 1$ and a is a constant (necessarily, non-positive). Then*

$$\varphi \ \leq\ U_g^\mu - c_\mu \ \leq\ U_g^\nu - a \quad \textit{on } [0,1]. \tag{3.5}$$

Proof We consider two cases.

Case I. $a \leq c_\mu$.

Let $h = U_g^\nu - U_g^\mu$. h is superharmonic in $D \setminus S(\mu)$, bounded from below ($U_g^\nu \geq 0$ while U_g^μ is bounded) and satisfies

$$\liminf_{z\to\partial D} h \ \geq\ 0.$$

On $S(\mu)$ we have

$$h \;=\; U_g^\nu - U_g^\mu \;=\; (U_g^\nu - \varphi) \;-\; (U_g^\mu - \varphi) \;\geq\; a - c_\mu .$$

Now, since U_g^μ is continuous, h is lower semicontinuous. Hence,

$$\liminf_{z \to x \,\in\, S(\mu)} h(z) \;\geq\; h(x) \;\geq\; a - c_\mu .$$

Applying the minimum principle for superharmonic functions, we conclude (as $a - c_\mu \leq 0$) that $h \geq a - c_\mu$ in D, and (3.5) follows.

Case II. $a > c_\mu$.

Then the same reasoning as above gives $h \geq 0$ in D. Next, since $a \leq 0$, we see that $c_\mu < 0$, so that $0 \notin S(\mu)$. By Lemma 2.1 (apply (2.12) to μ and (2.11) to ν) we obtain, as $h \geq 0$ and $\|\nu\| \leq \|\mu\|$:

$$0 \geq \|\nu\| - \|\mu\| \geq \int (U_g^\nu - U_g^\mu) d\sigma = \int h \, d\sigma \geq \int_{S(\mu)} h \, d\sigma \geq (a - c_\mu) \int_{S(\mu)} d\sigma > 0$$

and we have a contradiction. Thus, Case II is impossible, and the lemma is proved.

■

We now proceed with the

Proof of Theorem 1.2

(i) ⇒ (ii) Assume (1.5) holds. This means (see (1.11)), that with $\|\nu_n\| \leq 1$, the relation $U_g^{\nu_n}(x) - \varphi(x) \geq o(1)$ holds uniformly for $x \in [0,1]$. Let ν be any weak-star limit of ν_n. Then $U_g^\nu \geq \varphi$ on $[0,1]$, and Lemma 3.1 yields $U_g^\mu - c_\mu \leq U_g^\nu$. Integrating this against $d\sigma$ (see Lemma 2.1 (ii)), we obtain (1.13).

(ii) ⇒ (iii) This is obvious since otherwise the integral in (1.13) diverges.

(iii) $\Rightarrow$ (i). Here we know that $\varphi \le U_g^\mu$ on $[0,1]$, U_g^μ is continuous, $S(\mu) \subseteq [0,1]$ and $\|\mu\| = 1$. We need to construct ν_n, $\|\nu_n\| \le 1$, such that (1.11) will hold. We follow the reasoning in [9, p. 40-43]. Assume first that $\mu\{0\} = 0$, and define $0 = t_0 < t_1 < \cdots < t_n = 1$ by

$$\int_{t_{k-1}}^{t_k} d\mu = \frac{1}{n}, \quad k = 1, 2, \ldots, n. \tag{3.6}$$

Let ν_n be a measure having a mass $1/n$ at each t_j, $1 \le j \le n$. Fix $x \in [0,1]$ and let $t_{j-1} < x \le t_j$, for some j. Since $\log|(x+t)/(x-t)|$ is increasing on $[0, t_{j-1}]$ and is decreasing on $[t_j, 1]$, a simple estimation gives

$$U_g^{\nu_n}(x) \;\ge\; U_g^\mu(x) - \int_{t_{j-1}}^{t_j} \log\left|\frac{x+t}{x-t}\right| d\mu(t). \tag{3.7}$$

If $t \in [t_{j-1}, t_j]$ satisfies $|t - x| > n^{-1}$, we have

$$\log\left|\frac{x+t}{x-t}\right| \;<\; \log(1+2n),$$

so that the integration over such t's will contribute $O(\log n/n)$ to the integral in (3.7). If we can show that

$$\int_{|t-x| \le n^{-1}} \log\left|\frac{x+t}{x-t}\right| d\mu(t) \;=\; o(1), \quad n \to \infty \tag{3.8}$$

uniformly for $x \in [0,1]$, we are done because (3.7) then gives

$$U_g^{\nu_n} \;\ge\; U_g^\mu - o(1) \;\ge\; \varphi - o(1), \quad \text{on } [0,1]$$

as required.

Assuming (3.8) is false, one can find $x \in [0,1]$ and a sequence $x_n \to x$ such that

$$\int_{|t-x_n| \le n^{-1}} \log\left|\frac{x_n+t}{x_n-t}\right| d\mu(t) \;\ge\; \alpha \;>\; 0, \quad n \ge 1. \tag{3.9}$$

Now let $\delta > 0$. Since $x_n \to x$ and the integrand in (3.9) is nonnegative, we only strengthen (3.9) if we replace the range of integration by $| t - x | < \delta$, provided n is large enough, i.e.

$$\int_{|t-x|<\delta} \log\left|\frac{x_n + t}{x_n - t}\right| d\mu(t) \;\geq\; \alpha, \quad n \geq n_\delta. \tag{3.10}$$

Next, as μ has no point masses, we can select δ so that

$$\int_{|t-x|\geq\delta} \log\left|\frac{x + t}{x - t}\right| d\mu(t) \;\geq\; U_g^\mu(x) - \frac{\alpha}{2}.$$

For n large enough, we can replace x by x_n in the integrand, thereby changing the integral by $o(1)$. Then, on adding the resulting inequality to (3.10) we obtain

$$U_g^\mu(x_n) \;\geq\; U_g^\mu(x) + \frac{\alpha}{2} - o(1), \quad n \geq n_\delta,$$

contradicting the continuity of U_g^μ.

Finally, if $\mu\{0\} := 1 - \rho > 0$, define μ_ρ on $(0, 1]$ by $\mu_\rho := \mu/\rho$. Then we obtain a unit measure and, by the preceding argument, we have a stronger result with φ replaced by $U_g^{\mu_\rho}$.

■

4 Proof of Theorem 1.3

Let

$$\rho := \frac{2}{\pi^2} \int_0^1 \varphi(x) \, \frac{dx}{x\sqrt{1 - x^2}}. \tag{4.1}$$

Recall that the condition $\rho \leq 1$ is necessary for (1.5) to hold. Thus, in view of Theorem 1.2, we only need to show that if $\rho \leq 1$ and φ is increasing and concave, then $c_\mu = 0$. We will prove a stronger result, part (ii) of which will be used in Section 5.

Theorem 4.1 *Let φ, ρ be as above, and assume that $0 < \rho \leq 1$. Then*
(i) *the equilibrium potential U_g^μ satisfies*

$$U_g^\mu = \varphi, \quad \text{on } [0,1]. \tag{4.2}$$

Moreover, $S(\mu) = [0,1]$ and μ has a mass $1-\rho$ at 0.
(ii) *Let $\widetilde{\mu}$ denote the restriction of μ on $(0,1]$ (so that (4.2) holds with μ replaced by $\widetilde{\mu}$). Then $\widetilde{\mu}$ is absolutely continuous with respect to dt, and its density is given by*

$$v(t) = \frac{2}{\pi^2} t(1-t^2)^{-1/2} \, PV \int_0^1 \frac{\varphi'(s)}{t^2 - s^2} (1-s^2)^{1/2} \, ds, \tag{4.3}$$

for a.e. $t \in [0,1]$.

Proof (i) Proceeding as at the beginning of Section 3, we get a unit measure μ_ε on $[\varepsilon, 1]$ that satisfies

$$U_g^{\mu_\varepsilon} - \varphi = c_\varepsilon, \text{ on } S(\mu_\varepsilon) \tag{4.4}$$

$$U_g^{\mu_\varepsilon} - \varphi \geq c_\varepsilon, \text{ on } [\varepsilon, 1]. \tag{4.5}$$

Recall that we have shown that $\lim_{\varepsilon \to 0} c_\varepsilon = c_\mu \leq 0$.

Now, let I be an interval in the complement $[\varepsilon, 1] \setminus S(\mu_\varepsilon)$ with endpoints belonging to $S(\mu_\varepsilon)$. Then $U_g^{\mu_\varepsilon}$ is strictly convex on I while φ is concave. Since (4.4) holds at the endpoints of I, (4.5) is violated inside I. Thus, there is no such I, what means that $S(\mu_\varepsilon)$ is an interval, say $[a_\varepsilon, b_\varepsilon]$. Next, (4.4) holds at b_ε, and $U_g^{\mu_\varepsilon}$ is decreasing for $x > b_\varepsilon$. Since φ is increasing, b_ε must be equal to 1, since otherwise (4.5) is false on $(b_\varepsilon, 1]$. So, $S(\mu_\varepsilon) = [a_\varepsilon, 1]$. Now, by Theorem 2.2, c_ε is given by

$$-c_\varepsilon = F([a_\varepsilon, 1]) = \frac{1}{K'(a_\varepsilon)} \left\{ \int_{a_\varepsilon}^1 \varphi(t) \left\{ (1-t^2)(t^2 - \varepsilon^2) \right\}^{-1/2} dt \; - \pi K(a_\varepsilon) \right\},$$

where we used (2.3), (2.4).

Let I_ε denote the integral in { }. Putting $t^2 = s + a_\varepsilon^2(1-s)$ we obtain that

$$I_\varepsilon = \int_0^1 \psi(s + a_\varepsilon^2(1-s)) \frac{ds}{\sqrt{s(1-s)}}, \quad \psi(t) := \frac{\varphi(\sqrt{t})}{2\sqrt{t}}.$$

Since $\varphi(0) = 0$ and φ is concave, we have $\sqrt{t}\,\varphi'(\sqrt{t}) \le \varphi(\sqrt{t})$, which means that $\psi(t)$ is decreasing. Therefore,

$$I_\varepsilon \le \int_0^1 \psi(s) \frac{ds}{\sqrt{s(1-s)}} \;=\; \int_0^1 \varphi(t) \frac{dt}{t\sqrt{1-t^2}} \;\le\; \frac{\pi^2}{2},$$

by our assumption $\rho \le 1$. On the other hand, $\pi K(a_\varepsilon) > \pi^2/2$. Thus, $c_\varepsilon > 0$ and we have

$$0 \ge c_\mu = \lim_{\varepsilon\to 0} c_\varepsilon \ge 0.$$

Hence, $c_\mu = 0$, so that $U_g^\mu - \varphi = 0$ on $S(\mu)$. Since $U_g^\mu - \varphi = 0$ at 0 as well, the same reasoning used for μ_ε shows that $S(\mu) = [0,1]$. Thus, (4.2) holds. Recalling the definition of ρ we obtain from Lemma 2.1(ii):

$$\rho = \int \varphi d\sigma = \int U_g^\mu d\sigma = \|\mu\| - \mu(\{0\}) = 1 - \mu(\{0\}).$$

(ii) We use the method of [9], but care must be taken near 0, since φ' may not be in $L_p[0,1], p > 1$. So let us consider

$$\varphi_\varepsilon(x) := \begin{cases} \varphi(x), & x \in [\varepsilon, 1] \\ x\dfrac{\varphi(\varepsilon)}{\varepsilon}, & x \in [0, \varepsilon], \end{cases} \tag{4.6}$$

which is bounded. Next, extend φ_ε to $[-1,0]$ as an *odd* function. Then we may apply the known result (cf. [9] or [13, Ch.IV.3]), which asserts that the function

$$v_\varepsilon(t) := \frac{1}{\pi^2}\sqrt{\frac{1+t}{1-t}}\, PV \int_{-1}^1 \sqrt{\frac{1-s}{1+s}}\, \frac{\varphi_\varepsilon'(s)}{t-s} ds + \frac{A}{\sqrt{1-t^2}} \tag{4.7}$$

is defined a.e. on $[-1,1]$, belongs to $L_p[-1,1]$, $1<p<2$, and satisfies (with any choice of a constant A):

$$\int_{-1}^{1} \log \frac{1}{|x-t|} \, v_\varepsilon(t)dt \; = \; \varphi_\varepsilon(x) + c_A, \quad x \in [-1,1]. \tag{4.8}$$

Remark. It is assumed in the above references that the function φ_ε ($-f$, in their notation) is *even*, while our φ_ε is odd. However, this assumption was used for purposes other than proving (4.7), (4.8).

Since we want to return to Green potentials, we choose A to ensure that v_ε is odd. Set

$$A \; := \; \frac{1}{\pi^2} \, PV \, \int_{-1}^{1} \sqrt{\frac{1-s}{1+s}} \; \frac{\varphi_\varepsilon'(s)}{s} \, ds$$

(since $\varphi_\varepsilon' =$ const near 0, the PV-integral exists). With the above choice of A, we have

$$v_\varepsilon(t) = \frac{1}{\pi^2\sqrt{1-t^2}} \, PV \, \int_{-1}^{1} \sqrt{\frac{1-s}{1+s}} \; \varphi_\varepsilon'(s) \left(\frac{1+t}{t-s} + \frac{1}{s} \right) ds,$$

that is,

$$\begin{aligned} v_\varepsilon(t) &= \frac{t}{\pi^2\sqrt{1-t^2}} \, PV \, \int_{-1}^{1} \sqrt{1-s^2} \; \frac{\varphi_\varepsilon'(s)}{t-s} \; \frac{ds}{s} \\ &= \frac{2t}{\pi^2\sqrt{1-t^2}} \, PV \, \int_{0}^{1} \sqrt{1-s^2} \, \frac{\varphi_\varepsilon'(s)}{t^2-s^2} \; ds, \end{aligned} \tag{4.9}$$

where, in the last step, we used the evenness of φ_ε'. Thus, v_ε is odd. Then the integral in (4.8) is also odd in x, and so is φ_ε. Hence, $c_A = 0$ and applying the oddness of v_ε, we may rewrite (4.8) as:

$$\int_{0}^{1} \log \left| \frac{x+t}{x-t} \right| \, v_\varepsilon(t)dt \; = \; \varphi_\varepsilon(x), \quad x \in [0,1]. \tag{4.10}$$

Next, we verify that $v_\varepsilon > 0$ on $(0,1)$. A standard calculation shows that

$$PV \int_{0}^{1} \sqrt{1-s^2} \; \frac{ds}{t^2-s^2} \; = \; \frac{\pi}{2}, \quad t \in (-1,1) \setminus \{0\}. \tag{4.11}$$

Therefore v_ε can be written (use the second relation in (4.9)) as:

$$(4.12)\quad v_\varepsilon(t) \;=\; \frac{1}{\pi}\,\frac{t\varphi'_\varepsilon(t)}{\sqrt{1-t^2}} \;+\; \frac{2t}{\pi^2\sqrt{1-t^2}}\int_0^1 \sqrt{1-s^2}\,\frac{\varphi'_\varepsilon(s)-\varphi'_\varepsilon(t)}{t^2-s^2}\,ds.$$

Since φ'_ε is positive and nonincreasing on $[0,1]$, both terms in (4.12) are positive; hence $v_\varepsilon > 0$.

Now, define $\mu_\varepsilon \geq 0$ on $[0,1]$ by $d\mu_\varepsilon(t) := v_\varepsilon(t)dt$. (The present μ_ε should not be confused with the one that appeared before.) Then (4.10) and Lemma 2.1(ii) yield

$$\|\mu_\varepsilon\| \;=\; \int \varphi_\varepsilon d\sigma$$

Let $\varepsilon \to 0$, and let $\widetilde{\mu}$ be any weak-star limit of μ_ε. The usual reasoning gives (as $\varphi_\varepsilon \to \varphi$) that $U^{\widetilde{\mu}} = \varphi$ on $[0,1]$. Since $\varphi_\varepsilon \uparrow \varphi$, we obtain

$$(4.13)\qquad \|\widetilde{\mu}\| = \lim_{\varepsilon\to 0}\|\mu_\varepsilon\| = \int \varphi d\sigma = \rho$$

(see (4.1)). The uniqueness of the equlibrium measure then shows that $\widetilde{\mu}$ coincides with $\widetilde{\mu}$ as defined in Theorem 4.1 (the restriction of μ on $(0,1]$). To complete the proof, it remains to show that v, as given by (4.3), is the limit of v_ε, in $L_1[0,1]$. Note that the existence a.e of the integral (4.3) follows from the established existence of the corresponding integral (4.9), and as $\varphi_\varepsilon = \varphi$ on $[\varepsilon, 1]$.

Exactly as we deduced (4.12) from (4.9), we may write v in the form:

$$(4.14)\quad v(t) \;=\; \frac{1}{\pi}\,\frac{t\varphi'(t)}{\sqrt{1-t^2}} \;+\; \frac{2t}{\pi^2\sqrt{1-t^2}}\int_0^1 \sqrt{1-s^2}\,\frac{\varphi'(s)-\varphi'(t)}{t^2-s^2}ds.$$

Now, fix $0 < \delta < 1/2$ and rewrite (4.12), for $t \in [\delta, 1-\delta]$ and $\varepsilon < \frac{1}{2}\delta$, in the form

$$v_\varepsilon(t) = \frac{t\varphi'(t)}{\pi\sqrt{1-t^2}} + \frac{2t}{\pi^2\sqrt{1-t^2}}\left(\int_\varepsilon^1 \sqrt{1-s^2}\,\frac{\varphi'(s)-\varphi'(t)}{t^2-s^2}ds\right.$$

$$+ \quad \int_0^{\varepsilon} \frac{\varepsilon^{-1}\varphi(\varepsilon) - \varphi'(t)}{t^2 - s^2} \sqrt{1-s^2}\, ds \Bigg)$$

where we used the definition (4.6) of φ_ε.

As $\varepsilon \to 0$, the second integral is $O(\varphi(\varepsilon)) \to 0$, uniformly for $t \in [\delta, 1-\delta]$. The first integral is increasing to the integral in (4.14). Since $\int_0^1 v_\varepsilon = \|\mu_\varepsilon\|$ are uniformly bounded (see (4.13)) we conclude that $v \in L_1[\delta, 1-\delta]$, that

$$\int_\delta^{1-\delta} v \;=\; \lim_{\varepsilon \to 0} \int_\delta^{1-\delta} v_\varepsilon \;\leq\; C, \tag{4.15}$$

and that

$$\int_\delta^{1-\delta} |v - v_\varepsilon| \to 0, \quad \varepsilon \to 0. \tag{4.16}$$

Since (4.15) holds for any $\delta > 0$, we obtain that $v \in L_1[0,1]$. Therefore,

$$\left(\int_0^\delta + \int_{1-\delta}^1 \right) v \;\to 0, \quad \delta \to 0.$$

The same is true for v_ε, uniformly on ε, as (4.13) shows. Combined with (4.16) this gives: $v_\varepsilon \to v$ in $L_1[0,1]$, and the proof is complete.

■

5 Proof of Theorem 1.4

We first show that the extra conditions (a), (b) of Theorem 1.4 ensure that, for some $C_1, C_2 > 0$,

$$C_1 \leq v(t)\; t/\varphi(t) \leq C_2, \quad t \text{ small}, \tag{5.1}$$

where v is defined in (4.3). Extend φ to [-1, 1] as an odd function and rewrite (4.3), using the evenness of φ':

$$v(t) = \frac{t}{\pi^2\sqrt{1-t^2}}\, PV \int_{-1}^1 \sqrt{1-s^2}\, \frac{\varphi'(s)}{t-s}\, \frac{ds}{s}.$$

With the aid of

$$\frac{t}{(t-s)s} = \frac{1}{t-s} + \frac{1}{s}$$

we further rewrite v as follows:

$$v(t) = \frac{1}{\pi^2\sqrt{1-t^2}}\, PV \int_{-1}^{1} \sqrt{1-s^2}\, \frac{\varphi'(s)}{t-s}\, ds,$$

since the integral involving $1/s$ vanishes because the integrand is an odd function. Next, write

$$\sqrt{1-s^2} = \sqrt{1-t^2} + (\sqrt{1-s^2} - \sqrt{1-t^2}).$$

Then we obtain

$$v(t) = \frac{1}{\pi^2}\, PV \int_{-1}^{1} \frac{\varphi'(s)}{t-s} ds + \frac{B(t)}{\sqrt{1-t^2}}, \tag{5.2}$$

where

$$0 < B(t) < ct, \quad t \in (0,1].$$

Now express the integral in (5.2) as

$$\int_{-1}^{1} \frac{\varphi'(s) - \varphi'(t)}{t-s} + \varphi'(t) \log \frac{1+t}{1-t}.$$

The second term is positive. Also,

$$\int_{-1}^{-t} + \int_{t}^{1} = 2t \int_{t}^{1} \frac{\varphi'(s) - \varphi'(t)}{t^2 - s^2} ds \ > \ 0$$

as φ' is decreasing. Furthermore,

$$\begin{aligned}\int_{-t}^{t} \ &= \ 2t \int_{0}^{t} \frac{\varphi'(s) - \varphi'(t)}{t^2 - s^2} ds \ > \ \int_{0}^{t} \frac{\varphi'(s) - \varphi'(t)}{t-s} ds \\ &> \ \frac{1}{t} \int_{0}^{t} [\varphi'(s) - \varphi'(t)] ds \ = \ \frac{\varphi(t)}{t} - \varphi'(t) \ \geq \ \delta \frac{\varphi(t)}{t}\end{aligned}$$

by the assumption (b), provided t is small. Collecting all estimates, we get the left-hand inequality in (5.1).

Now we return to the integral in (5.2). We have

$$\left(\int_{-1}^{-2t} + \int_{2t}^{1}\right) \frac{\varphi'(s)}{t-s}ds \;=\; 2t\int_{2t}^{1} \frac{\varphi'(s)}{t^2-s^2}ds < 0.$$

Also,

$$\int_{-2t}^{0} \frac{\varphi'(s)}{t-s}ds \;<\; \frac{1}{t}\int_{-2t}^{0} \varphi'(s)ds \;=\; \frac{\varphi(2t)}{t} \;<\; \frac{2\varphi(t)}{t},$$

by concavity. Finally,

$$PV\int_{0}^{2t} \frac{\varphi'(s)}{t-s}\,ds \;=\; \int_{0}^{2t} \frac{\varphi'(s)-\varphi'(t)}{t-s}\,ds \;=\; \frac{1}{t}\int_{0}^{2t} \frac{(t-s+s)\varphi'(s)-t\varphi'(t)}{t-s}\,ds$$

$$= \; \frac{1}{t}\varphi(2t) \;+\; \frac{1}{t}\int_{0}^{2t} \frac{s\varphi'(s)-t\varphi'(t)}{t-s} < \frac{1}{t}\varphi(2t),$$

since the integrand is negative, by the assumption (a). Thus we have the second inequality in (5.1).

We can now proceed with the

Proof of Theorem 1.4

We have to construct a measure ν_n having a mass $\dfrac{1}{n}$ at some $t_1, \ldots, t_n$ and such that (see (1.12))

$$U_g^{\nu_n} \;\geq\; \varphi - \frac{c}{n}, \quad \text{on } [0,1].$$

Consider the function $(1+\varepsilon)\varphi$. Since strict inequality holds in (1.14'), there is $\varepsilon > 0$ such that

$$\frac{2}{\pi^2}\int_{0}^{1} (1+\varepsilon)\,\varphi(x)\,\frac{dx}{x\sqrt{1-x^2}} \;=\; 1.$$

By Theorem 4.1, the equilibrium measure μ for $(1+\varepsilon)\varphi$ has no mass at 0 and it satisfies $U_g^{\mu} = (1+\varepsilon)\varphi$ on $[0,1]$. Let $0 \;=\; t_0 < t_1 < \cdots < t_n \;=\; 1$ and

ν_n be as in the proof of Theorem 1.2. The relation (3.7) now becomes

$$U_g^{\nu_n}(x) \ \geq \ (1+\varepsilon)\varphi(x) - \int_{t_{j-1}}^{t_j} \log\left|\frac{x+t}{x-t}\right| d\mu(t), \quad x \in (t_{j-1}, t_j).$$

Thus, our task is to show that

$$\int_{t_{j-1}}^{t_j} \log\left|\frac{x+t}{x-t}\right| d\mu(t) \ \leq \ \varepsilon\varphi(x) + \frac{c}{n} \tag{5.3}$$

for some C and for all $x \in (t_{j-1}, t_j)$, $j = 1, \ldots, n$. We have already shown that the above integral is $o(1)$ uniformly in $x \in (t_{j-1}, t_j)$ and in j. Therefore, (5.3) is obvious for $x \geq \delta$ (any $\delta > 0$) provided $n \geq n(\delta)$, and we may only consider the case $t_{j-1} < x < t_j < \delta$, where (a), (b) hold.

By the lower bound in (5.1) and by concavity of φ,

$$\frac{1}{n} = \int_{t_{j-1}}^{t_j} v(t)dt \ \geq \ \frac{1}{c_1}\int_{t_{j-1}}^{t_j} \frac{\varphi(t)}{t}\, dt \ \geq \ \frac{1}{c_1}\int_{t_{j-1}}^{t_j} \varphi'(t)dt \ = \ \frac{1}{c_1}(\varphi(t_j) - \varphi(t_{j-1})).$$

Therefore, for $t_{j-1} < x < t_j$,

$$\max_{[t_{j-1}, t_j]} \varphi \ = \ \varphi(t_j) \ \leq \ \varphi(x) + \frac{c_1}{n}, \tag{5.4}$$

uniformly in j, provided $t_j \leq \delta$. Next,

$$\int_{t_{j-1}}^{t_j} \log\left|\frac{x+t}{x-t}\right| v(t)dt \ \leq \ \left\{\int_{t_{j-1}}^{t_j} \log^2\left|\frac{x+t}{x-t}\right| \frac{dt}{t}\right\}^{\frac{1}{2}} \left\{\int_{t_{j-1}}^{t_j} tv^2(t)dt\right\}^{\frac{1}{2}}.$$

The first integral is bounded by an absolute constant (put $t = \tau x$). The second is bounded by

$$\left\{\int_{t_{j-1}}^{t_j} v(t)dt\right\}^{\frac{1}{2}} \max_{[t_{j-1}, t_j]}\{tv(t)\}^{\frac{1}{2}} \ \leq \ c_2 n^{-1/2} \max_{[t_{j-1}, t_j]} \varphi(t)^{1/2} \ \leq \ c_3\left[n^{-1/2}\varphi(x)^{1/2} + n^{-1}\right],$$

where we used the definition of t_j's, the upper bound in (5.1) and, in the last step, (5.4). Since for any $\varepsilon > 0$,

$$c_3\sqrt{\frac{\varphi}{n}} \ \leq \ \varepsilon\varphi + \frac{c_3^2}{4\varepsilon n},$$

the bound (5.3) follows.

■

6 Rational Approximation of sgn x and $|x|$.

Throughout this section, r_n denotes a function of the form (1.1), while R_n denotes some rational function of order n, not necessarily of the form (1.1).

We start with a lemma that is due to D. Newman [11], except for a minor modification.

Lemma 6.1 *Let δ_n denote a positive function on* [0,1].

(i) *Assume that*

$$\Big| |x| - R_n(x) \Big| \leq \delta_n(|x|), \ x \in [-1, 1]. \tag{6.1}$$

Then there exists r_n that satisfies

$$x|r_n(x)| \leq 2\delta_n(x), \ x \in [0, 1]. \tag{6.2}$$

Moreover, we may assume that all zeros of r_n lie on (0,1].

(ii) *Assume that*

$$|\operatorname{sgn} x - R_n(x)| \leq \delta_n(|x|), \ x \in [-1, 1]. \tag{6.3}$$

Then there exists r_n such that

$$|r_n(x)| \leq 2\delta_n(x), \ x \in [0, 1]. \tag{6.4}$$

(iii) *Assume that r_n satisfies* (6.2) *and, additionally,*

$$1 + r_n(x) \geq \frac{1}{2}, \ x \in [0, 1]. \tag{6.5}$$

Then there exists R_n that satisfies (6.1) *with δ_n replaced by $4\delta_n$. Similarly, if r_n satisfies* (6.4) *and* (6.5), *then there is R_n satisfying* (6.3) *with δ_n replaced by $4\delta_n$.*

Proof (i) If R_n is even, (6.2) follows from [11], with factor 2 dropped. For arbitrary $R_n := p_n/q_n$, we may assume $q_n > 0$ on [-1,1]. Then (6.1) implies

$$\left|\frac{xq_n(\pm x) - p_n(\pm x)}{q_n(x) + q_n(-x)}\right| \leq \left|\frac{xq_n(\pm x) - p_n(\pm x)}{q_n(\pm x)}\right| \leq \delta_n(x),$$

for $x \in [0,1]$. Therefore, with

$$\widetilde{R}_n(x) := \frac{p_n(x) + p_n(-x)}{q_n(x) + q_n(-x)}$$

we have $|x - \widetilde{R}_n(x)| \leq 2\delta_n(x),\ x \in [0,1]$. Since $\tilde{R}_n$ is even, we may continue as in [11].

Next, let $|(x-\xi)/(x+\xi)|$ be one of the factors on $|r_n|$. In view of (2.5), (2.6) we only strengthen (6.2) if we replace $\xi \in D$ by a suitable $\tilde{\xi} \in (0,1]$. For $\xi \notin D$, the above factor is ≥ 1 and we may drop it.

(ii) This follows easily from (i).

(iii) See [11].

■

To ensure (6.5) for r_n satisfying (1.4), a small adjustment is needed. Let

$$|r_n(x)| \leq Ce^{-n\varphi(x)},\ x \in [0,1].$$

Assuming φ is increasing, define $0 < \alpha_n < 1$ by $n\varphi(\alpha_n) = \log 2C$. Now, if r_n has a zero, ξ, on $(0, \alpha_n)$ we replace it by α_n. Since

$$\left|\frac{x-\alpha_n}{x+\alpha_n}\right| < \left|\frac{x-\xi}{x+\xi}\right|,\ x > \alpha_n,\ \xi < \alpha_n$$

we see that a new r_n still satisfies (1.4) on $[\alpha_n, 1]$. On $[0, \alpha_n]$, $|r_n| \leq 1$ while $C\ \exp(-n\varphi(x)) \geq \frac{1}{2}$. Thus (1.4) still holds with C replaced by $2C$. Therefore a new r_n satisfies (1.4) and also (6.5). With these preliminaries, the following result is an immediate consequence of the necessary condition (1.14) and Theorem 1.4.

Theorem 6.2 *Let φ be continuous and increasing on $[0,1]$ with $\varphi(0)=0$. Assume there exist $R_n, n \ge 1$, such that*

$$|\operatorname{sgn} x - R_n(x)| \le C \exp(-n\varphi(x)),\ x \in [-1,1]. \tag{6.6}$$

Then φ satisfies (1.14).

Conversely, if strict inequality holds in (1.14) and φ is as in Theorem 1.4, then there exist $R_n, n \ge 1$ satisfying (6.6).

We now turn to the approximation of $|x|$.

Proof of the second part of Theorem 1.5

Given $\varepsilon > 0$, let n be large enough. Since φ satisfies (1.17) with strict inequality, (1.14') holds for $\varepsilon^{-1}\varphi$. Hence there is an $r_{[n\varepsilon]}$ such that

$$|r_{[n\varepsilon]}| \le c e^{-[n\varepsilon]\varepsilon^{-1}\varphi} \le c_1 e^{-n\varphi},\ \text{on } [0,1].$$

Next, put $p := [n(1-\varepsilon)]$. Then (cf. [19]) there is an r_p such that $1 + r_p \ge \frac{1}{2}$ on [0,1] and

$$x|r_p(x)| \le c_2 e^{-\pi\sqrt{p}}.$$

Note that c_1, c_2 are independent of n. Since $|r_{[n\varepsilon]}| \le 1$, we have $1 + r_p r_{[n\varepsilon]} \ge \frac{1}{2}$ on $[0,1]$. Then Lemma 6.1(iii) yields R_n of order $p + [n\varepsilon] \le n$ that satisfies (1.16), with a constant independent of n.

■

Proof of the first part of Theorem 1.5

Assume (1.16) holds. Since φ is increasing, $\varphi > 0$ on $(0,1]$. Then, applying Lemma 6.1(i), we get r_n that satisfy:

$$x|r_n(x)| \le c \exp\left(-\pi\sqrt{n(1-\varepsilon)}\right),\ x \in [0,1] \tag{6.7}$$

and also:

$$x|r_n(x)| \le ce^{-n\varphi(x)},\ x \in [0,1]. \tag{6.8}$$

Fix $\eta > 0$, and let $t_1, \dots, t_n$ be zeros of r_n on $(0,1]$. We show below that

$$N := \#\{t_j : t_j > \frac{1}{n}\eta\} \ \le\ \varepsilon n + O(\sqrt{n}\,\log n). \tag{6.9}$$

Note that for other zeros we have

$$\left|\frac{x-t_j}{x+t_j}\right| \ \ge\ 1 - \frac{2}{n},\quad t_j \le \frac{1}{n}\eta,\ x \in [\eta,1],$$

so that

$$\prod_{t_j \le \frac{1}{n}\eta} \left|\frac{x-t_j}{x+t_j}\right| \ge e^{-2},\quad x \in [\eta,1].$$

Therefore, we obtain from (6.8):

$$|r_N(x)| \le c\eta^{-1}e^2e^{-n\varphi(x)},\ x \in [\eta,1],$$

where r_N is of degree N. On taking logarithms and integrating against $d\sigma$ (see Lemma 2.1) we obtain

$$N \ =\ \int_0^1 \ \ge\ \int_\eta^1 \log|r_N^{-1}|d\sigma \ \ge\ c_1\int_\eta^1 d\sigma + n\int_\eta^1 \varphi d\sigma.$$

Dividing by n and letting $n \to \infty$ yields, by (6.9),

$$\varepsilon \ge \int_\eta^1 \varphi d\sigma.$$

Since this holds for any $\eta > 0$, (1.17) follows.

It remains to prove (6.9). The same reasoning as above shows that

$$\prod_{t_j > \frac{1}{n}\eta} \left|\frac{x-t_j}{x+t_j}\right| \ge e^{-2},\quad x \in [0, n^{-2}\eta].$$

Thus, we get from (6.7) that

$$x|r_{n-N}(x)| \le ce^2 \exp\left(-\pi\sqrt{n(1-\varepsilon)}\right), \quad x \in [0, n^{-2}\eta].$$

On taking logarithms, dividing by x and integrating from $\varepsilon_n := \eta n^{-2} \exp(-\pi\sqrt{n-N})$ to ηn^{-2}, we obtain first, by Newman's inequality, and then dividing by $\int_{\varepsilon_n}^{\eta n^{-2}} dx/x$ that

$$\frac{1}{2}\,\frac{\log^2 n^{-2}\eta - \log^2 \varepsilon_n}{\log n^{-2}\eta - \log \varepsilon_n} - \frac{\pi^2}{2}\,\frac{n-N}{\log n^{-2}\eta/\varepsilon_n} \le c_1 - \pi\sqrt{n(1-\varepsilon)}.$$

With our choice of ε_n, we get

$$\log n^{-2}\eta - \pi\sqrt{n-N} \le c_1 - \pi\sqrt{n(1-\varepsilon)}$$

and (6.9) follows.

■

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