CONSTRAINED ENERGY PROBLEMS WITH APPLICATIONS TO ORTHOGONAL POLYNOMIALS OF A DISCRETE VARIABLE

By

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Abstract. Given a positive measure σ with $\|\sigma\| > 1$, we write $\mu \in \mathcal{M}^\sigma$ if μ is a probability measure and $\sigma - \mu$ is a positive measure. Under some general assumptions on the constraining measure σ and a weight function w, we prove existence and uniqueness of a measure λ_w^σ that minimizes the weighted logarithmic energy over the class $\mathcal{M}^\sigma$. We also obtain a characterization theorem, a saturation result and a balayage representation for the measure λ_w^σ. As applications of our results, we determine the (normalized) limiting zero distribution for ray sequences of a class of orthogonal polynomials of a discrete variable. Explicit results are given for the class of Krawtchouk polynomials.

1. Introduction

In this paper we shall investigate constrained energy problems in the presence of an external field. Before defining the problem, we briefly recall the classical and the weighted energy problems of potential theory. In so doing, we introduce the terminology that will be needed for stating our results.

The classical approach starts with the concepts of energy of measures and capacity of sets (cf. [T], [L]). Let E be a compact subset of the complex plane $\mathbf{C}$ and $\mathcal{M}_E$ be the collection of all probability measures with support in E. The logarithmic energy of a measure μ is defined by

$$I(\mu) := \int\int \log \frac{1}{|z-t|} d\mu(z) d\mu(t) \tag{1.1}$$

and the quantity

$$V_E := \inf\{I(\mu) \mid \mu \in \mathcal{M}_E\}$$

is called the Robin's constant. If $V_E < \infty$, there exists a unique measure $\mu_E \in \mathcal{M}_E$ such that $I(\mu_E) = V_E$. The extremal measure μ_E is called the *equilibrium measure*

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of the set E. The logarithmic capacity of E is given by

$$\operatorname{cap}(E) := e^{-V_E}$$

For an arbitrary finite Borel measure μ with compact support, the logarithmic potential of μ is defined by

$$U^{\mu}(z) := \int \log \frac{1}{|z-t|}\, d\mu(t) \tag{1.3}$$

We denote the support of μ by S_μ.

The following properties of the equilibrium measure were established by O. Frostman [F]: If $\operatorname{cap}(E) > 0$, then

$$U^{\mu_E}(z) \le V_E \quad \text{for all } z \in \mathbf{C}$$

$$U^{\mu_E}(z) = V_E \quad \text{q.e. on } E,$$

where by a property satisfied q.e. (quasi-everywhere) we mean everywhere except on a subset of capacity zero.

For the weighted energy problem we shall follow the scheme given in [ST]. Let E be a closed subset of $\mathbf{C}$ (we no longer require boundedness of E).

Definition 1.1 A weight function $w : E \to [0, \infty)$ is said to be *admissible* if it satisfies the following conditions:

(a) w is upper semi-continuous;

(b) $E_0 := \{z \in E | w(z) > 0\}$ has positive capacity;

(c) If E is unbounded, then $|z|w(z) \to 0$ as $|z| \to \infty, z \in E$.

We define the external field $Q = Q_w$ by $w(z) = e^{-Q(z)}$. Then $Q : E \to (-\infty, \infty]$ is lower semi-continuous, $Q(z) < \infty$ on a set of positive capacity and, if E is unbounded, then

$$\lim_{|z| \to \infty,\, z \in E} \{Q(z) - \log|z|\} = \infty$$

For $\mu \in \mathcal{M}_E$ the weighted energy integral is defined by

$$\begin{aligned} I_w(\mu) &:= \int\int \log\,[|z-t|w(z)w(t)]^{-1}\, d\mu(z)d\mu(t) \\ &= I(\mu) + 2\int Q\, d\mu. \end{aligned} \tag{1.5}$$

From the admissibility of w we have that the first integral is well-defined. The second equality holds when both integrals are finite. Furthermore, the quantity

$$V_w = \inf\{I_w(\mu) | \mu \in \mathcal{M}_E\}$$

is finite and there exists a unique measure $\mu_w \in \mathcal{M}_E$ such that $I_w(\mu_w) = V_w$. The measure μ_w is called the *extremal measure associated with* w and has the following properties:

$$U^{\mu_w}(z) + Q(z) \geq F_w \quad \text{q.e. on } E,$$

$$U^{\mu_w}(z) + Q(z) \leq F_w \quad \text{for all } z \in S_{\mu_w}$$

where $F_w := V_w - \int Q(t)\, d\mu_w(t)$.

In the unweighted case ($w \equiv 1$) on the interval $[-1, 1]$, Rakhmanov [R] introduced and studied the problem of minimal energy of measures that are subject to a constraint. Such measures arise in connection with the zero distribution of the Chebyshev polynomials of a discrete variable (cf. [Sz, §2.8]),

$$t_n(x) = n!\Delta^n \binom{x}{n}\binom{x-N}{n} \qquad n = 0, \qquad N -$$

Here our goal is to study the weighted analogue of the constrained minimal energy problem in a more general geometric setting. In so doing, we also obtain new general results for the unweighted case.

Definition 1.2 Let $w : E \to [0, \infty)$ be an admissible weight function. A finite Borel measure σ is called *an admissible constraint* for w if

(i) $S_\sigma = E$;

(ii) $\sigma(E_0) > 1$, where E_0 is defined in Definition 1(b);

(iii) σ has finite logarithmic energy over compact sets, i.e. for any compact set K,

$$\int\int_{K \times K} \log \frac{1}{|z - t|} d\sigma(z) d\sigma(t) < \infty.$$

Next we define the class of measures

$$\mathcal{M}^\sigma := \{\mu \mid \|\mu\| = 1 \text{ and } 0 \leq \mu \leq \sigma\},$$

where by $\mu \leq \sigma$ we mean that $\sigma - \mu$ is a positive Borel measure. As a consequence we have that if $\mu \in \mathcal{M}^\sigma$, then $S_\mu \subset S_\sigma = E$. The analogue of the Robin's constant for this problem is given by

$$V_w^\sigma := \inf\{I_w(\mu) \mid \mu \in \mathcal{M}^\sigma\}.$$

where $I_w(\mu)$ is defined in .5).

Definition 1.3 A measure $\mu \in \mathcal{M}^\sigma$ such that $I_w(\mu) = V_w^\sigma$ is called a *σ-constrained extremal measure for the weight w.* Such a measure will be denoted by λ_w^σ. In the unweighted case ($w \equiv 1$) we omit the subscript w.

The constrained energy problem deals with the existence and uniqueness of the extremal measure in question, its characterization and the investigation of its properties. The following electrostatical interpretation justifies our interest in this type of problem. Let E be a plane conductor and place a positive unit charge on E. If the force between two charged particles is proportional to the reciprocal of their distance, then the distribution of the charge with minimal energy will be μ_E. If, in addition, an external field $Q = \log(1/w)$ is present, then the solution will be μ_w. Now what if we relax the condition that E is a conductor and assume, more generally, that E consists of matter with varying conductivity, what is known as *continuous media* (see [LL])? Then one way to accommodate that generality in our mathematical model is to invoke a constraint σ on E.

Remark 1.4 Notice that if $\sigma \geq \mu_w$, then the extremal measure will not depend on the constraint and will coincide with μ_w. In this sense, the constrained energy problem generalizes the weighted energy problem.

The outline of this paper is the following. In Section 2 we state our main results, the analogue of Frostman's characterization theorem, the saturation property, and a balayage representation of the extremal measure, as well as some corollaries. The proofs are presented in Section 5. Section 3 is devoted to studying the asymptotic behavior of the zeros of L_p-extremal polynomials generated by a distribution with jumps at N points. In particular we obtain explicit results for ray sequences for the Krawtchouk polynomials. In Section 4 we find explicitly the constrained extremal measure for several special cases.

2. Statements of main results

In this section we state our main results, deferring their proofs to Section 5. We start with the fundamental theorem, which asserts the existence and uniqueness of the constrained extremal measure and also provides us with a criterion for recognizing that measure.

Theorem 2.1 *Let $w = e^{-Q}$ be an admissible weight on a closed set E and σ be an admissible constraint for w. Let V_w^σ be defined as in* (1.8). *Then the following properties hold*:

(a) *V_w^σ is finite.*

(b) *There exists a unique measure* $\lambda = \lambda_w^\sigma \in \mathcal{M}^\sigma$ *such that*

$$I_w(\lambda) = V_w^\sigma$$

(c) *There exists a constant* F_w^σ *such that*

$$U^\lambda(z) + Q(z) \geq F_w^\sigma \ \textit{holds} \ (\sigma - \lambda) \ \textit{a.e}$$

and

$$U^\lambda(z) + Q(z) \leq F_w^\sigma \quad \textit{holds for all } z \in S_\lambda \tag{2.2}$$

(d) *If* $\mu \in \mathcal{M}^\sigma$ *has compact support and there exists a constant c such that*

$$U^\mu(z) + Q(z) \geq c \ \textit{holds} \ (\sigma - \mu) \ \textit{a.e.}$$

and

$$U^\mu(z) + Q(z) \leq c \ \textit{holds} \ \mu \ \textit{a.e.}, \tag{2.4}$$

then $\mu = \lambda$.

(e) *For every measure* $\mu \in \mathcal{M}^\sigma$ *with compact support we define*

$$\begin{aligned} \mathcal{F}_\mu &= \operatorname{ess\,inf}_{\sigma-\mu}(U^\mu(x) + Q(x)) \\ &= \max\{K \in \mathbf{R} \mid U^\mu(x) + Q(x) \geq K \textit{ holds } (\sigma - \mu) \textit{ a.e.}\} \end{aligned}$$

Then $\mathcal{F}_\mu \leq \mathcal{F}_\lambda$. *Moreover, if the polynomial convex hull of* $S_\mu \cup S_\lambda$ *has two-dimensional Lebesgue measure zero, then* $\mathcal{F}_\mu = \mathcal{F}_\lambda$ *implies* $\mu = \lambda$. *(In particular, the last assertion holds if* $E \subset \mathbf{R}$.)

Remark 2.2 From the proof we shall see that $S_{\lambda_w^\sigma} \subset E_\epsilon$ for some $\epsilon > 0$, where

$$E_\epsilon := \{z \mid w(z) \geq \epsilon\}$$

Further, since $w(z)$ is admissible, E_ϵ is a compact set and so the support of the constrained extremal measure is also compact. Moreover, λ_w^σ has finite logarithmic energy since $I_w(\lambda_w^\sigma) < \infty$ and Q is lower bounded on compact sets.

Remark 2.3 Lemma 5.2 below shows that if U^σ is continuous on $\mathbf{C}$, so is $U^{\lambda_w^\sigma}$, and if Q is also continuous, inequality (2.1) holds everywhere on $S_{\sigma-\lambda_w^\sigma}$. If, in addition, $S_{\sigma-\lambda_w^\sigma} \cap S_{\lambda_w^\sigma} \neq \emptyset$, which is the case in many of the applications, the constant F_w^σ is uniquely determined and equals $\mathcal{F}_{\lambda_w^\sigma}$. The next example shows that if $S_{\sigma-\lambda_w^\sigma} \cap S_{\lambda_w^\sigma} = \emptyset$, then the constant F_w^σ need not be unique.

Example 2.4 Let $E = [-1, 1] \cup [2, 3]$, and let $w = \exp(-x^2)$. Then $d\mu_w = 2\sqrt{1-t^2}dt/\pi$, $t \in [-1, 1]$ (cf. [ST, Theorem IV.5.1]). Let $\sigma := \mu_w + \nu$, where ν is the Lebesgue measure restricted to the interval $[2, 3]$. Then clearly $\lambda_w^\sigma = \mu_w$ and $S_{\lambda_w^\sigma} = [-1, 1]$, $S_{\sigma-\lambda_w^\sigma} = [2, 3]$. Now it is easy to see that

$$\sup_{[-1,1]} (U^{\lambda_w^\sigma}(z) + Q(z)) < \inf_{[2,3]}(U^{\lambda_w^\sigma}(z) + Q(z$$

nd any number between the sup and the inf can serve as a value for F_w^σ

Remark 2.5 The following example shows that the assumption in part (e) of Theorem 2.1 that the polynomial convex hull of $S_\lambda \cup S_\mu$ have two-dimensional Lebesgue measure zero is essential to conclude the equality of the measures. Let E be the union of the two circles $|z| = 1$ and $|z| = 1/2$ and $\sigma := \omega_{|z|=1} + \omega_{|z|=1/2}$, where $\omega_{|z|=r}$ denotes the normalized one-dimensional Lebesgue measure on the circle $|z| = r$. Let $w \equiv 1$. Then clearly $\lambda = \lambda^\sigma = \omega_{|z|=1}$, and for $\mu = \omega_{|z|=1/2}$, we have $\mathcal{F}_\mu = \mathcal{F}_\lambda = 0$, but $\mu \neq \lambda$.

The next theorem gives more information on the location of the support of the extremal measure. Hereafter we shall assume that $Q(z)$ and $U^\sigma(z)$ are continuous on E. Let μ_w be the unconstrained extremal measure associated with the weight w on E. Then the following theorem holds.

Theorem 2.6 (Saturation Property) *Let σ be an admissible constraint for the admissible weight $w = e^{-Q}$ on E. If U^σ and Q are continuous on E, then $S_{\mu_w} \subset S_{\lambda_w^\sigma}$. Moreover,*

$$(\lambda_w^\sigma)_{|S_{\sigma-\lambda_w^\sigma}} \geq (\mu_w)_{|S_{\sigma-\lambda_w^\sigma}}. \tag{2.6}$$

Consequently, $\lambda_w^\sigma = \sigma$ on any positive subset for a Hahn decomposition of the signed measure $\mu_w - \sigma$.

Remark 2.7 Theorem 2.6 has the following interpretation: Assume that the measures μ_w and σ have continuous densities, ϕ and ψ respectively. Then the σ-constrained extremal measure λ_w^σ saturates the constraint σ on the set B where ϕ is greater than or equal to ψ, i.e. $(\lambda_w^\sigma)_{|B} = \sigma_{|B}$.

It is natural to ask whether a theorem of this type holds if we consider $\lambda_w^{\sigma_1}$ and $\lambda_w^{\sigma_2}$, where $\sigma_1 \leq \sigma_2$. We shall prove the following theorem.

Theorem 2.8 *Let $E_1 \subset E_2$ and w be an admissible weight on E_2, such that $w|_{E_1}$ is also admissible and let σ_1, σ_2 be admissible constraints for w on E_1 and E_2,* *respectively (this implies, by definition, $S_{\sigma_i} = E_i$, $i = 1, 2$). Suppose that $\sigma_1 \leq \sigma_2$, and let λ_1 and λ_2 be the corresponding σ_1- and σ_2-constrained extremal measures, respectively. Assume further that U^{σ_2} and Q are continuous on E. Then*

$$\lambda_2|_{S_{\sigma_1 - \lambda_1}} \leq \lambda_1|_{S_{\sigma_1 - \lambda_1}} \tag{2.7}$$

Next we derive some corollaries from the fundamental theorem. We first introduce the following representation:

$$\lambda_w^\sigma = (\lambda_w^\sigma)|_{S_\sigma \setminus S_{\sigma - \lambda_w^\sigma}} + (\lambda_w^\sigma)|_{S_{\sigma - \lambda_w^\sigma}} =: \lambda_1 + \lambda_2$$

Then $\lambda_1 = \sigma_{|S_\sigma \setminus S_{\sigma - \lambda_w^\sigma}} =: \sigma_1$ and we can easily deduce the following fact about λ_2.

Corollary 2.9 *If U^σ and Q are continuous on E and $\lambda_2(S_{\sigma - \lambda_w^\sigma}) > 0$, then $\lambda_2 = c\nu$, where $c = 1 - \sigma(S_\sigma \setminus S_{\sigma - \lambda_w^\sigma}) > 0$ and ν is the weighted equilibrium measure on $S_{\sigma - \lambda_w^\sigma}$ with external field*

$$Q_0 := \{U^{\sigma_1} + Q\}/c. \tag{2.9}$$

We now introduce the dual problem. For $0 < t < \|\sigma\|$ we define

$$\mathcal{M}_t^\sigma := \{\mu \mid \|\mu\| = t \text{ and } 0 \leq \mu \leq \sigma\} \tag{2.10}$$

and let $\lambda^\sigma(t, w)$ be the solution of

$$I_w(\lambda^\sigma(t, w)) = \inf\{I_w(\mu) \mid \mu \in \mathcal{M}_t^\sigma\}.$$

Note that $\lambda^\sigma(t, w) = t\lambda_w^{\sigma/t}$.

Theorem 2.1 can be reformulated for this case and then we obtain the following corollary.

Corollary 2.10 *With the above notations, if U^σ and $Q = \log(1/w)$ are continuous on E and E is compact, then*

$$\sigma - \lambda^\sigma(t, w) = \lambda^\sigma(s, v),$$

where $t + s = \|\sigma\|$ and $v = \exp((U^\sigma + tQ)/s)$.

Remark 2.11 The proof of this corollary follows by simply rewriting the equations (2.1) and (2.2) for $\sigma - \lambda^\sigma(t, w)$. The assumptions on U^σ, Q, and E guarantee that the weight v is admissible.

Example 2.12 Let μ be a probability measure with continuous logarithmic potential and compact support E and let w_c denote generically the weight $w_c(x) := \exp(U^{c\mu/2})$. Then if c, c' are positive real numbers with $(1/c) + (1/c') = 1$ we obtain from Corollary 2.10 the formula

$$\mu = \frac{1}{c}\lambda_{w_c}^{c\mu} + \frac{1}{c'}\lambda_{w_{c'}}^{c'\mu}$$

(Consider $\lambda_{w_c}^{c\mu} = c\lambda^{\mu}(1/c, w_c)$ and $\lambda_{w_{c'}}^{c'\mu} = c'\lambda^{\mu}(1/c', w_{c'})$ and use (2.11).)

Now suppose $1 < c < 2$ and set $\lambda_c := \lambda_{w_c}^{c\mu}$. It is easy to show that in this case the solution of the unconstrained problem on E with weight w_c is given by (see (1.6))

$$\mu_{w_c} = \frac{c\mu}{2} + \left(\quad - \frac{c}{2}\right)\mu_E, \tag{2.13}$$

where μ_E is the equilibrium measure on E (note that the continuity of U^{μ} implies $\operatorname{cap}(E) > 0$, because μ has a finite logarithmic energy). Therefore, we have $\mu_{w_c} \geq c\mu/2$ and by the saturation theorem we obtain $\lambda_c \geq c\mu/2$. Thus, if $\sigma' := (c/(2-c))\mu$, then $\lambda_c = c\mu/2 + (1 - c/2)\lambda^{\sigma'}$, where $\lambda^{\sigma'}$ is the σ'-extremal measure on E with weight $w \equiv 1$. This fact, when combined with (2.12) in the case $c > 2$, yields the general formula

$$\lambda_c = \frac{c\mu}{2} + \frac{2-c}{2}\lambda^{c\mu/|c-2|} \tag{2.14}$$

for every $c > 1, c \neq 2$, and $\lambda_c = \mu$ when $c = 2$.

The following theorem provides a practical method for determining λ_w^{σ} from knowledge of the set

$$S_{\mu_w}^{*} := \{z \mid U^{\mu_w}(z) + Q(z) \leq F_w$$

Notice that $S_{\mu_w} \subset S_{\mu_w}^{*}$ (see (1.6)).

Theorem 2.13 (Balayage Representation) *Let E be a compact set and σ be an admissible constraint for the admissible weight $w = e^{-Q}$ on E. Suppose U^{σ} and Q are continuous on E and let μ_v be the solution to the unconstrained extremal problem for probability measures on E with weight $v = \exp((U^{\sigma} + Q)/(\|\sigma\| - 1))$. Assume further that $S_{\mu_v} \subset S_{\mu_w}^{*}$ (in particular this is true if $S_{\mu_w}^{*} = S_{\sigma} = E$). Then $\tau := \sigma - \lambda_w^{\sigma} = (\|\sigma\| - 1)\mu_v$. Moreover, the σ-constrained extremal measure λ_w^{σ} has* *the representation*

$$\lambda_w^\sigma = \sigma - \widehat{\sigma} + \widehat{\mu_w} \tag{2.16}$$

where $\widehat{\sigma}$ *and* $\widehat{\mu_w}$ *are the balayage measures* (see [L; Chapter IV) *of* σ *and* μ_w *onto* S_τ.

Remark 2.14 This theorem completely settles the case when $S^*_{\mu_w} = S_\sigma$. In the case when $w \equiv 1$, the set $S^*_{\mu_w}$ is the whole complex plane and the following corollary holds.

Corollary 2.15 *Let* σ *be an admissible constraint with respect to* $w \equiv 1$ *on the compact set* E *and* U^σ *be continuous on* E. *Then the measure* $\tau := \sigma - \lambda_w^\sigma = (\|\sigma\| - 1)\mu_v$, *where* μ_v *is the solution to the unconstrained weighted energy problem on* S_σ *with weight* $v := \exp(U^{\sigma/(\|\sigma\|-1)})$. *The* σ*-constrained extremal measure* λ^σ *has the representation*

$$\lambda^\sigma = \sigma - \widehat{\sigma} + \mu_S, \tag{2.17}$$

Now we pay special attention to the support of λ_w^σ when E is a subset of the real line.

Theorem 2.16 *Let* $E \subset \mathbf{R}$ *be closed and* σ *be admissible constraint for* $w = e^{-Q}$ *on* E. *Suppose* I *is an open interval and* $I \subset E$.

(a) *If* Q *is convex on* I, *then* $S_{\lambda_w^\sigma} \cap I$ *is an interval.*

(b) *If* $E \subset [0, \infty)$ *and* $xQ'(x)$ *increases on* I, *then* $S_{\lambda_w^\sigma} \cap I$ *is an interval.*

3. Zero distribution of minimal polynomials in the discrete L_p norm

In this section we present the main application — the weighted analogue of Rakhmanov's result [R] on the zero distribution of orthogonal polynomials in a discrete variable. In particular, we obtain weak* asymptotics of "ray sequences" of Krawtchouk polynomials.

Definition 3.1 A triangular scheme of points $\{\eta_{0,N} < \eta_{1,N} < \cdots < \eta_{N,N}\}_{N=1}^\infty$ is called *admissible* for the finite interval $[a, b]$ if:

(i) The measures $\nu_N := (1/N)\sum_{i=0}^N \delta(\eta_{i,N})$ (here $\delta(z)$ is the discrete measure with mass one at z) converge weak* to a measure ν with support $S_\nu = [a, b]$ and continuous logarithmic potential on $\mathbf{C}$.

(ii) The polynomials

$$R_N(x) := \prod_{i=0}^{N}(x - \eta_{i,N} \tag{3.1}$$

associated with the scheme, satisfy the property

$$|R'_N(\eta_{k,N})|^{1/N} \quad e^{-U^{\nu}(\eta)} \quad \text{as } N \to \infty \text{ and } \eta_{k,N} \to \eta,\ k = k(N). \tag{3.2}$$

Now we provide two important examples of admissible triangular schemes.

Lemma 3.2 *The scheme* $\{\eta_{i,N}\}_{i=0,N=1}^{N,\ \infty}$ *is admissible for the finite interval* $[a, b]$ *if either*

(A) *condition* (i) *of Definition* 3.1 *holds and*

(ii′) *there exists a constant* $\rho > 0$ *such that* $\min_i |\eta_{i+1,N} - \eta_{i,N}| > \rho/N$ *for every* N;

or

(B) $\{\eta_{i,N}\}_{i=0}^{N}$ *are the zeros of a system of orthogonal polynomials with respect to a weight* $W(x)$, *where* $W > 0$ *a.e.* (*with respect to Lebesgue measure*) *on* $[a, b]$ *and has finite moments* (*for example, the Chebyshev polynomials of first or second kind*), *in which case* $d\nu = 1/\pi\sqrt{(b - x)(x - a)}dx$ *is the equilibrium distribution for* $[a, b]$.

Observe that the distance between two consecutive zeros of the N-th degree Chebyshev polynomials near the endpoints -1 and 1 is of order $1/N^2$ and so their admissibility is not handled by (A). It is an interesting question to characterize the admissibility of a triangular scheme, but we leave this for a future work.

In what follows we shall assume that $\{w_N\}$ is a sequence of positive continuous weights on $[a, b]$ that converge uniformly to a positive continuous weight w. Fix $p > 0$ and, for each $N \geq 1$ and $0 \leq n \leq N$, let $p^*_{n,N}$ be the monic polynomial of degree n that has minimal discrete L_p norm on the set $\tau_N := \{\eta_{i,N}\}_{i=0}^{N}$ with respect to the weight w^n_N, that is

$$\|w^n_N p^*_{n,N}\|_{p,\tau_N} = \left(\sum_{i=0}^{N}[w_N(\eta_{i,N})]^{pn}|p^*_{n,N}(\eta_{i,N})|^p\right)^{1/p}$$

$$\min_{p_n = x^n + \cdots} \|w^n_N p_n\|_{p,\tau_N}$$

In particular, when $p = 2$, we obtain the orthogonal polynomials with varying weight w^{2n}_N on the discrete set τ_N. In the case $p = \infty$ the norm above is the sup norm and the extremal polynomials are the extremal Chebyshev polynomials.

The next theorem deals with the asymptotics of the polynomials $p^*_{n,N}$ for pairs (n_j, N_j), where $n_j \to \infty$, $N_j \to \infty$ and $(N_j/n_j) \to C > 1$ as $j \to \infty$. Note that then $\sigma_j := (N_j/n_j)\nu_{N_j} \to C\nu =: \sigma$, where the measure σ is an admissible constraint on $[a,b]$ for the weight w and has a continuous logarithmic potential.

Theorem 3.3 *Let $\{\eta_{i,N}\}_{i=0,N=1}^{N\ \infty}$ be an admissible triangular scheme and let $\{w_N\}$ be a sequence of continuous positive weights on $[a,b]$ uniformly converging to a positive weight w. Let $\{p^*_{n,N}\}$ be the associated monic L_p-extremal polynomials and (n_j, N_j) be a sequence such that $n_j \to \infty$, and $N_j/n_j \to C > 1$ as $j \to \infty$. Then (with $n = n_j$, $N = N_j$) we have*

$$\lim_{j\to\infty} \|w_N^n p^*_{n,N}\|_{p,T_N}^{1/n} = e^{-F_w^\sigma} \tag{3.4}$$

$$\chi_{p^*_{n,N}} \xrightarrow{*} \lambda_w^\sigma, \quad \text{as } j \to \infty$$

where $\chi_{p^*_{n,N}} := (1/n)\sum_{p^*_{n,N}(z)=0} \delta(z)$ is the normalized zero counting measure of $p^*_{n,N}$, the measure $\sigma = C\nu$ (see Definition 3.1), and λ_w^σ, F_w^σ are as in Theorem 2.1.

We now investigate the asymptotic zero distribution and the n-th root of the discrete norms of the Krawtchouk polynomials (cf. [Sz, §2.82])

$$k_n(x,p,N) = \binom{N}{n}^{-1/2} (pq)^{-n/2} \sum_{s=0}^{n} (-1)^{n-s} \binom{N-x}{n-s}\binom{x}{s} p^{n-s} q^s$$

where $p, q > 0$, $p + q =$ These polynomials are orthonormal with respect to

$$\sum_{i=0}^{N} \binom{N}{i} p^i q^{N-i} \delta(i),$$

where $\delta(i)$ is the unit measure with mass point at i, $i = 0, 1, \ldots, N$. As is well known the zeros of k_n are all simple and lie in the interval $(0, N)$. They are also separated by the mass points of the measure of orthogonality. We shall convert the problem to the interval $[0,1]$ by scaling the polynomials. Let $P_n(x) = P_n(x,p,N) := A_{n,N} k_n(Nx, p, N)$, where the factor

$$A_{n,N} = \binom{N}{n}^{1/2} (pq)^{n/2} n! N^{-n}$$

is chosen so that P_n has leading coefficient one. Then the polynomials $P_n(x)$ satisfy the orthogonality relation

$$(3.7) \qquad \sum_{i=0}^{N} \binom{N}{i} p^i q^{N-i} P_n\left(\frac{i}{N}\right) P_m\left(\frac{i}{N}\right) = A_{n,N}^2 \delta_{n,m}, \quad n, m = 0, 1, \quad N.$$

Also P_n is the discrete L_2-minimal polynomial, i.e.

$$(3.8) \quad \|P_n\|_{\tau_N}^2 = \min_{p_n = x^n + \cdot} \left\{ \sum_{i=0}^{N} \binom{N}{i} p^i q^{N-i} p_n^2\left(\frac{i}{N}\right) \right\} = A_{n,N}^2, \quad n = 0, \quad , N.$$

Here the triangular scheme of points is $\tau_N = \{i/N\}_{i=0}^N$. We shall find the asymptotics of P_n when $N_j/n_j \to C > 1$ as $j \to \infty$. Consider the weights $w_N(x)$ to be piecewise linear with nodes at i/N and $w_N(i/N) = [\binom{N}{i} p^i q^{N-i}]^{1/(2n)}$. It is not difficult to show that $w_N(x) \to w(x)$ uniformly on $[0, 1]$, where

$$w(x) = \exp[-(C/2)(x \log x + (1-x)\log(1-x) + x\log(q/p) - \log q)].$$

Since the triangular scheme τ_N and the sequence of weights $\{w_N\}$ satisfy the assumptions of Theorem 3.3, we obtain that $\chi_{P_n} \to \lambda_w^\sigma$ in the weak* sense, and moreover, $\lim_{j\to\infty} \|P_n\|_{\tau_N}^{1/n} = \exp(-F_w^\sigma)$, where $d\sigma = Cdx$.

In the case when $p = q = 1/2$, known as the binary case, the constrained energy problem that arises is analyzed in Example 4.2 of the next section. Thereby we obtain the following theorem.

Theorem 3.4 *Let $k_n(x, p, N)$ be the Krawtchouk polynomials and $P_n(x) := A_n k_n(Nx, p, N)$ be the associated normalized monic polynomials. Consider the sequence $N = N_j, n = n_j$ and $N_j/n_j \to C > 1$ as $j \to \infty$. Then the normalized zero counting measures of P_n and the n-th root of the discrete norms satisfy*

$$\chi_{P_n} \xrightarrow{*} \lambda_w^\sigma \quad \text{as } j \to \infty$$

$$(3.10) \qquad \lim_{j\to\infty} \|P_n\|_{\tau_N}^{1/n} = e^{-F_w^\sigma}$$

where

$$w(x) = [x^x(1-x)^{1-x}/p^x q^{1-x}]^{-C/2}$$

and $d\sigma = Cdx$ on $[0, 1]$.

In particular, if $p = q = 1/2$, then

$$(3\ \ 1)\qquad \chi_{P_n} \xrightarrow{*} \frac{C}{2}dx + \frac{2-C}{2}\lambda^{Cdx/|C-2|} \quad \text{as } j \to \infty$$

and

$$(3.12)\qquad \lim_{j\to\infty} \|P_n\|_{T_N}^{1/n} = \frac{C^{(C-2)/2}}{2e(1-C)^{(C-1)/2}},$$

where $\lambda^{Cdx/|C-2|} =: \psi_C(x)\,dx$ is the solution to the constrained energy problem on $[0,1]$ with weight $w \equiv 1$ and constraint measure $Cdx/|C-2|$. The density ψ_C is given by

$$\psi_C(x) := \begin{cases} \dfrac{C}{|C-2|} & x \in [0,r] \cup [1-r,1] \\[2ex] \dfrac{2C}{\pi|C-2|}\arctan\left(\dfrac{\sqrt{r(1-r)}}{\sqrt{(r-x)(1-r-x)}}\right) & x \in [r,\ \ -r] \end{cases}$$

where $r = 1/2 - \sqrt{C-1}/C$. If $C = 2$, the limit measure in (3.11) is the Lebesgu measure dx and the constant on the right in (3.12) is $1/(2e)$.

Remark 3.5 Notice that the limit measure in (3.11) has support $[r, 1-r] =$ $/2 - \sqrt{C-1}/C, 1/2 + \sqrt{C-1}/C]$ when $C > 2$, and has support $[0,1]$ for $< C \leq 2$. Thus, if $x_1^{(n,N)}$ denotes the smallest zero of $k_n(x, 1/2, N)$ we have

$$\limsup_{j\to\infty} \frac{x_1^{(n,N)}}{N} \leq \frac{1}{2} - \frac{\sqrt{C-1}}{C} \qquad \text{for } C > 2,$$

and

$$\lim_{j\to\infty} \frac{x_1^{(n,N)}}{N} = 0, \qquad \text{for } 1 < C \leq 2.$$

This fact was also proved, using elementary methods, by McEliece et al. [MRRW] and plays an important role in their analysis of the rate of a binary code. Recently it was shown (see [Le]) that the lim sup in the above estimate can be replaced with the ordinary limit.

Remark 3.6 Note that we can find the n-th root limit in (3.10) directly from the expression for $A_{n,N}$, namely

$$\lim_{j\to\infty} A_{n,N}^{1/n} = \frac{C^{(C-2)/2}\sqrt{pq}}{e(C-1)^{(C-1)/2}},$$

which in the case $p = q = 1/2$ is precisely what we obtain in (3.12).

The asymptotic analysis of the normalized Krawtchouk polynomials for $p \neq 1/2$ is investigated by the authors in [DS].

4. Examples

In this section we provide some examples to illustrate the results and the ideas introduced in the previous sections. We begin with the example given by Rakhmanov in [R], which has application to the zero distribution of the Chebyshev orthogonal polynomials of a discrete variable.

Example 4.1 (Rakhmanov) Let $E = S_\sigma = [-1, 1]$, $w \equiv 1$ and $d\sigma = C\,dt/2$ on $[-1, 1]$, where $\|\sigma\| = C > 1$. Obviously σ is an admissible constraint and U^σ is continuous.

From Corollary 2.15 we have that $(\sigma - \lambda^\sigma)/(C - 1)$ is a solution to the weighted energy problem on $[-1, 1]$ with weight $\exp(U^\sigma/(C-1))$. The corresponding external field $\tilde{Q}$ is

$$\begin{aligned}\tilde{Q}(x) &= -U^\sigma(x)/(C-1) = \frac{C}{2(C-1)}\int_{-1}^{1} \log|x-y|\,dy \\ &= \frac{C}{2(C-1)}\{(1-x)\log|1-x| + (1+x)\log|1+x| - 2\}\end{aligned}$$

Since $\tilde{Q}''(x) = C/[(C-1)(1-x^2)] > 0$, the function $\tilde{Q}(x)$ is convex on $(-1, 1)$ and therefore $S_{\sigma-\lambda^\sigma}$ is an interval (cf. [ST, Theorem IV.1.10]). By the symmetry of Q, the support $S_{\sigma-\lambda^\sigma}$ must be of the form $[-r, r]$, where r maximizes the Mhaskar–Saff functional (see [MS])

$$\begin{aligned}F(r) &= \log\operatorname{cap}([-r, r]) + \frac{1}{C-1}\int_{-r}^{r} U^\sigma(x)\frac{1}{\pi\sqrt{r^2-x^2}}\,dx \\ &= \log(r/2) + \frac{C}{2(C-1)}\int_{-}^{1} U^{\mu_{[-r,r]}}(y)\,dy \\ &= \log(r/2) + \frac{C}{C-} \ \{\log 2 - \log(1+\sqrt{1-r^2}) + \sqrt{1-r^2}\}.\end{aligned}$$

Now solving

$$F'(r) = \frac{1}{r} - \frac{Cr}{(C-1)(1+\sqrt{1-r^2})} = 0,$$

we get that the maximum of $F(r)$ is achieved when $r = \sqrt{1-C^{-2}}$.

From Corollary 2.15 we obtain that with $r = r_C := \sqrt{1-C^{-2}}$

$$\lambda^\sigma = \sigma - \widehat{\sigma} + \mu_{[-r,r]},$$

where $\widehat{\sigma}$ is the balayage of σ on $[-r,r]$ and $\mu_{[-r,r]} = dx/(\pi\sqrt{r^2-x^2})$ is the equilibrium measure for the interval $[-r,r]$. For the density of $\widehat{\sigma}$ we have (see [ST, Section II.4])

$$\begin{aligned}\frac{d\widehat{\sigma}}{dx} &= \frac{1}{\pi}\left\{\int_{-1}^{-r} + \int_{r}^{1}\right\} \frac{\sqrt{t^2-r^2}}{|x-t|\sqrt{r^2-x^2}}\, d\sigma(t) + \frac{C}{2} \\ &= \frac{C}{2\pi\sqrt{r^2-x^2}} \int_r^1 \frac{\sqrt{t^2-r^2}}{t^2-x^2} 2t\, dt + \frac{C}{2} \qquad (4.4)\\ &= \frac{C\sqrt{1-r^2}}{\pi\sqrt{r^2-x^2}} - \frac{C}{\pi}\arctan\frac{\sqrt{1-r^2}}{\sqrt{r^2-x^2}} + \frac{C}{2}, \quad x \in [-r,r].\end{aligned}$$

Using (4.3) and the fact that $C\sqrt{1-r^2} =$, we obtain that $d\lambda^{\sigma} = f_C(x)dx$, where

$$(4.5) \qquad f_C(x) := \begin{cases} \dfrac{C}{2}, & x \in [-1,-r] \cup [r,1], \\[2ex] \dfrac{C}{\pi}\arctan\left(\dfrac{\sqrt{1-r^2}}{\sqrt{r^2-x^2}}\right) & x \in [-r,r]. \end{cases}$$

To compute F^{σ} we observe first that for $r = r_C$

$$(4.6) \qquad F(r) = F_v,$$

where $v = \exp(-\tilde{Q})$ (see [ST, Theorem IV.1.5(b)]). On the other hand $U^{(\sigma-\lambda^{\sigma})/(C-1)}(x) + \tilde{Q}(x) = F_v$ on $[-r,r]$, i.e. $-U^{\lambda^{\sigma}}(x)/(C-1) = F_v$ on $[-r,r]$, from which we derive

$$(4.7) \qquad F^{\sigma} = -(C-1)F_v.$$

Using (4.2), (4.6), and (4.7) we obtain (recall that for $r = r_C$, we have $C\sqrt{1-r^2} = 1$)

$$(4.8) \qquad F^{\sigma} = \log\frac{2}{r} - \frac{1}{\sqrt{1-r^2}}\log\left(\frac{1+\sqrt{1-r^2}}{r}\right) + 1 \qquad \square$$

The example below illustrates the application of our results for finding explicitly the weak* asymptotics of the Krawtchouk polynomials in the binary case ($p = q = 1/2$). The general case requires finer analysis and will be considered in a future paper [DS].

Example 4.2 Let $E = [0,1]$, the constraint measure $d\sigma = Cdx$, $C > 1$, and the weight function $w(x) = \exp[-(C/2)(x\log x + (1-x)\log(1-x) - 1)]$. Then

$$(4.9) \qquad Q(x) = (C/2)[x\log x + (1-x)\log(1-x) - 1] = -U^{\sigma/2}(x).$$

From Example 2.12 with $d\mu = dx|_{[0,1]}$ we obtain that

$$\lambda_w^\sigma = \frac{C}{2}dx + \frac{2-C}{2}\lambda^{Cdx/|2-C|} \tag{4.10}$$

and

$$F_w^\sigma = \frac{2-C}{2}F^{Cdx/|2-C|},$$

where $\lambda^{Cdx/|C-2|}$ is the solution of the constrained problem on $[0,1]$ with constraint measure $Cdx/|C-2|$ and weight $w \equiv 1$. The constant $F^{Cdx/|C-2|}$ is the corresponding extremal constant. To find this constant we observe that

$$F_{[0,1]}^{Cdx} = \log 2 + F_{[-1,1]}^{(C/2)dx}$$

where $F_{[a,b]}^\sigma$ denotes the extremal constant of the σ-constrained energy problem on $[a,b]$ with $w \equiv 1$. Now using (4.8) with $r = 2\sqrt{C-1}/C$ we compute that

$$\begin{aligned} F_w^\sigma &= \frac{2-C}{2}\left\{\log 2 + \log\frac{C}{\sqrt{C-1}} - \frac{C}{|C-2|}\log\frac{C+|C-2|}{2\sqrt{C-1}} + 1\right\} \\ &= \frac{2-C}{2}(1 + \log 2 + \log C) + \frac{C-1}{2}\log(C-1), \end{aligned} \tag{4.11}$$

for the case when $C \neq 2$. In the case when $C = 2$ formula (4.10) reduces to $\lambda_w^\sigma = dx|_{[0,1]}$ and the constant $F_w^\sigma = 0$. □

The next example deals with the case when the constraint is the well-known Ullman distribution, with normalized support on $[-1,1]$.

Example 4.3 Let $E = S_\sigma = [-1,1]$, $w \equiv 1$,

$$\frac{d\sigma}{dt} = C\frac{l}{\pi}\int_{|t|}^1 \frac{u^{l-1}}{\sqrt{u^2-t^2}}du \quad \text{for } t \in [-1,1],$$

where $l > 0$ and $\|\sigma\| = C > 1$. Then σ is an admissible constraint with continuous logarithmic potential. Hence by Corollary 2.15 the dual extremal measure $(\sigma - \lambda^\sigma)/(C-1)$ is the solution to the unconstrained weighted energy problem on $[-1,1]$ with external field

$$\begin{aligned} \frac{1}{C-1}U^\sigma(x) &= \frac{C}{C-1}\int_{-1}^1 \log|x-t|\left(\frac{l}{\pi}\int_{|t|}^1 \frac{u^{l-1}}{\sqrt{u^2-t^2}}du\right)dt \\ &= -\frac{C}{C-1}U^{\mu_w}(x), \end{aligned} \tag{4.12}$$

where μ_w is the weighted equilibrium measure for the weight $w = \exp(-\gamma_l |x|^l)$ with

$$\gamma_l := \frac{\Gamma\left(\frac{l}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{l}{2}+\frac{1}{2}\right)}$$

(cf. [ST, Theorem IV.5.1]). Therefore, by the equilibrium conditions (1.6), $-U^\sigma/(C-1)$ coincides with $[C/(C-1)]\gamma_l|x|^l$+const. everywhere on $[-1,1]$. So, on replacing x by x/r, it can be seen (cf. [ST, Theorem I.3.3]) that

$$\frac{d(\sigma-\lambda^\sigma)}{dt} = C\frac{l}{\pi}\int_{|t|}^{r} \frac{u^{l-1}}{\sqrt{u^2-t^2}}\,du \quad \text{for } t \in [-r,r], \tag{4.13}$$

where $C(1-r^l) = 1$. Consequently the density of the σ-extremal measure is given by

$$\frac{d\lambda^\sigma}{dt} = \begin{cases} C\frac{l}{\pi}\int_{|t|}^{1} \frac{u^{l-1}}{\sqrt{u^2-t^2}}\,du & r < |t| \le 1, \\ C\frac{l}{\pi}\left(\int_{|t|}^{1} \frac{u^{l-1}}{\sqrt{u^2-t^2}}\,du - \int_{|t|}^{r} \frac{u^{l-1}}{\sqrt{u^2-t^2}}\,du\right) & |t| \le r. \end{cases} \tag{4.14}$$

For the extremal potential we have

$$U^{\lambda^\sigma}(x) \begin{cases} = F_w - Cr^l \log\frac{1}{r}, & x \in [-r,r], \\ \le F_w - Cr^l \log\frac{1}{r}, & x \in [\cdot\ \ ,-r] \cup [r,1], \end{cases} \tag{4.15}$$

where $F_w = \log 2 + 1/l$ (cf. [ST, Theorem IV.5.1]).

In particular, when $l = 2$ we obtain

$$\frac{d\lambda^\sigma}{dx} = \begin{cases} \frac{2C}{\pi}\sqrt{1-x^2}, & x \in [\cdot\ \ ,-r_C] \cup [r_C,1], \\ \frac{2C}{\pi}\left(\sqrt{1-x^2} - \sqrt{r_C^2-x^2}\right), & x \in [-r_C,r_C] \end{cases}$$

where r_C satisfies $C(1-r_C^2) = 1$. The constant F^σ is given by

$$F^\sigma = \log 2 + \frac{1}{l} - \frac{C-1}{2}\log\frac{C}{C-1} \qquad \square$$

The next example deals with a case where the support of the constraining measure is the unit disk.

Example 4.4 Let $\sigma = (C/\pi)m_2$, where m_2 is 2-dimensional Lebesgue measure on the unit disk $\{z \in \mathbf{C} \mid |z| \le 1\}$ and $C > 1$, i.e.

$$d\sigma = \frac{C}{\pi}\rho\, d\rho\, d\theta, \quad 0 \le \rho \le \qquad -\pi \le \theta < \pi.$$

Clearly σ is an admissible constraint with respect to $w \equiv 1$ and its logarithmic potential is continuous in $\mathbf{C}$. Let λ^σ be the extremal measure. Then by Corollary 2.15 we get that $\tau := \sigma - \lambda^\sigma$ is a solution of the unconstrained weighted energy problem with weight $\exp(U^\sigma)$ for measures with mass $C - 1$. The external field $-U^\sigma$ can be computed as follows:

$$\begin{aligned}
-U^\sigma(z) &= -\frac{C}{\pi}\int_0^1 \left\{\int_{-\pi}^{\pi} \log\frac{1}{|z-\rho e^{i\theta}|}\, d\theta\right\}\rho\, d\rho \\
&= -2C\int_0^{|z|} \log\frac{1}{|z|}\rho\, d\rho - 2C\int_{|z|}^1 \rho\log\frac{1}{\rho}\, d\rho \\
&= -C|z|^2\log\frac{1}{|z|} + C|z|^2\log\frac{1}{|z|} - C\int_{|z|}^1 \rho\, d\rho \\
&\quad -\frac{C}{2} + \frac{C|z|^2}{2}
\end{aligned} \tag{4.18}$$

for $|z| \le 1$. Note that $-U^\sigma = C\log|z|$ for $|z| > 1$.

Now from [ST, Theorem IV.6.1], we obtain that $\sigma - \lambda^\sigma = (C/\pi)m_2$ on $|z| \le r := \sqrt{(C-1)/C}$. So $\lambda^\sigma = \sigma - \tau = \sigma_{|D_{r,1}}$, where $D_{r,1}$ is the annulus $\{z : r \le |z| \le 1\}$. The constant F^σ is given by

$$\begin{aligned}
F^\sigma &= U^{\lambda^\sigma}(r) = U^{\sigma|_{D_{r,1}}}(r) \\
&= 2C\int_r^1 \rho\log\frac{1}{\rho}\, d\rho \\
&= \frac{C}{2} - Cr^2\log\frac{1}{r} - \frac{Cr^2}{2} = \frac{1}{2} + \frac{r^2}{1-r^2}\log r.
\end{aligned}$$

□

In the next example we solve another constrained extremal problem in the presence of a weight $w \not\equiv 1$.

Example 4.5 Let $w^* = e^{-x^2}$ and $d\sigma^* = Cdx/(\pi\sqrt{1-x^2})$, $C > 1$ on $E^* = 1, 1]$. Then $d\mu_{w^*} = 2\sqrt{1-x^2}dx/\pi$ (cf. [ST, Theorem IV.5.1]). We shall transfer the problem to $[0,1]$. The new weight becomes $w = e^{-2x}$, the weighted equilibrium measure is $d\mu_w = (2/\pi)(\sqrt{(1-x)/x})dx$, and the constrained measure is $d\sigma = C/(\pi\sqrt{x(1-x)})dx$ on $E = [0,1]$ (see [ST, Theorem IV.1.10]). If $C \geq 2$, the example is trivial; therefore we shall consider the case $1 < C < 2$. By Theorem 2.13, we have that $\tau = (C-1)\mu_v$, where μ_v is the solution to the unconstrained problem on $[0,1]$ with weight $v = \exp((U^\sigma + Q)/(C-1))$. The corresponding external field Q_1 is

$$Q_1 = -(U^\sigma + Q)/(C-1) = -(C\log 4 + 2x)/(C -$$

Since $Q_1'' = 0$, the support S_τ is an interval. For $P_n \equiv 1$ we have $\|v^n P_n\|_{[0,1]} = v(1)^n$, which implies by the peaking theorem (cf. [ST, Theorem IV.1.3) that $1 \in S_\tau$. So the interval is of the form $[a,1]$. Now [ST, Theorem IV.1.11] gives us

$$\frac{1}{\pi}\int_a^1 \left(-\frac{2}{C-1}\right)\sqrt{\frac{1-x}{x-a}}\,dx = - \tag{4.19}$$

Using the substitution $x = (1+a)/2 - ((1-a)\cos\theta)/2$, we obtain

$$\frac{-a}{2\pi} \qquad + \cos\theta)\,d\theta = \frac{C-1}{}$$

from which we find that $a = 2 - C$. So $S_\tau = [2-C, 1]$ and by (2.16) we obtain

$$\lambda_w^\sigma = \sigma - \widehat{\sigma} + \widehat{\mu_w}, \tag{4.20}$$

where the balayage measures are taken onto $[2-C, 1]$.

Now we find the balayage measures in question. Since σ is the equilibrium measure on $[0,1]$ times the constant C, its balayage onto $[a,1]$ will be the equilibrium on $[a,1]$ times C, therefore

$$\frac{d\widehat{\sigma}}{dx} = \frac{C}{\pi\sqrt{(x-a)(1-x)}} \quad \text{on } [a,1]. \tag{4.21}$$

Let $\mu_w = \mu_1 + \mu_2$, where $\mu_1 = (\mu_w)|_{[0,a]}$. Then $\widehat{\mu_w} = \widehat{\mu_1} + \mu_2$. So for $x \in [a,1]$ we find

$$\frac{d\widehat{\mu_w}}{} = \frac{1}{\pi}\int_0^a \frac{\sqrt{(a-t)(1-t)}}{(x-t)\sqrt{(x-a)(1-x)}}\cdot\frac{2}{\pi}\sqrt{\frac{1-t}{t}}\,dt + \frac{2}{\pi}\sqrt{\frac{1-x}{x}}$$

$$= \frac{2}{\pi\sqrt{(x-a)(1-x)}}\cdot\frac{1}{\pi}\int_0^a \frac{1-t}{x-t}\sqrt{\frac{a-t}{t}}\,dt + \frac{2}{\pi}\sqrt{\frac{1-x}{}} \tag{4.22}$$

$$\frac{2}{\pi}\sqrt{\frac{1-x}{x-a}} + \frac{a}{\pi\sqrt{(x-a)(1-x)}}.$$

Here we used the facts that

$$\frac{1}{\pi}\int_0^a \frac{1}{x-t}\sqrt{\frac{a-t}{t}}\,dt = -\sqrt{\frac{x-a}{x}},$$

and

$$\frac{1}{\pi}\int_0^a \sqrt{\frac{a-t}{t}}\,dt = \frac{a}{2}.$$

Now substituting (4.21) and (4.22) in (4.20) (recall that $a = 2 - C$), we obtain the density of the σ-constrained extremal measure λ_w^σ as follows:

$$\frac{d\lambda_w^\sigma}{dx} = \begin{cases} \dfrac{C}{\pi\sqrt{x(1-x)}} & x \in [0,a], \\[2ex] \dfrac{C}{\pi\sqrt{x(1-x)}} - \dfrac{2}{\pi}\sqrt{\dfrac{x-a}{1-x}} & x \in [a,1]. \end{cases}$$

If we transfer back the problem to the interval $[-1,1]$, the solution $\lambda^* := \lambda_{w^*}^{\sigma^*}$ is

$$\frac{d\lambda^*}{dx} = \begin{cases} \dfrac{C}{\pi\sqrt{1-x^2}}, & x \in [-r,r], \\[2ex] \dfrac{C - 2x\sqrt{x^2-r^2}}{\pi\sqrt{1-x^2}} & x \in [-1,-r]\bigcup[r,1], \end{cases}$$

where $r = \sqrt{a} = \sqrt{2-C}$. We see that if $C \to 2^-$, then λ^* tends to μ_{w^*}, which is the solution of the unconstrained weighted problem. Indeed, if $C \geq 2$, the density of the constraint measure lies above the density of μ_{w^*}, and that is why it provides no restriction. □

5. Proofs of main results

Proof of Theorem 2.1 (a) First we note that the function $\log[|z-t|w(z)w(t)]^{-1}$ is bounded from below on $E \times E$, which is an easy consequence of the definition of an admissible weight; therefore $I_w(\mu)$ is well-defined for any $\mu \in \mathcal{M}^\sigma$ and $V_w^\sigma > -\infty$.

From the definition of E_ϵ in Remark 2.2 we have

$$E_0 = \bigcup_{n=1}^{\infty} E_{1/n}.$$

Since $\sigma(E_0) = C > 1$ and σ is regular Borel measure, there exists n_0 such that $\sigma(E_{1/n_0}) = C_0 > 1$. Then

$$\nu = \frac{\sigma|_{E_{1/n_0}}}{\sigma(E_{1/n_0})} \in \mathcal{M}^\sigma$$

and

$$I_w(\nu) = I(\nu) + \int Q\, d\nu$$

is finite, because $I(\sigma_{|E_{1/n_0}})$ and $\int Q\, d\nu$ are finite. Therefore V_w^σ is finite.

(b) The uniqueness follows exactly as in the unrestricted case (see [ST, Theorem I.1.3 (b)]). Thus we need only prove existence.

First we verify the following:

Claim For sufficiently small $\epsilon > 0$

$$V_w^\sigma = \inf\{I_w(\mu) \mid \mu \in \mathcal{M}^{\sigma_{|E_\epsilon}}$$

Since E_ϵ is compact the existence will then follow from standard compactness arguments.

Proof of the Claim First we observe that for every $K > 0$ there exists an $\epsilon > 0$ such that

$$\log\left[|z-t|w(z)w(t)\right]^{-1} > K \quad \text{if } (z,t) \notin E_\epsilon \times E_\epsilon. \tag{5.3}$$

Indeed, assume to the contrary that there exist $K > 0$ and a sequence $\{(z_n, t_n)\}_{n=}^{\infty}$ such that

$$\lim_{n\to\infty} \min\,(w(z_n), w(t_n)) = 0$$

$$\log\left[|z_n - t_n|w(z_n)w(t_n)\right]^{-1} \le K \quad \text{for all } n.$$

Now choose any convergent subsequence, say $\{(z_n, t_n)\}_{n\in\mathcal{N}_0}$,

$$(z_n, t_n) \to (z, t) \quad \text{as } n \to \infty, \quad n \in \mathcal{N}_0$$

where z or t could be ∞. If they are both finite, we get a contradiction with (5.5) because of (5.4). If one or both of z, t are infinite, then the contradiction with (5.5) follows from the admissibility of w; see Definition 1.1(c).

Next we observe that for each $K > 0$ there exists an $\epsilon > 0$ such that

$$\log\left[|z-t|w(z)\right]^{-1} > K \quad \text{if } t \in E_\epsilon \text{ and } z \in E \setminus E_\epsilon$$

Indeed, it is enough to prove that if the sequence $\{(z_n, t_n)\}$ is such that $w(z_n) \to 0$ as $n \to \infty$ and $w(t_n) \ge w(z_n)$, then

$$\log[|z_n - t_n|w(z_n)]^{-} \to \infty \quad \text{as } n \quad \infty.$$

To see this, choose a convergent subsequence $\{(z_n, t_n)\}_{n \in \mathcal{N}}$ such that $z_n \to z$ and $t_n \to t$, where z or t could be ∞. If both z and t are finite, then (5.7) follows because $w(z_n) \to 0$. If t is finite and $z = \infty$, then the result follows from the admissibility of w (see Definition 1.1(c)), because

$$|z_n - t_n| w(z_n) \leq |z_n| w(z_n) + |t_n| w(z_n)$$

and the right-hand side tends to zero. If $t = \infty$ and z is finite, then

$$|z_n - t_n| w(z_n) \leq |z_n - t_n| w(t_n)$$

and the argument is the same. And finally, if both z and t are ∞, then

$$|z_n - t_n| w(z_n) \leq |z_n| w(z_n) + |t_n| w(t_n)$$

and we again use the admissibility of w.

Now we choose n_0 as in part (a), i.e. $\sigma(E_{1/n_0}) = C_0 > 1$. The constant $K = K(n_0) > 0$ will be chosen later. It will actually depend on n_0, the set E_{1/n_0}, w, and σ. Next we choose $0 < \epsilon < 1/n_0$ so that (5.3) and (5.6) hold. In order to verify (5.2) for this choice of ϵ, it suffices to show that if $\mu \in \mathcal{M}^\sigma$ is a measure with $\mu(E_\epsilon^c) = \delta > 0$ (here $E_\epsilon^c := \mathbf{C} \setminus E_\epsilon$), then there is a measure $\eta \in \mathcal{M}^{\sigma_{|E_\epsilon}}$ with $I_w(\eta) < I_w(\mu)$.

Now let μ be such a measure. We decompose μ as $\mu = \mu_\epsilon + \mu_\delta$, where $\mu_\epsilon := \mu_{|E_\epsilon}$ and $\mu_\delta := \mu_{|E_\epsilon^c}$. Note that $\|\mu_\delta\| = \delta > 0$. Since $(\sigma - \mu_\epsilon)(E_{1/n_0}) > C_0 - 1 + \delta > 0$, the measure

$$\nu := \frac{(\sigma - \mu)_{|E_{1/n_0}}}{(\sigma - \mu)(E_{1/n_0})}$$

is a positive unit measure supported on E_{1/n_0}, such that

$$\nu \leq \frac{\sigma_{|E_{1/n_0}}}{C_0 - 1}.$$

Then the measure $\eta := \mu_\epsilon + \delta\nu \in \mathcal{M}^{\sigma_{|E_\epsilon}}$. We will show that $I_w(\eta) < I_w(\mu)$ which will complete the proof of (5.2). For this purpose we introduce the notation

$$< \mu, \nu >_w := \int \int \log [|z - t| w(z) w(t)]^{-1} \, d\mu(z) d\nu(t).$$

Consider the difference

$$
\begin{aligned}
I_w(\eta) - I_w(\mu) &=< \mu_\epsilon + \delta\nu, \mu_\epsilon + \delta\nu >_w - < \mu_\epsilon + \mu_\delta, \mu_\epsilon + \mu_\delta >_w \\
&= \delta^2 < \nu, \nu >_w + 2\delta < \mu_\epsilon, \nu >_w - < \mu_\delta, \mu_\delta >_w - 2 < \mu_\epsilon, \mu_\delta >_w \\
&\le \delta^2 < \nu, \nu >_w + 2\delta \int\int \log \frac{1}{|z-t| w(z)} d\mu_\epsilon(t) d\nu(z) \\
&\quad -\delta^2 K - 2\delta(1-\delta)K,
\end{aligned} \tag{5.10}
$$

where the $w(t)$ term in the integral is eliminated on combining the terms $2\delta < \mu_\epsilon, \nu >_w$ and $2 < \mu_\epsilon, \mu_\delta >_w$. Next we find upper bounds independent of ϵ and μ for the two integrals above. Note that

$$
I_1 :=< \nu, \nu >_w \le 2N + \int\int \log \frac{1}{|z-t|} d\nu(z) d\nu(t)
$$

where $N > 0$ is an upper bound for $Q(z) = -\log w(z)$ on E_{1/n_0}. Such a bound exists because $Q(z)$ is lower semi-continuous and E_{1/n_0} is a compact set. Now, if $d := \operatorname{diam}(E_{1/n_0})$, the inequality can be continued as

$$
\begin{aligned}
I_1 &\le 2N + \int\int \log \frac{d}{|z-t|} d\nu(z) d\nu(t) - \log(d) \\
&\le 2N - \log(d) + \frac{1}{(C_0-1)^2} \int_{E_{1/n_0}} \int_{E_{1/n_0}} \log \frac{d}{|z-t|} d\sigma(z) d\sigma(t) \\
&K_1,
\end{aligned} \tag{5.11}
$$

where K_1 depends only on the choice of n_0

For the second integral we have

$$
\begin{aligned}
I_2 &:= \int\int \log \frac{1}{|z-t| w(z)} d\mu_\epsilon(t) d\nu(z) \\
&\le (1-\delta)N + \int\int \log \frac{1}{|z-t|} d\mu_\epsilon(t) d\nu(z) \\
&\le N + \int_{A\cap E_\epsilon} \int_{E_{1/n_0}} \log \frac{1}{|z-t|} d\mu_\epsilon(t) d\nu(z),
\end{aligned}
$$

where $A := \{t : \operatorname{dist}(t, E_{1/n_0}) < 1\}$. This is so because for $t \in E_\epsilon \setminus A$ the logarithm s negative. Since $\operatorname{diam}(A) \leq d+2$, we have

$$
\begin{aligned}
I_2 &\leq N-(1-\delta)\log(d+2)+\int_{A\cap E_\epsilon}\int_{E_{1/n_0}}\log\frac{d+2}{|z-t|}\,d\mu_\epsilon(t)d\nu(z) \\
&\leq N-\log(d+2)+\frac{1}{C_0-1}\int_A\int_A\log\frac{d+2}{|z-t|}\,d\sigma(t)d\sigma(z) \qquad (5.12)\\
&=: K_2.
\end{aligned}
$$

We see that K_2 like K_1 depends only on n_0 and the set E_{1/n_0}. Therefore by choosing $K > 3\cdot\max(K_1, K_2)$ we guarantee that the upper bound in (5.10) is negative which completes the proof of the claim.

By the definition of V_w^σ, there exists a sequence of measures $\{\mu_n\} \subset \mathcal{M}^{\sigma|E_\epsilon}$ such that $I_w(\mu_n) \to V_w^\sigma$ as $n\to\infty$. Since $\operatorname{supp}(\mu_n) \subset E_\epsilon$ and E_ϵ is compact, Helly's theorem asserts that there is a subsequence, which we again denote by μ_n, that converges weak* to a probability measure μ. We claim that $\mu \in \mathcal{M}^\sigma$. Indeed, if f is positive continuous function with compact support, then

$$
\int f\,d(\sigma-\mu_n) \geq 0
$$

for all n. Letting $n\to\infty$, we obtain from the weak* convergence that $\int f\,d(\sigma-\mu) \geq 0$, which proves our claim.

Now since w is upper semi-continuous, there exists a sequence of positive continuous functions $w_m(z)$ on E_ϵ such that $w_m \searrow w$. The functions

$$
G_m(z,t) := \log\left[\qquad {}_{m}, |z-t|)w_m(z)w_m(t)\right]^{-1}
$$

converge monotone increasingly on $E_\epsilon \times E_\epsilon$ to $\log[|z-t|w(z)w(t)]^{-}$ and so, from the monotone convergence theorem, we have

$$
\begin{aligned}
I_w(\mu) &= \lim_{m\to\infty}\int\int G_m(z,t)\,d\mu(z)d\mu(t) \\
&= \lim_{m\to\infty}\left(\lim_{n\to\infty}\int\int G_m(z,t)\,d\mu_n(z)d\mu_n(t)\right) \qquad (5.13)\\
&\leq \lim_{m\to\infty}\ \liminf_{n\to\infty}\int\int\log[|z-t|w(z)w(t)]^{-1}\,d\mu_n(z)d\mu_n(t) \\
&= \lim_{n\to\infty} I_w(\mu_n) = V_w^\sigma.
\end{aligned}
$$

Thus μ is an extremal measure subject to the constraint σ and the weight w, which proves the existence assertion.

(c) Assume to the contrary that there exists an $x_0 \in S_\lambda$ and a set $K_2 \subset S_{\sigma-\lambda}$, such that $(\sigma - \lambda)(K_2) > 0$ and

$$U^\lambda(x_0) + Q(x_0) > l_1 > l_2 > U^\lambda(x) + Q(x) \quad \text{for all } x \in K_2$$

for some constants l_1 and l_2. Since $U^\lambda(x) + Q(x)$ is lower semi-continuous we obtain that there is a disk $B_\epsilon(x_0) = \{x : |x - x_0| < \epsilon\}$, such that $U^\lambda(x) + Q(x) > l_1$ for all $x \in K_1$, where $K_1 := B_\epsilon(x_0) \cap S_\lambda$. Clearly, $\lambda(K_1) > 0$ and $K_1 \cap K_2 = \emptyset$. Now we define a signed measure η to be $(-\alpha\lambda)$ on K_1, $\beta(\sigma - \lambda)$ on K_2 and zero elsewhere. If we choose $0 < \alpha < 1$ and $0 < \beta < 1$ such that $\alpha\lambda(K_1) = \beta(\sigma - \lambda)(K_2)$, we guarantee that η is a signed measure with total mass zero and $(\lambda + \delta\eta) \in \mathcal{M}^\sigma$ for all $\delta > 0$ small enough.

For the weighted energy of $(\lambda + \delta\eta)$ we have

$$\begin{aligned} I_w(\lambda + \delta\eta) - I_w(\lambda) &= 2\delta \int \left(U^\lambda(x) + Q(x)\right) d\eta(x) + \delta^2 I(\eta) \\ &< 2\delta\,[l_2\beta(\sigma - \lambda)(K_2) - l_1\alpha\lambda(K_1)] + \delta^2 I(\eta) \\ &\quad -2\delta\alpha(l_1 - l_2)\lambda(K_1) + \delta^2 I(\eta) < 0, \end{aligned} \tag{5.14}$$

for sufficiently small $\delta > 0$, which contradicts the fact that λ has minimal weighted energy in the class $\mathcal{M}^\sigma$.

(d) Let $\mu \in \mathcal{M}^\sigma$ have compact support and satisfy the assumptions (2.3) and (2.4), and let ν be any element of $\mathcal{M}^\sigma$. Then we have the representation

$$\begin{aligned} I_w(\nu) - I_w(\mu) &= 2\int [U^\mu(x) + Q(x)]\; d(\nu - \mu)(x) + I(\nu - \mu) \\ &\geq 2\int [U^\mu(x) + Q(x)]\; d(\nu - \mu)(x). \end{aligned} \tag{5.15}$$

Here we use the fact that $I(\nu - \mu) \geq 0$ (see [H] or [ST, Lemma I.1.8]). To complete the proof it suffices to show that

$$\int (U^\mu + Q)\, d(\nu - \mu) \geq 0. \tag{5.16}$$

Indeed, if (5.16) is true, then $I_w(\mu) \leq I_w(\nu)$ for all $\nu \in \mathcal{M}^\sigma$ and therefore $\mu = \lambda_w^\sigma$.

Set

$$f(z) := U^\mu(z) + Q(z) - c,$$

where c is the constant appearing in (2.3) and (2.4). Then, since $(\nu-\mu)(\mathbf{C})=0$, inequality (5.16) is equivalent to

$$\int f(z)\,d(\nu-\mu)(z)\geq 0. \tag{5.17}$$

To establish this inequality we write

$$\int f\,d(\nu-\mu)=\Big(\int_{E^-}+\int_{E^+}\Big)f\,d(\nu-\mu),$$

where

$$E^-:=\{z\in E\,|\,f(z)<0\},\quad E^+:=\{z\in E\,|\,f(z)>0\}$$

Since $\sigma\geq\nu$ and $(\sigma-\mu)(E^-)=0$, we have

$$\int_{E^-}f\,d(\nu-\mu)=\int_{E^-}f\,d(\nu-\sigma+\sigma-\mu)=\int_{E^-}f\,d(\nu-\sigma)\geq 0.$$

Also, because $\mu(E^+)=0$,

$$\int_{E^+}f\,d(\nu-\mu)=\int_{E^+}f\,d\nu\geq 0.$$

Hence (5.17) holds.

(e) Assume that there exists $\mu\in\mathcal{M}^\sigma$ with compact support such that $\mathcal{F}_\mu>\mathcal{F}_\lambda$. By (2.2) we have that $U^\lambda(x)+Q(x)\leq\mathcal{F}_\lambda$ for every $x\in S_\lambda$. Let $\lambda_1:=\lambda|_{S_{\sigma-\mu}}$, $\mu_1:=\mu|_{S_{\sigma-\mu}}$, and $\sigma_1:=\sigma|_{S_{\sigma-\mu}}$. Set $\lambda_2:=\lambda-\lambda_1$, $\mu_2:=\mu-\mu_1$, and $\sigma_2:=\sigma-\sigma_1$. Then, by Lemma 5.1 below, $\lambda_1(S_\lambda\cap S_{\sigma-\mu})=\lambda(S_{\sigma-\mu})>0$. We have also that $\mu_2=\sigma_2$; therefore $\mu_2-\lambda_2$ is a positive measure. The assumption $\mathcal{F}_\lambda<\mathcal{F}_\mu$ implies that there exists an $\epsilon>0$, so that

$$U^{\lambda_1}(x)\leq U^{\mu_1+\mu_2-\lambda_2}(x)-\epsilon,\quad (\sigma-\mu)\ \text{ a.e. on } S_\lambda\cap S_{\sigma-\mu}=S_{\lambda_1}$$

Let A be the set of points $x\in S_{\lambda_1}$ such that (5.18) is not satisfied. Then $(\sigma-\mu)(A)=0$, i.e. $\mu_1|_A=\sigma|_A$. This implies that $\lambda_1|_A\leq\mu_1|_A$. Define $\lambda_1':=\lambda_1|_{S_{\lambda_1}\setminus A}$ and $\lambda_1'':=\lambda_1|_A$. Observe that $\mu_1-\lambda_1''$ is a positive measure. If λ_1' is the zero measure, then we are done because $\mu-\lambda$ will be a positive measure with total mass zero, which leads to the contradiction $\lambda=\mu$. Thus, we assume that $\|\lambda_1'\|>0$. The inequality (5.18) can be rewritten as

$$U^{\lambda_1'}(x)\leq U^{(\mu_1-\lambda_1'')+(\mu_2-\lambda_2)}(x)-\epsilon,\quad \lambda_1'\ \text{ a.e.} \tag{5.19}$$

The norms of the two measures in (5.19) are equal and the logarithmic energy of λ_1' is finite; therefore by the principal of domination the inequality holds everywhere on $\mathbf{C}$, i.e. $U^{\lambda}(x) \leq U^{\mu}(x) - \epsilon$ for all x. Letting $x \to \infty$ we obtain the desired contradiction. This proves the first part of (e).

If $\mathcal{F}_{\mu} = \mathcal{F}_{\lambda}$, we obtain in the same way that $U^{\lambda}(x) \leq U^{\mu}(x)$ everywhere in $\mathbf{C}$. Then the function $U^{\mu-\lambda}(z)$, which is nonnegative and harmonic everywhere in $\mathbf{C}\backslash(S_{\lambda} \cup S_{\mu})$, even at ∞, achieves its minimum at ∞. Therefore it is identically zero in the unbounded component of $\mathbf{C} \setminus (S_{\lambda} \cup S_{\mu})$. The unicity theorem (see [ST, II.2.1]) then implies that $\mu = \lambda$. This completes the proof of the theorem. □

Lemma 5.1 *For any $\nu, \mu \in \mathcal{M}^{\sigma}$, the equality $\nu(S_{\sigma-\mu}) = 0$ implies $\nu = \mu$.*

Proof Assume that the measures ν and μ are in the class $\mathcal{M}^{\sigma}$, and that $\nu(S_{\sigma-\mu}) = 0$. Let A be any Borel subset of S_{σ}. Then

$$\begin{aligned}\nu(A) &= \nu(A \cap S_{\sigma-\mu}) + \nu(A \cap S^c_{\sigma-\mu}) = \nu(A \cap S^c_{\sigma-\mu}\\ &\leq \sigma(A \cap S^c_{\sigma-\mu}) = \mu(A \cap S^c_{\sigma-\mu}) \leq \mu(A).\end{aligned}$$

This shows that $\mu - \nu$ is a positive measure with total mass zero Therefore, $\mu = \nu$. □

Now we introduce the following simple, but very useful lemma.

Lemma 5.2 *If σ is positive measure whose logarithmic potential $U^{\sigma}(z)$ is continuous on $\mathbf{C}$ and λ is a positive measure such that $\lambda \leq \sigma$, then $U^{\lambda}(z)$ is continuous on $\mathbf{C}$.*

Proof We shall use the fact that a potential of a positive measure is lower semi-continuous on $\mathbf{C}$. Then, by the representation

$$U^{\lambda}(z) = U^{\sigma}(z) - U^{\sigma-\lambda}(z),$$

we have that $U^{\lambda}(z)$ is simultaneously lower and upper semi-continuous, and therefore continuous. □

Proof of Theorem 2.6 Let λ^{σ}_{w} be the extremal measure subject to the constraint σ. By the continuity of U^{σ} and Q, and Lemma 5.2, the equilibrium conditions become:

$$U^{\lambda^{\sigma}_{w}}(z) + Q(z) \geq F^{\sigma}_{w} \quad \text{for } z \in S_{\sigma-\lambda^{\sigma}_{w}}, \tag{5.20}$$

$$U^{\lambda^{\sigma}_{w}}(z) + Q(z) \leq F^{\sigma}_{w} \quad \text{for } z \in S_{\lambda^{\sigma}_{w}} \tag{5.21}$$

The equilibrium conditions for μ_w are (see (1.6)):

$$U^{\mu_w}(z) + Q(z) \geq F_w \quad \text{for q.e. } z \in S_\sigma,$$

$$U^{\mu_w}(z) + Q(z) \leq F_w \quad \text{for } z \in S_{\mu_w}.$$

Combining (5.21) and (5.22) we get

$$U^{\lambda_w^\sigma}(z) - U^{\mu_w}(z) + F_w \leq F_w^\sigma \quad \text{for q.e. } z \in S_{\lambda_w^\sigma}$$

By the principle of domination (see [L, Theorem 1.27]) the last inequality is true everywhere. On the other hand, for $z \in S_{\mu_w} \cap S_{\sigma - \lambda_w^\sigma}$, we get from (5.20) and (5.23)

$$U^{\lambda_w^\sigma}(z) - U^{\mu_w}(z) + F_w \geq F_w^\sigma$$

.e., equality holds on this set. From [ST, Theorem IV.4.5], we get

$$(\mu_w)|_{S_{\sigma-\lambda_w^\sigma} \cap S_{\mu_w}} \leq (\lambda_w^\sigma)|_{S_{\sigma-\lambda_w^\sigma} \cap S_{\mu_w}}$$

Since on $S_{\sigma-\lambda_w^\sigma} \setminus S_{\mu_w}$ the measure $\mu_w \equiv 0$, we can extend the inequality to the whole set $S_{\sigma-\lambda_w^\sigma}$ as stated in (2.6). This also proves that $S_{\mu_w} \subset S_{\lambda_w^\sigma}$.

To prove the last assertion, assume to the contrary that there is a positive set U for $(\mu_w - \sigma)$ such that $\sigma(U) > \lambda_w^\sigma(U)$. Let $V := U \cap S_{\sigma-\lambda_w^\sigma}$. Then $(\sigma - \lambda_w^\sigma)(V) > 0$ and by (2.6) we have $\mu_w(V) \leq \lambda_w^\sigma(V)$. But $\sigma(V) \leq \mu_w(V)$ (since $V \subset U$), which yields a contradiction. □

Proof of Theorem 2.8 The proof is a slight modification of the proof of Theorem 2.6. By the continuity of U^{σ_2} and Q, and Lemma 5.2, the equilibrium conditions for λ_1 and λ_2 are

$$U^{\lambda_1}(z) + Q(z) \geq F_1 \quad \text{for } z \in S_{\sigma_1-\lambda_1}, \tag{5.24}$$

$$U^{\lambda_1}(z) + Q(z) \leq F_1 \quad \text{for } z \in S_{\lambda_1} \tag{5.25}$$

and

$$U^{\lambda_2}(z) + Q(z) \geq F_2 \quad \text{for } z \in S_{\sigma_2-\lambda_2} \tag{5.26}$$

$$U^{\lambda_2}(z) + Q(z) \leq F_2 \quad \text{for } z \in S_{\lambda_2} \tag{5.27}$$

First we note that since $\lambda_1 \leq \sigma_1 \leq \sigma_2$ we have $\lambda_1 \in \mathcal{M}^{\sigma_2}$ and by Lemma 5.1 $\lambda_1(S_{\sigma_2-\lambda_2}) > 0$ (unless $\lambda_1 = \lambda_2$ in which case the conclusion of the theorem is trivial). Let $\lambda_i' := \lambda_i|_{S_{\lambda_1}\cap S_{\sigma_2-\lambda_2}}$ and $\lambda_i'' := \lambda_i|_{S_{\lambda_1}\cap S^c_{\sigma_2-\lambda_2}}$, $i = 1,2$. It is clear that $\lambda_1'' \le \lambda_2''$ and $\|\lambda_1'\| = \lambda_1(S_{\sigma_2-\lambda_2}) > 0$. From (5.25) and (5.26) we get

$$U^{\lambda_1'}(z) \le U^{\lambda_2-\lambda_1''}(z) - F_2 + F_1 \quad \text{for } z \in S_{\lambda_1} \cap S_{\sigma_2-\lambda_2}$$

Since $S_{\lambda_1'} \subset S_{\lambda_1} \cap S_{\sigma_2-\lambda_2}$, by the principle of domination we can extend the above inequality to the whole complex plane $\mathbf{C}$. (Here we used the fact that λ_1' has a finite energy.)

On the other hand, from (5.24) and (5.27), we get

$$U^{\lambda_2}(z) \le U^{\lambda_1}(z) - F_1 + F_2 \quad \text{for } z \in S_{\lambda_2} \cap S_{\sigma_1-\lambda_1}$$

$$U^{\lambda_1}(z) \le U^{\lambda_2}(z) - F_2 + F_1 \quad \text{for } z \in \mathbf{C},$$

$$U^{\lambda_1}(z) = U^{\lambda_2}(z) - F_2 + F_1 \quad \text{for } z \in S_{\lambda_2} \cap S_{\sigma_1-\lambda_1} \tag{5.29}$$

Again, by [ST, Theorem IV.4.5], we obtain

$$\lambda_2|_{S_{\sigma_1-\lambda_1}} \le \lambda_1|_{S_{\sigma_1-\lambda_1}}$$ □

Proof of Corollary 2.9 By Lemma 5.2 we have that $U^{\lambda_w^\sigma}$ is continuous on E, and since Q is also continuous on E the conditions (2.1) and (2.2) yield

$$\begin{aligned} \frac{1}{c}U^{\lambda_2}(z) + Q_0(z) &\ge \frac{1}{c}F_w^\sigma \quad \text{on } S_{\sigma-\lambda_w^\sigma} \\ &= \frac{1}{c}F_w^\sigma \quad \text{on } S_{\lambda_2}, \end{aligned} \tag{5.30}$$

that is, the probability measure λ_2/c, which has a finite logarithmic energy, is the equilibrium measure on $S_{\sigma-\lambda_w^\sigma}$ with external field Q_0 (cf. [ST, Theorem I.3.3]). □

We now proceed with the proof of the Balayage Representation Theorem.

Proof of Theorem 2.13 Let $\mu := \tau/\|\tau\|$ (recall that $\|\tau\| = \|\sigma\| - 1$). The representation

$$I_w(\lambda_w^\sigma) = \|\tau\|^2\{I(\mu) + 2\int(-U^\sigma - Q)/\|\tau\|\, d\mu\} + I(\sigma)$$

shows that

$$(5.32) \qquad I_v(\mu) = \min\{I_v(\nu) \mid \nu \in \mathcal{M}^{\sigma/\|\tau\|}\}$$

From the equilibrium conditions for μ_v we get

$$U^{\|\tau\|\mu_v}(z) - U^{\sigma}(z) - Q(z) \geq \|\tau\|F_v \quad \text{q.e. on } S_\sigma$$

$$U^{\|\tau\|\mu_v}(z) - U^{\sigma}(z) - Q(z) \leq \|\tau\|F_v \quad \text{on } S_{\mu_v}$$

Furthermore, by the equilibrium conditions for μ_w (see (1.6)) and the assumption that $S_{\mu_v} \subset S^*_{\mu_w}$, we can extend these inequalities to

$$(5.33) \qquad U^{\|\tau\|\mu_v}(z) - U^{\sigma}(z) + U^{\mu_w}(z) - F_w \geq \|\tau\|F_v \quad \text{q.e. on } S_\sigma$$

and

$$(5.34) \qquad U^{\|\tau\|\mu_v}(z) - U^{\sigma}(z) + U^{\mu_w}(z) - F_w \leq \|\tau\|F_v \quad \text{on } S_{\mu_v}$$

Applying the principle of domination to (5.33), we obtain that this inequality holds everywhere (notice that all measures involved have finite energy, and therefore q.e. implies a.e. with respect to the corresponding measure). Since (5.34) becomes equality, we deduce from [ST, Theorem IV.4.5] that

$$(\|\tau\|\mu_v + \mu_w)|_{S_{\mu_v}} \leq \sigma|_{S_{\mu_v}},$$

so that $\|\tau\|\mu_v \leq \sigma$. This shows that $\mu_v \in \mathcal{M}^{\sigma/\|\tau\|}$, which implies that $I_v(\mu_v) = I_v(\mu)$. The uniqueness of the measure μ_v now gives that $\mu_v = \mu$ and $S_{\mu_v} = S_\tau$.

To verify the representation (2.16), we take the balayage of σ and of μ_w onto S_τ. We have q.e. on S_τ that $U^{\widehat{\sigma}} = U^{\sigma} + \text{const.}$ and $U^{\widehat{\mu_w}} = U^{\mu_w} + \text{const.}$ Thus, from the equality in (5.34), we can write

$$(5.36) \qquad U^{\tau}(z) - U^{\widehat{\sigma}}(z) + U^{\widehat{\mu_w}}(z) = \text{const.} \quad \text{q.e. on } S_\tau$$

Finally, applying the principle of domination in both directions, we extend this equality to the whole complex plane, and by the unicity theorem (see [L, Theorem 1.12′]) we obtain $\widehat{\sigma} = \tau + \widehat{\mu_w}$, which yields (2.16). □

Proof of Theorem 2.16 The proof proceeds as in the unconstrained case.

(a) Assume to the contrary that $S_{\lambda_w^\sigma} \cap I$ is not an interval. Then there exist points x_1 and x_2 in the support $S_{\lambda_w^\sigma}$, such that $(x_1, x_2) \cap S_{\lambda_w^\sigma} = \emptyset$. From (2.2) we get

$$U^{\lambda_w^\sigma}(x_i) + Q(x_i) \leq F_w^\sigma, \quad i = \ ,2$$

The weighted potential $U^{\lambda_w^\sigma} + Q$ is strictly convex on (x_1,x_2); therefore $(U^{\lambda_w^\sigma} + Q)(\alpha x_1 + (1-\alpha)x_2) < F_w^\sigma$ for $0 < \alpha < 1$. The last inequality contradicts (2.1), thus proving part (a) of the theorem.

(b) We proceed as in part (a), but instead of a convexity argument, we observe that $x(U^{\lambda_w^\sigma}(x) + Q(x))'$ is increasing on (x_1,x_2). Therefore the derivative of the weighted potential $U^{\lambda_w^\sigma} + Q$ cannot change sign from positive to negative, which means that $U^{\lambda_w^\sigma} + Q$ will be either strictly monotone on (x_1,x_2) or will decrease and then increase. But both cases lead to a contradiction with (2.1). □

7

We now proceed with the proof of Theorem 3.3, which requires two lemmas. The proof of Lemma 3.2 is given at the end of this section.

Lemma 5.3 *With the assumptions of Theorem* 3.3 *we have*

$$\limsup_{j\to\infty} \|w_N^n p_{n,N}^*\|_{p,\tau_N}^{1/n} \le e^{-F_w^\sigma} \tag{5.37}$$

Proof Set $\lambda := \lambda_w^\sigma$. Since U^σ is continuous on $\mathbf{C}$ (recall from Definition 3.1 that U^ν is continuous and that $\sigma = C\nu$), so is U^λ by Lemma 5.2. Furthermore, by our assumption on w, the function $Q = \log(1/w)$ is also continuous on $[a,b]$. Now fix $\epsilon > 0$ and choose $A \supset S_{\sigma-\lambda}$ to be a union of finitely many closed intervals such that $U^\lambda(x) + Q(x) > F_w^\sigma - \epsilon$ for $x \in A$ and $\lambda(A^c) < 1$, where $A^c := [a,b]\setminus A$. Let $\lambda^{(1)} := \lambda|_A$ and $\lambda^{(2)} := \lambda|_{A^c}$. Also define $\sigma_j^{(1)} := \sigma_j|_A$ and $\sigma_j^{(2)} := \sigma_j|_{A^c}$. Then $\sigma_j^{(2)} \xrightarrow{*} \lambda^{(2)}$ as $j\to\infty$. Let $m_j := |\{\eta_{i,N} \in A^c\}|$. Since $(m_j/n) \to \lambda(A^c) < 1$, there exists a j_0, such that $m_j < n$ for every $j > j_0$. Now let p_n be a monic polynomial with zeros $\{y_k\}_{k=1}^n$ that consist of all $\eta_{i,N} \in A^c$ and $n - m_j$ zeros in A, chosen so that the weak* convergence $\chi_{p_n} \to \lambda$ as $n\to\infty$ takes place (it is enough to discretize λ on A). Then with $Q_N(x) := \log(1/w_N(x))$ we have for $j > j_0$

$$\begin{aligned}\|w_N^n p_{n,N}^*\|_{p,\tau_N}^{1/n} &\le \|w_N^n p_n\|_{p,\tau_N}^{1/n} = \left(\textstyle\sum_{\eta_{k,N}\in A} w_N^{pn}(\eta_{k,N})|p_n(\eta_{k,N})|^p\right)^{1/(pn)}\\ &\le N^{1/(pn)} \exp\{-\min_{\eta_{k,N}\in A}[U^{\chi_{p_n}}(\eta_{k,N}) + Q_N(\eta_{k,N})]\}\\ &= N^{1/(pn)} \exp\{-[U^{\chi_{p_n}}(\eta_N^*) + Q_N(\eta_N^*)]\}\end{aligned} \tag{5.38}$$

for some $\eta_N^* \in \tau_N \cap A$. Let $\eta \in S_{\sigma-\lambda}$ be any limit point of η_N^*, say $\eta_{N_l}^* \to \eta$ as $l \quad \infty$. Then

$$\liminf_{l\to\infty}[U^{\chi_{p_n}}(\eta_{N_l}^*) + Q_N(\eta_{N_l}^*)] \ge U^\lambda(\eta) + Q(\eta) \ge F_w^\sigma - \epsilon. \tag{5.39}$$

Here we used the principal of descent (see [L, Theorem 1.3]) and the uniform convergence of Q_N. Now from (5.38) and (5.39) we see that

$$\limsup_{j\to\infty} \|w_N^n p_{n,N}^*\|_{p,\tau_N}^{1/n} \le e^{-F_w^\sigma + \epsilon}$$

Letting $\epsilon \to 0$ we obtain the assertion of the lemma

Remark 5.4 We can modify the proof of this lemma, so that all the zeros $\{y_k\}_{k=1}^n$ of p_n lie in τ_N (see [D] for details). This shows that the restricted L_p-minimal polynomials (with zeros in τ_N) will also satisfy (5.37). Since any weak* limit of the normalized zero counting measures of these polynomials will be in the class $\mathcal{M}^\sigma$, we can use the argument in the proof of Theorem 3.3 to conclude that (3.4) is also true for these restricted polynomials.

Lemma 5.5 *With the assumptions of Theorem* 3.3, *let* $P_n(x) = x^n + \cdots$ *be a polynomial of degree n with real simple zeros* $\{x_{k,n}\}_{k=1}^n$ *separated by the points of* τ_N. *Assume that the normalized zero counting measures* χ_{P_n} *associated with* P_n *have weak* limit* μ. *Then*

$$\liminf_{j\to\infty} \|w_N^n P_n\|_{p,\tau_N}^{1/n} \ge e^{-\mathcal{F}_\mu} \tag{5.40}$$

where $\mathcal{F}_\mu$ *is the constant defined in Theorem* 2.1(e) *for the limiting weight w*

Proof First we note that by the separation assumption on the zeros of P_n, the limit measure $\mu \in \mathcal{M}^\sigma$. By Lemma 5.2 and the continuity of U^σ we obtain that U^μ is continuous. The weight w is positive and continuous, so $Q = \log(1/w)$ is also continuous. Let $x_0 \in S_{\sigma-\mu}$ be such that $U^\mu(x_0) + Q(x_0) = \mathcal{F}_\mu$. Let $\epsilon > 0$ and set $\triangle_\epsilon := (x_0 - \epsilon, x_0 + \epsilon)$. Furthermore, set $\sigma^{(2)} := \sigma|_{\triangle_\epsilon^c}$, $\mu^{(2)} := \mu|_{\triangle_\epsilon^c}$, and $\chi_{P_n}^{(2)} := \chi_{P_n}|_{\triangle_\epsilon^c}$. Now we choose $0 < \delta < \epsilon$ such that the following are true:

(i) if $j > j_0$, then $|Q_N(y) - Q(y)| < \epsilon$ for every $|y - x_0| < \delta, y \in [a,b]$;

(ii) if $y \in [a,b]$ and $|y - x_0| < \delta$, then $|Q(y) - Q(x_0)| < \epsilon$; and

(iii) if $y \in [a,b]$ and $|y - x_0| < \delta$, then $|U^{\mu^{(2)}}(y) - U^{\mu^{(2)}}(x_0)| < \epsilon$ and $|U^{\sigma^{(1)}}(y) - U^{\sigma^{(1)}}(x_0)| < \epsilon$, where $\sigma^{(1)} := \sigma - \sigma^{(2)}$.

We can guarantee (i) because of the uniform convergence of $Q_N = \log(1/w_N)$; (ii) is satisfied from the continuity of Q; and (iii) follows from the continuity of $U^{\mu^{(2)}}$, $U^{\sigma^{(1)}}$ obtained by Lemma 5.2.

Let $\triangle_\delta := (x_0 - \delta, x_0 + \delta)$, and write

$$P_{n,1}(x) := \prod_{x_{k,n}\in\triangle_\epsilon} (x - x_{k,n}), \quad P_{n,2}(x) := \prod_{x_{k,n}\in\triangle_\epsilon^c} (x - x_{k,n}).$$

Then $P_n(x) = P_{n,1}(x)P_{n,2}(x)$.

Since $x_0 \in S_{\sigma-\mu}$ we have that $q := (\sigma - \mu)(\Delta_\delta) > 0$. Let l_n be the number of zeros of P_n in Δ_δ and m_n be the number of points in $\tau_N \cap \Delta_\delta$. Then $\lim_{j\to\infty}(m_n - l_n)/n = q > 0$. Since the intervals $((\eta_{k,N}+\eta_{k-1,N})/2, (\eta_{k,N}+\eta_{k+1,N})/2)$ around the mass points $\eta_{k,N}$ are disjoint, there exists, for j sufficiently large, at least one interval containing no zeros of P_n, whose corresponding mass point is in Δ_δ. Denote this point by η_N^* and let its adjacent mass points be η_1 and η_2. Then

$$\liminf_{j\to\infty} \|w_N^n P_n\|_{p,\tau_N}^{1/n} \geq \liminf_{j\to\infty} |w_N^n(\eta_N^*)P_n(\eta_N^*)|^{1/n}$$

Now we obtain (assuming $0 < \epsilon <$ '2)

$$|P_{n,1}(\eta_N^*)|^{1/n} \quad \left(\frac{|\eta_N^* - \eta_1| \cdot |\eta_N^* - \eta_2|}{4}\right)^{1/n} \quad \left(\prod_{\eta_N^* \neq \eta_{i,N} \in \Delta_\epsilon} |\eta_N^* - \eta_{i,N}|\right)$$

$$\geq (1/4)^{1/n}\Big(\prod\nolimits_{\eta_N^* \neq \eta_{i,N} \in \Delta_\epsilon} |\eta_N^* - \eta_{i,N}| \quad {}^{2/n}$$

Letting $j \to \infty$, passing to subsequences if necessary (so that $\eta_N^* \to \eta$), we derive using condition (ii) of Definition 3.1 that

$$\text{(5.43)} \qquad \begin{aligned} \liminf\nolimits_{j\to\infty} |P_{n,1}(\eta_N^*)|^{1/n} &\geq \exp\{-2U^{\sigma^{(1)}}(\eta)\} \\ &\geq \exp\{-2U^{\sigma^{(1)}}(x_0) - 4\epsilon\} \end{aligned}$$

Here we also used the uniform convergence of $U^{\beta_{N,2}}$ to $U^{\sigma^{(2)}}$ on Δ_δ, where

$$\beta_{N,2} := \frac{1}{n} \sum_{\eta_{i,N} \in \Delta_\epsilon^c} \delta(\eta_{i,N})$$

Next we find a lower bound for

$$\liminf_{j\to\infty} |w_N^n(\eta_N^*)P_{n,2}(\eta_N^*)|^{1/n}.$$

We have that $\chi_{P_n}^{(2)} \to \mu^{(2)}$ in the weak* topology, and moreover $U^{\chi_{P_n}^{(2)}}$ converges uniformly on Δ_δ to $U^{\mu^{(2)}}$. Therefore, there exists j_1 such that for $j > j_1$ we have $|U^{\chi_{P_n}^{(2)}}(y) - U^{\mu^{(2)}}(y)| < \epsilon$ for every $y \in \Delta_\epsilon$. But then for $j > \max(j_0, j_1)$, using (i), (ii), and (iii) we deduce that

$$\text{(5.44)} \qquad \liminf_{j\to\infty} |w_N^n(\eta_N^*)P_{n,2}(\eta_N^*)|^{1/n} \geq e^{-Q(x_0) - U^{\mu^{(2)}}(x_0) - 4\epsilon}$$

Finally, combining (5.41), (5.43) and (5.44), letting $\epsilon \to 0$ and using the fact that $U^{\mu^{(2)}}(x_0) \to U^{\mu}(x_0)$ and $U^{\sigma^{(1)}}(x_0) \to 0$ (this follows by the dominated convergence theorem), we obtain

$$\liminf_{j\to\infty} \|w_N^n P_n\|_{p,\tau_N}^{1/n} \geq e^{-\mathcal{F}_\mu}$$

which proves the lemma

Proof of Theorem 3.3 To prove the statement of the theorem we observe first that the zeros of $p_{N,n}^*$ are separated by the points of τ_N. Let μ be a weak* limit of the normalized counting zero measures χ_n of $p_{n,N}^*$. Then combining Lemmas 5.3 and 5.4 we get

$$e^{-\mathcal{F}_\mu} \leq \liminf_{n\to\infty} \|w_N^n p_{n,N}^*\|_{p,\tau_N}^{1/n} \leq \limsup_{n\to\infty} \|w_N^n p_{n,N}^*\|_{p,\tau_N}^{1/n} \leq e^{-F^\sigma} \tag{5.45}$$

Since $\mu \in \mathcal{M}^\sigma$ and $\mathcal{F}_\mu \geq F_w^\sigma$, by Theorem 2.1(e) we obtain $\mu = \lambda_w^\sigma$. The measure μ was any weak* limit, so we deduce (3.5). The equality in (5.45) implies (3.4). This completes the proof of the theorem. □

We now prove Lemma 3.2, which provides two sufficient conditions for admissibility of a triangular scheme of points.

Proof of Lemma 3.2 We assume first that the triangular scheme satisfies (i) and (ii ′) and let $\eta_{k,N} \to \eta$. We want to prove that this implies property (3.2) for the associated polynomials R_N given in (3.1). Let $s_{i,N}, i = 1, \dots, N$ be the N zeros of the polynomial R_N' (by Rolle's theorem between every two zeros of R_N there is a zero of R_N'). Since the $\{s_{i,N}\}$'s are separated by the $\{\eta_{i,N}\}$'s, the normalized zero counting measures $\chi_{R_N'}$ of R_N' converge to ν in the weak* topology. Then by the principle of descent (cf. [L, Theorem 1.3]), we obtain

$$\begin{aligned}
\limsup_{N\to\infty} |R_N'(\eta_{k,N})|^{1/N} &= \limsup_{N\to\infty} |R_N'(\eta_{k,N})/(N+1)|^{1/N} \\
&= \exp(-\liminf_{N\to\infty} U^{\chi_{R_N'}}(\eta_{k,N})) \\
&\leq \exp(-U^\nu(\eta))
\end{aligned} \tag{5.46}$$

We now derive the lower bound. Let $\epsilon > 0$ and choose $\delta > 0$, such that $0 < \delta < \epsilon$. Let $\Delta_\epsilon := [\eta - \epsilon, \eta + \epsilon] \cap [a, b]$ and similarly define Δ_δ. Let $l_N := |\{s_{i,N} \in \Delta_\epsilon\}|$. Then $l_N/N \to \nu(\Delta_\epsilon)$. We now write

$$\begin{aligned}
|R_N'(\eta_{k,N})| &= \prod_{\eta_{i,N}\in\Delta_\epsilon, i\neq k} |\eta_{k,N} - \eta_{i,N}| \times \prod_{\eta_{i,N}\in\Delta_\epsilon^c} |\eta_{k,N} - \eta_{i,N}| \\
&=: |R_{N,1}(\eta_{k,N})| \times |R_{N,2}(\eta_{k,N})|.
\end{aligned}$$

From the condition (ii we can easily obtain

$$|R_{N,1}(\eta_{k,N})|^{1/N} \geq \left(\frac{\rho}{N}\frac{2\rho}{N}\cdots\frac{l_N\rho}{N}\right)^{1/N} = \frac{\rho^{l_N/N}(l_N!)^{1/N}}{N^{l_N/N}}$$
$$\left(\frac{\rho}{e}\right)^{l_N/N}\left(\frac{l_N}{N}\right)^{l_N/N} l_N^{1/2N} O(1)^{1/N}$$

from which we derive

$$\liminf_{N\to\infty} |R_{N,1}(\eta_{k,N})|^{1/N} \geq \left(\frac{\rho}{e}\right)^{\nu(\Delta_\epsilon)} \nu(\Delta_\epsilon)^{\nu(\Delta_\epsilon)} =: K_\epsilon \tag{5.47}$$

Observe that $K_\epsilon \to 1$ as $\epsilon \to 0$, because ν has no point masses.

To obtain lower estimate for the other polynomials, we define $\nu^{(2)} := \nu|_{\Delta_\epsilon^c}$ and

$$\beta_{N,2} := \frac{1}{N} \sum_{\eta_{i,N}\in\Delta_\epsilon^c} \delta(\eta_{i,N})$$

Note that $\beta_{N,2} \xrightarrow{*} \nu^{(2)}$ and the corresponding potentials $U^{\beta_{N,2}}$ uniformly converge to $U^{\nu^{(2)}}$ on Δ_δ. Therefore,

$$\lim_{N\to\infty} |R_{N,2}(\eta_{k,N})|^{1/N} = \exp(-U^{\nu^{(2)}}(\eta)). \tag{5.48}$$

Combining (5.47) and (5.48), we get

$$\liminf_{N\to\infty} |R'_N(\eta_{k,N})|^{1/N} \geq K_\epsilon \exp(-U^{\nu^{(2)}}(\eta))$$

We now let $\epsilon \to 0$ and use the dominated convergence theorem to conclude that $U^{\nu^{(2)}}(\eta) \to U^\nu(\eta)$. Since $K_\epsilon \to 1$, we derive

$$\liminf_{N\to\infty} |R'_N(\eta_{k,N})|^{1/N} \geq \exp(-U^\nu(\eta)). \tag{5.49}$$

This, together with (5.46) proves property (3.2) for the triangular scheme in question.

We now prove the second assertion of the lemma. Let $\{R_N\}$ be monic orthogonal polynomials with respect to a weight $W(x)$ on $[a,b]$, where $W > 0$ a.e. on $[a,b]$ and has finite moments. Then we have that (see [ET])

$$\chi_{R_N} \xrightarrow{*} \nu \quad \text{and} \quad \|R_N\|_2^{1/N} \to \mathrm{cap}([a,b]) = (b-a)/4. \tag{5.50}$$

where $\nu = \mu_{[a,b]}$ is the equilibrium measure on $[a, b]$. The same argument used to deduce (5.46) yields

$$\limsup_{N\to\infty} |R'_N(\eta_{k,N}|^{1/N} \le \exp\left(-U^{\nu}(\eta)\right), \tag{5.51}$$

whenever $\eta_{k,N} \to \eta$, $k = k(N)$.

We now derive the lower estimate. For this purpose recall that the Christoffel numbers $\lambda_{k,N}$ are positive and their sum $\lambda_{0,N} + \lambda_{1,N} + \cdots + \lambda_{N,N} = \int_a^b W(x)dx =: K > 0$. We also have the representation (see [Sz], Theorem 3.4.2)

$$\lambda_{k,N} = \int_a^b \left(\frac{R_N(x)}{R'_N(\eta_{k,N})(x - \eta_{k,N})}\right)^2 W(x)dx$$

from which we find

$$\begin{aligned} |R'_N(\eta_{k,N})|^{1/N} &= \left\{\frac{1}{\lambda_{k,N}} \int_a^b \left(\frac{R_N(x)}{x - \eta_{k,N}}\right)^2 W(x)dx\right\}^{1/(2N)} \\ &\ge \left(\frac{1}{K(b-a)^2}\right)^{1/(2N)} \|R_N\|_2^{1/N} \end{aligned}$$

From (5.50) we thus obtain

$$\liminf_{N\to\infty} |R'_N(\eta_{k,N})|^{1/N} \ge \frac{b-a}{4} = \exp\left(-U^{\nu}(\eta)\right) \quad \text{for all } \eta \in [a, b].$$

By (5.51) and (5.52) we deduce the assertion for (B) of the lemma. □

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