Journal of Approximation Theory **92**, 331–338 (1998)
Article No. AT973127

# Rational Approximation with Varying Weights, II

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*Communicated by Vilmos Totik*

Received August 19, 1996; accepted in revised form January 28, 1997

We consider two problems concerning uniform approximation by weighted rational functions $\{w^n r_n\}_{n=1}^{\infty}$, where $r_n = p_n/q_n$ has real coefficients, $\deg p_n \leqslant [\alpha n]$ and $\deg q_n \leqslant [\beta n]$, for given $\alpha > 0$ and $\beta \geqslant 0$. For $w(x) := e^x$ we show that on any interval $[0, a]$ with $a \in (0, \hat{a}(\alpha, \beta))$, every real-valued function $f \in C([0, a])$ is the uniform limit of some sequence $\{w^n r_n\}$. An implicit formula for $\hat{a}(\alpha, \beta)$ was given in the first part of this series of papers; in particular, $\hat{a}(1, 1) = 2\pi$. For $w(x) := x^\theta$ with $\theta > 1$ we show that uniform approximation of real-valued $f \in C([b, 1])$ on $[b, 1]$ by weighted rationals $w^n r_n$ is possible for any $b \in (\hat{b}(\theta; \alpha, \beta), 1)$, where $\hat{b}(\theta; \alpha, \beta)$ was also found in Part I; in particular, $\hat{b}(\theta; 1, 1) = \tan^4((\pi/4)((\theta - 1)/\theta))$. Both of the mentioned results are sharp in the sense that approximation is no longer possible if $\hat{a}$ is replaced by $\hat{a} + \varepsilon$ or $\hat{b}$ is replaced by $\hat{b} - \varepsilon$ with $\varepsilon > 0$. We use potential theoretic methods to prove our theorems. © 1998 Academic Press

* The research of these authors was supported, in part, by the National Science Foundation under Grant DMS 9501130.

† The research done by this author was in partial fulfillment of the Ph.D. Requirements at the University of South Florida.



0021-9045/98 $25.00

# 1. INTRODUCTION AND MAIN RESULTS

## 1.1. *Exponential Weight*

We first consider the approximation problem for the weight $w(x)=e^x$. Let $\mathcal{P}_m$ be the space of algebraic polynomials of degree at most $m$ having complex coefficients. Let $\alpha, \beta \geqslant 0$ with $\alpha+\beta>0$. Assume that for some $a>0$ there are $p_n \in \mathcal{P}_{[\alpha n]}$ and $q_n \in \mathcal{P}_{[\beta n]}$, such that $e^{nx}p_n(x)/q_n(x) \to 1$, as $n \to \infty$, uniformly on $[0, a]$. Here and throughout the paper $[\cdot]$ denotes the greatest integer function. It was shown in [1, Theorem 3] that $a \leqslant \hat{a} = \hat{a}(\alpha, \beta)$, where

$$\hat{a} = 2\pi\alpha, \qquad \text{if} \quad \alpha=\beta, \tag{1.1}$$

$$\hat{a} = 2(\alpha-\beta)/(1-2\hat{y}), \qquad \text{if} \quad \alpha \neq \beta, \tag{1.2}$$

and $\hat{y}=\hat{y}(\alpha,\beta)$ is the root in $[0, 1]$ of the equation

$$(y(1-y))^{1/2}/(1-2y) - \sin^{-1}\sqrt{y} = (\pi/2)(\beta/(\alpha-\beta)). \tag{1.3}$$

Thus uniform approximation of the constant function 1 by $\{e^{nx}p_n(x)/q_n(x)\}$ is not possible on any interval $[0, \hat{a}+\varepsilon]$, $\varepsilon>0$. The purpose of this paper is to prove, as claimed in [1], that such weighted rational approximation of the constant function 1 and, moreover, of any continuous function on the interval $[0, a]$ with $a \in (0, \hat{a})$ is indeed possible.

THEOREM 1.1. *Let $\alpha>0$ and $\beta \geqslant 0$. For $a \in (0, \hat{a}(\alpha,\beta))$, where $\hat{a}$ is defined in* (1.1)–(1.3), *every real-valued function $f \in C([0, a])$ is the uniform limit on $[0, a]$ of a sequence of weighted real rational functions of the form $\{e^{nx}p_n(x)/q_n(x)\}$ with $p_n \in \mathcal{P}_{[\alpha n]}$ and $q_n \in \mathcal{P}_{[\beta n]}$.*

*If $\alpha=0$, $\beta>0$, and $a \in (0, \hat{a}(0,\beta))$, then $f \in C([0, a])$ is uniformly approximable on $[0, a]$ by weighted real rationals $\{e^{nx}/q_n(x)\}$ with $q_n \in \mathcal{P}_{[\beta n]}$, if and only if $f$ does not change sign on $(0, a)$.*

In the case $\alpha=\beta=1$ we have $\hat{a}(\alpha,\beta)=2\pi$, and so Theorem 1.1 and [1, Theorem 3] immediately yield the following.

COROLLARY 1.2. *Let $a^*$ be the supremum of all numbers $a$ such that every real-valued $f \in C([0, a])$ is the uniform limit on $[0, a]$ of weighted real rationals of the form $\{e^{nx}p_n(x)/q_n(x)\}$, where $p_n, q_n \in \mathcal{P}_n$. Then $a^*=2\pi$.*

*Remark* A. It was shown by P. C. Simeonov and V. Totik that in the case $\alpha=\beta=1$, approximation of the constant function 1 is not possible on the whole interval $[0, \hat{a}]$, and A. B. J. Kuijlaars found a class of functions that are approximable on $[0, \hat{a}]$. These results will appear in another paper.

## 1.2. *Incomplete Rational Functions*

We next consider the approximation problem for the weight $w(x)=x^{\theta}$, where $\theta>1$. Let $\alpha\geqslant 0$, $\beta\geqslant 0$, and $\alpha+\beta>0$. Assume that for some $b\in(0,1)$ there are $p_n\in\mathcal{P}_{[\alpha n]}$ and $q_n\in\mathcal{P}_{[\beta n]}$, such that $x^{n\theta}p_n(x)/q_n(x)\to 1$, as $n\to\infty$, uniformly on $[b,1]$. It was shown in [1, Theorem 4] that $b\geqslant\hat{b}=\hat{b}(\theta;\alpha,\beta)$, where

$$\hat{b}=0, \qquad \text{if} \quad \beta/\theta>1, \tag{1.4}$$

$$\hat{b}=\text{root of} \qquad h(b)=1-\beta/\theta, \qquad \text{if} \quad \beta/\theta\leqslant 1, \tag{1.5}$$

where

$$h(b)=\frac{1}{\pi}\int_0^b \frac{\sqrt{(\xi-\sqrt{t})(1-\xi\sqrt{t})}}{t^{3/4}(1-t)}\,dt; \qquad \xi:=1+\frac{\alpha}{\theta}-\frac{\beta}{\theta}. \tag{1.6}$$

Thus uniform approximation of the constant function 1 by $\{x^{n\theta}p_n(x)/q_n(x)\}$ is not possible on any interval $[\hat{b}-\varepsilon,1]$ with $\varepsilon>0$. Here we prove, as claimed in [1], that for any $f\in C([b,1])$ such weighted rational approximation on $[b,1]$ is possible whenever $b\in(\hat{b},1)$.

THEOREM 1.3. *Let $\alpha>0$, $\beta\geqslant 0$, and $\theta>1$. Let $b\in(\hat{b}(\theta;\alpha,\beta),1)$, where $\hat{b}$ is defined in* (1.4)–(1.6). *Then every real-valued function $f\in C([b,1])$ is the uniform limit on $[b,1]$ of a sequence of weighted real rational functions of the form $\{x^{n\theta}p_n(x)/q_n(x)\}$ with $p_n\in\mathcal{P}_{[\alpha n]}$ and $q_n\in\mathcal{P}_{[\beta n]}$.*
*If $\alpha=0$, $\beta>0$, and $b\in(\hat{b}(\theta;0,\beta),1)$, then $f\in C([b,1])$ is uniformly approximable on $[b,1]$ by weighted real rationals $\{x^{n\theta}/q_n(x)\}$, $q_n\in\mathcal{P}_{[\beta n]}$, if and only if $f$ does not change sign on $(b,1)$.*

In the case $\alpha=\beta=1$ we have $\hat{b}(\theta;1,1)=\tan^4((\pi/4)((\theta-1)/\theta))$ and so by Theorem 1.3 and [1, Theorem 4] we obtain the following.

COROLLARY 1.4. *Let $b^*$ be the infimum of all numbers $b\in(0,1)$ such that every real-valued function $f\in C([b,1])$ is the uniform limit on $[b,1]$ of a sequence of weighted real rational functions $\{x^{n\theta}p_n(x)/q_n(x)\}$, with $\theta>1$ and $p_n,q_n\in\mathcal{P}_n$. Then $b^*=\tan^4((\pi/4)((\theta-1)/\theta))$.*

*Remark* B. Concerning approximation by incomplete rationals of the form $x^{n\theta}p_n(x)/q_n(x)$, $p_n\in\mathcal{P}_{[\alpha n]}$, $q_n\in\mathcal{P}_{[\beta n]}$ we do not know the class of functions for which uniform convergence on $[\hat{b},1]$ is possible. For the special case of incomplete polynomial approximation ($\alpha=1$, $\beta=0$) we have $\hat{b}=\hat{b}(\theta;1,0)=(\theta/(1+\theta))^2$ and it has been shown by A. B. J. Kuijlaars (see [2, Theorem 1.2]) that a necessary and sufficient condition for $f\in C([\hat{b},1])$ to be approximable is that $f(\hat{b})=0$.

### 1.3. *The Main Approximation Theorem*

Theorems 1.1 and 1.3 are consequences of the following result which concerns logarithmic potentials. For a finite Borel measure $\mu$ with compact support we denote by $V(z, \mu)$ its logarithmic potential

$$V(z, \mu) := \int \log \frac{1}{|z-t|} \, d\mu(t).$$

For a positive Borel measure $\mu$, the total mass of $\mu$ is denoted by $\|\mu\|$.

THEOREM 1.5. *Let $[a, b] \subset \mathbf{R}$ be a finite interval and $w: [a, b] \to [0, \infty)$ be a weight such that*

$$w(u) = \exp(V(u, \mu^+) - V(u, \mu^-) + F), \tag{1.7}$$

*where $F$ is a constant and $\mu^{\pm} = s^{\pm}(t)\, dt$ are absolutely continuous measures whose densities $s^{\pm}$ are nonnegative and continuous on $(a, b)$ and satisfy at each endpoint $c \in \{a, b\}$, $s^+(t)\, |t-c|^{1/2} \to l_c^+ (<\infty)$ as $t \to c$, $t \in (a, b)$, and the same holds for $s^-$. Then for each $\alpha > \|\mu^+\|$ and $\beta > \|\mu^-\|$, every real-valued function $f \in C([a, b])$ is uniformly approximable on $[a, b]$ by weighted real rationals of the form $w^n p_n / q_n$, with $p_n \in \mathscr{P}_{[\alpha n]}$ and $q_n \in \mathscr{P}_{[\beta n]}$.*

*If $\|\mu^-\| = 0$, then the last statement is also true for $\beta = 0$.*

*If $\|\mu^+\| = 0$, then $f \in C([a, b])$ is uniformly approximable by weighted real rationals of the form $w^n / q_n$ with $q_n \in \mathscr{P}_{[\beta n]}$, $\beta > \|\mu^-\|$, if and only if $f$ does not change sign on $(a, b)$. The latter condition on $f$ can be removed if the $q_n \in \mathscr{P}_{[\beta n]}$ are allowed to have complex coefficients.*

## 2. PROOFS OF THE THEOREMS

First we will prove Theorem 1.5 and then use it to establish Theorems 1.1 and 1.3.

### 2.1. *Proof of Theorem* 1.5

For the proof we need the following lemma which easily follows from results of Totik [4] and Kuijlaars [2] (see also [3, Chapter VI]) regarding weighted polynomial approximation with varying weights.

LEMMA 2.1. *Suppose $w(x) = C \exp(V(x, \mu))$, $x \in [a, b]$, where $\mu$ is a positive measure on $[a, b]$ of total mass $\|\mu\| > 0$ and has the form*

$$d\mu(t) = \frac{v(t)}{\sqrt{(t-a)(b-t)}} \, dt, \qquad t \in [a, b],$$

*where $v$ is positive and continuous on $[a, b]$. Then every real-valued $f \in C([a, b])$ is uniformly approximable on $[a, b]$ by weighted polynomials $w^n p_n$ with $p_n \in \mathcal{P}_{[\|\mu\| n]}$.*

*Proof.* With the transformation used by A. B. J. Kuijlaars (see [2, p. 298]) we turn each endpoint into an interior point. Then we apply Theorem 4.2 from [4] to the transformed weight for which the extremal measure has continuous density at the transformed point and use the inverse transformation to complete the proof of the lemma. ∎

*Proof of Theorem* 1.5. First let $\alpha > \|\mu^+\|$, $\beta > \|\mu^-\|$, and $f \in C([a, b])$ be real-valued. We assume, without loss of generality, that $[a, b] = [0, 1]$. Let

$$v(t)\, dt := \frac{dt}{\pi \sqrt{t(1-t)}}, \qquad t \in [0, 1]$$

denote the equilibrium distribution for the interval $[0, 1]$ and choose a number $\gamma \in (0, \min(\alpha - \|\mu^+\|, \beta - \|\mu^-\|))$. We consider the measures $\nu^\pm$ defined by $d\nu^\pm(t) = v^\pm(t)\, dt$, $t \in [0, 1]$, where $v^\pm(t) := s^\pm(t) + \gamma v(t)$. Then $v^\pm(t) > 0$ on $(0, 1)$ and, at each endpoint $c \in \{0, 1\}$, we have $v^\pm(t)\, |t - c|^{1/2} \to l_c^\pm + \gamma/\pi > 0$ as $t \to c$, $t \in (0, 1)$. Furthermore $\|\nu^+\| < \alpha$ and $\|\nu^-\| < \beta$ by the choice of $\gamma$. Next we define the weights

$$w^\pm(u) := e^{V(u,\, \nu^\pm/\|\nu^\pm\|)}, \qquad u \in [0, 1].$$

By Lemma 2.1 there exist $\tilde{p}_n \in \mathcal{P}_n$ and $\tilde{q}_n \in \mathcal{P}_n$ such that

$$w^+(u)^n\, \tilde{p}_n(u) \to f(u) \qquad \text{and} \qquad w^-(u)^n\, \tilde{q}_n(u) \to 1,$$

as $n \to \infty$, uniformly on $[0, 1]$; that is,

$$e^{(n/\|\nu^+\|)\, V(u,\, \nu^+)} \tilde{p}_n(u) \to f(u), \tag{2.1}$$

$$e^{(n/\|\nu^-\|)\, V(u,\, \nu^-)} \tilde{q}_n(u) \to 1. \tag{2.2}$$

We can write (2.1) and (2.2) in the forms

$$e^{(nV(u,\, \nu^+) + ([n\, \|\nu^+\|]/\|\nu^+\| - n)\, V(u,\, \nu^+))} \tilde{p}_{[n\, \|\nu^+\|]}(u) \to f(u), \tag{2.3}$$

$$e^{(nV(u,\, \nu^-) + ([n\, \|\nu^-\|]/\|\nu^-\| - n)\, V(u,\, \nu^-))} \tilde{q}_{[n\, \|\nu^-\|]}(u) \to 1, \tag{2.4}$$

as $n \to \infty$, uniformly on $[0, 1]$. Since the sets $\{e^{\tau V(u,\, \nu^\pm)} : \tau \in [0, \|\nu^\pm\|^{-1}]\}$ are compact subsets of $C([0, 1])$, there are polynomials $r_n \in \mathcal{P}_{[\alpha n] - [n\, \|\nu^+\|]}$ and $s_n \in \mathcal{P}_{[\beta n] - [n\, \|\nu^-\|]}$ such that

$$r_n(u)\, e^{(n-[n\,\|v^+\|]/\|v^+\|)\, V(u,\, v^+)} \to 1, \tag{2.5}$$

$$s_n(u)\, e^{(n-[n\,\|v^-\|]/\|v^-\|)\, V(u,\, v^-)} \to 1, \tag{2.6}$$

as $n \to \infty$, uniformly on $[0, 1]$. Next from (2.3)–(2.6) it follows that

$$p_n := e^{-nF} \tilde{p}_{[n\,\|v^+\|]} r_n \in \mathscr{P}_{[\alpha n]} \qquad \text{and} \qquad q_n := \tilde{q}_{[n\,\|v^-\|]} s_n \in \mathscr{P}_{[\beta n]},$$

satisfy

$$e^{nV(u,\, v^+)} e^{nF} p_n(u) \to f(u), \qquad e^{nV(u,\, v^-)} q_n(u) \to 1, \tag{2.7}$$

as $n \to \infty$, uniformly on $[0, 1]$, where $F$ is the constant appearing in (1.7). Then from (2.7) and the relation $v^+ - v^- = \mu^+ - \mu^-$ it follows that

$$\begin{aligned} w(u)^n \frac{p_n(u)}{q_n(u)} &= e^{nV(u,\, \mu^+ - \mu^-)} e^{nF} \frac{p_n(u)}{q_n(u)} = \frac{e^{nV(u,\, v^+)} e^{nF} p_n(u)}{e^{nV(u,\, v^-)} q_n(u)} \\ &= f(u) + o(1), \end{aligned}$$

as $n \to \infty$, uniformly on $[0, 1]$.

Next, suppose that $\alpha > \|\mu^+\|$ and $\beta = \|\mu^-\| = 0$. Choose a number $\lambda \in (0, 1)$ so that $\|\mu^+\| + (1-\lambda)/\lambda < \alpha$, and consider the weight $w(u)^\lambda = 4^{\lambda - 1} \exp(V(u, \mu_\lambda) + \lambda F)$, $u \in [0, 1]$, where $\mu_\lambda := \lambda \mu^+ + (1-\lambda)\, v\, dt$. Then $\mu_\lambda$ satisfies the conditions of Lemma 2.1; hence every real-valued $f \in C([0, 1])$ is uniformly approximable on $[0, 1]$ by weighted polynomials $w^{\lambda n} p_n$ with $p_n \in \mathscr{P}_{[\|\mu_\lambda\|\, n]}$. By the choice of $\lambda$ we have $\|\mu_\lambda\|/\lambda < \alpha$. Using an argument like that for (2.3)–(2.7) one can show that every such $f$ is uniformly approximable on $[0, 1]$ by $w^n p_n$ with $p_n \in \mathscr{P}_{[\alpha n]}$.

Finally, suppose that $\alpha = \|\mu^+\| = 0$. It is obvious that uniform approximation of $f \in C([0, 1])$ on $[0, 1]$ by weighted real rationals of the form $w^n/q_n$, $q_n \in \mathscr{P}_{[\beta n]}$, is not possible if $f$ changes sign on $(0, 1)$. So let $f \in C([0, 1])$ be nonnegative on $[0, 1]$. Define $f_k(u) := f(u) + k^{-1}$ for $k \in \mathbf{N}$. By what we have already proved in the previous case, it follows that for every $k \in \mathbf{N}$ there is a sequence of real polynomials $\{p_{n,k} \in \mathscr{P}_{[\beta n]}\}_{n \in \mathbf{N}}$ such that $(w^{-1})^n\, p_{n,k} \to f_k^{-1}$, as $n \to \infty$, uniformly on $[0, 1]$. Now we define the sequence $\{n_k\}$ as follows: $n_1 = 1$, and for $k \geqslant 2$, $n_k > n_{k-1}$ is chosen so that for $n \geqslant n_k$

$$|(w(u)^{-1})^n\, p_{n,k}(u) - f_k(u)^{-1}| < k^{-2}, \qquad u \in [0, 1].$$

Then the polynomials $q_n := p_{n,k}$ for $n \in \{n_k, ..., n_{k+1} - 1\}$ and $k \in \mathbf{N}$ satisfy $w(u)^n/q_n(u) \to f(u)$, as $n \to \infty$, uniformly on $[0, 1]$.

Now let $f \in C([0, 1])$ be an arbitrary real-valued function. For $k \in \mathbf{N}$ we define the complex-valued function $f_k := f + ik^{-1}$. Then $|f_k| \geqslant k^{-1}$

on $[0, 1]$. As we have already shown, for every $k \in \mathbf{N}$ there are real polynomials $p_{n,k,r}$ and $p_{n,k,i} \in \mathscr{P}_{[\beta n]}$ such that

$$(w^{-1})^n \, p_{n,k,r} \to \operatorname{Re}(f_k^{-1}) \qquad \text{and} \qquad (w^{-1})^n \, p_{n,k,i} \to \operatorname{Im}(f_k^{-1}),$$

as $n \to \infty$, uniformly on $[0, 1]$. Then, as above, it follows that for a suitable choice of indices, the sequence $\{w^n/(p_{n,k,r} + i p_{n,k,i})\}$ tends to $f$ uniformly on $[0, 1]$. ▮

2.2. *Proofs of Theorems* 1.1 *and* 1.3

*Proof of Theorem* 1.1. Let $\alpha \geqslant 0$, $\beta \geqslant 0$ and $\alpha + \beta > 0$. For fixed $a > 0$ and $x \in \mathbf{R}$ we define the function

$$\sigma(t, x) := \frac{a - t - x}{\pi \sqrt{t(a-t)}}, \qquad t \in [0, a],$$

and let $\sigma(t, x) = \sigma^+(t, x) - \sigma^-(t, x)$ be its Jordan decomposition on $[0, a]$. We set $p(x, a) = \|\sigma^+(t, x)\|$ and $n(x, a) = \|\sigma^-(t, x)\|$. By (2.2)–(2.3) in [1] we also have

$$e^u = \exp(V(u, -\sigma(t, x)\, dt) + \text{const}), \qquad u \in [0, a].$$

First let $\alpha > 0$ and $\beta > 0$. It follows from (1.1)–(1.3) that $\hat{a}(\alpha, \beta)$ is a continuous function of $\alpha$ and $\beta$. Thus if $a < \hat{a}(\alpha, \beta)$ there is some $\varepsilon > 0$ such that $a < \hat{a}(\alpha - \varepsilon, \beta - \varepsilon)$. Then by Lemmas 7 and 9 in [1] there is an $\bar{x}$ such that $p(\bar{x}, a) < \beta$ and $n(\bar{x}, a) < \alpha$. Hence Theorem 1.1 follows from Theorem 1.5 with $s^{\pm}(t) = \sigma^{\mp}(t, \bar{x})$.

If $\beta = 0$, then $\hat{a}(\alpha, 0) = 2\alpha$ by (1.3) and in this case Theorem 1.1 again follows from Theorem 1.5 with $w(u) = e^u = C \exp(V(u, -\sigma(t, 0)\, dt))$, $u \in [0, a]$, and the fact that $\|\sigma(t, 0)\| = a/2 < \alpha$ for $a \in (0, \hat{a}(\alpha, 0))$.

Finally if $\alpha = 0$ and $\beta > 0$, the last assertion of Theorem 1.1 follows in a similar fashion from Theorem 1.5. ▮

*Proof of Theorem* 1.3. Let $\theta > 1$, $\alpha \geqslant 0$, $\beta \geqslant 0$ and $\alpha + \beta > 0$. As in [1], for fixed $b \in (0, 1)$ and $x \in \mathbf{R}$ we define the function

$$\tilde{\sigma}(t, x) := \frac{(\sqrt{b}/t - x)}{\pi \sqrt{(t-b)(1-t)}}, \qquad t \in [b, 1].$$

Let $\tilde{\sigma}(t, x) = \tilde{\sigma}^+(t, x) - \tilde{\sigma}^-(t, x)$ be the Jordan decomposition of the measure $\tilde{\sigma}(t, x)\, dt$ in $[b, 1]$ and set

$$p(x, b) := \|\tilde{\sigma}^+(t, x)\| \qquad \text{and} \qquad n(x, b) := \|\tilde{\sigma}^-(t, x)\|.$$

By (3.1)–(3.5) in [1] we have

$$u^{\theta} = \exp(V(u, -\theta\tilde{\sigma}(t, x)\, dt) + \text{const}), \qquad u \in [b, 1].$$

Assume first that $\alpha > 0$ and $\beta > 0$. From Lemmas 11 and 12 in [1] it follows that for $b \in (\hat{b}(\theta; \alpha, \beta), 1)$ there exists an $x \in (\sqrt{b}, 1/\sqrt{b})$ such that $p(x, b) < \beta/\theta$ and $n(x, b) < \alpha/\theta$. Then Theorem 1.3 follows from Theorem 1.5 with $s^{\pm}(t) = \theta\sigma^{\mp}(t, x)$.

Next assume that $\beta = 0$. By Lemma 12 in [1], $\hat{b}(\theta; \alpha, 0) = (1 + \alpha/\theta)^{-2}$. Let $b \in (\hat{b}, 1)$. Then $\alpha + \theta > \theta/\sqrt{b}$ and for fixed $x \in [\theta/\sqrt{b}, \alpha + \theta)$, the function

$$s(t) := -\theta\tilde{\sigma}(t, x/\theta), \qquad t \in [b, 1]$$

is nonnegative on $[b, 1]$, satisfies $\int_b^1 s(t)\, dt = x - \theta < \alpha$, and so Theorem 1.3 again follows from Theorem 1.5 with $s^{+}(t) = s(t)$, $s^{-}(t) = 0$.

Finally if $\alpha = 0$ and $\beta > 0$, the last assertion of Theorem 1.3 follows in a similar fashion from Theorem 1.5. ∎

## ACKNOWLEDGMENTS

The authors are grateful for helpful suggestions from V. Totik and A. B. J. Kuijlaars.

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*Printed in Belgium*