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GIBBS PHENOMENON FOR BEST L_p APPROXIMATION BY POLYGONAL LINES

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We show that for a class of functions f having a jump discontinuity at $x = 0$, best L_p-polygonal line approximants with equally spaced knots $\{i/n\}_{i=-n}^{n}$ over $[-1, 1]$ exhibit a Gibbs phenomenon behavior as $n \to \infty$ when $p > 1$. Moreover, we prove that the overshoot involved is an increasing function of p that tends to zero as p tends to one.

1. Introduction

It is well known that if a function f has a jump discontinuity, the partial sums of its Fourier series (the best L_2-approximations of the function

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by trigonometric polynomials) tend to overshoot f (Gibbs phenomenon) [1]. This phenomenon also occurs when the function f is approximated by best trigonometric polynomials in the L_1 - metric [2] and when f is approximated by best spline functions in the L_2 - metric [3]. For a recent survey of the subject, see [4].

In this paper we study Gibbs phenomenon in the case of approximation by polygonal lines in the L_p-metric, where $p \geq 1$. We shall show that an overshoot occurs for every $p > 1$ but disappears when $p = 1$. We also show that the amount of overshoot is a monotone increasing function of p that tends to zero as p tends to one and tends to half of the jump as p tends to infinity.

The outline of the paper is as follows. In Section 2 we investigate in detail the Gibbs phenomenon for best polygonal line approximants to the signum function. Section 3 is devoted to strong uniqueness properties for best polygonal line approximants. Finally, in Section 4 we describe how the Gibbs phenomenon for the signum function extends to a class of functions having a jump discontinuity at $x = 0$.

2. Best approximation of the signum function

In this section we study the best L_p-approximation of the function $\lambda(x) := \operatorname{sgn}(x)$ over the interval $[-nh, nh]$, $h > 0$, $n \in \mathbb{Z}^+$, by continuous splines of order 1 (polygonal lines) with knots at $x_i = ih$, $i = -n, -n+1, \ldots, n$.

Let $S_\triangle$ be the set of all polygonal lines with knots

$$\triangle = \{x_i\}_{i=-n}^{n}, \; x_i = ih,$$

and for $i = -n, -n+1, \ldots, n$, let B_i denote the cardinal spline of order 1:

$$B_i(x) := \begin{cases} 0, & x \leq x_{i-1} \\ (x - x_{i-1})/h, & x_{i-1} \leq x \leq x_i \\ (x_{i+1} - x)/h, & x_i \leq x \leq x_{i+1} \\ 0, & x_{i+1} \leq x \; . \end{cases}$$

Then, any polygonal line $s \in S_\triangle$ can be represented as $s(x) = \sum_{i=-n}^{n} \alpha_i B_i(x)$ with $\alpha_i = s(x_i)$. Let $s_{\lambda,p} \in S_\triangle$, $p \geq 1$, be a best $L_p[x_{-n}, x_n]$ approximation of λ over the interval $[x_{-n}, x_n]$, i.e.,

$$\int_{x_{-n}}^{x_n} |\lambda(x) - s_{\lambda,p}(x)|^p \, dx = \inf_{s \in S_\triangle} \int_{x_{-n}}^{x_n} |\lambda(x) - s(x)|^p \, dx.$$

For $p > 1$, the strict convexity of the norm implies that $s_{\lambda,p}$ is unique. Consequently, since the function λ is odd, so is the function $s_{\lambda,p}$. Then, $s_{\lambda,p}(0) = 0$ because of the continuity of $s_{\lambda,p}$. Hence the functions $s_{\lambda,p}$ and λ cannot coincide in the interval $[x_{-1}, x_0]$ or in the interval $[x_0, x_1]$. Lemma 2.1 states that these functions do not coincide in *any* subinterval $[x_i, x_{i+1}]$.

In the case $p = 1$, it turns out that $s_{\lambda,1}$ is not unique. The values of $s_{\lambda,1}(0)$ are the whole interval $[-1, 1]$, when $s_{\lambda,1}$ passes through the set of all best approximants. If for some best approximant $s_{\lambda,1}$ the inequality $s_{\lambda,1}(0) < 1$ is true, then Lemma 2.1 states that $s_{\lambda,1}$ and λ do not coincide in any subinterval of the interval $[x_0, x_n]$. In the extreme case $s_{\lambda,1}(0) = 1$ (or $s_{\lambda,1}(0) = -1$) the functions $s_{\lambda,1}$ and λ coincide in the interval $[x_0, x_n]$, ($[x_{-n}, x_0]$) but these functions do not coincide in any subinterval of the interval $[x_{-n}, x_0]$ ($[x_0, x_n]$ respectively). Usually, we shall consider only the interval $[x_0, x_n]$, (the considerations for the interval $[x_{-n}, x_0]$ are analogous) so in the case $p = 1$, we shall assume that $s_{\lambda,1}(0) < 1$.

Lemma 2.1. *Let $p \geq 1$. If for some i, $0 \leq i < n$, we have $s_{\lambda,p}(x_i) \neq 1$, then $s_{\lambda,p}(x_{i+1}) \neq 1$. Similarly if for some i, $-n < i \leq 0$, we have $s_{\lambda,p}(x_i) \neq -1$, then $s_{\lambda,p}(x_{i-1}) \neq -1$.*

Proof. Suppose to the contrary that for some i, $0 \leq i < n$, $s_{\lambda,p}(x_i) \neq 1$, but $s_{\lambda,p}(x_{i+1}) = 1$. Then, because $s_{\lambda,p}$ is a best polygonal line approximant we have $s_{\lambda,p}(x) = 1$ for $x_{i+1} \leq x \leq x_n$ and

$$\int_{x_i}^{x_n} |\lambda(x) - s_{\lambda,p}(x)|^p \, dx = \int_{x_i}^{x_{i+1}} |\lambda(x) - s_{\lambda,p}(x)|^p \, dx = \frac{|H_i|^p}{p+1} h, \tag{2.1}$$

where $H_i := s_{\lambda,p}(x_i) - 1$. We shall now construct another polygonal line $s(x)$, with $s(x_i) = 1 + H_i$, that gives better L_p-approximation of $\lambda(x)$ over the interval $[x_i, x_n]$. This will contradict the fact that $s_{\lambda,p}$ is the best approximant.

Let H_k, $k = i+1, i+2, \dots, n$, be chosen such that $H_k H_{k+1} < 0$, $k = i, i+1, \dots, n-1$,

$$s(x_k) = 1 + H_k, \quad k = i+1, i+2, \dots, n,$$

and let

$$z_{k+1} = x_k + h t_{k+1}, \quad 0 < t_{k+1} < 1,$$

be the intercept points of $\lambda(x)$ and $s(x)$. Obviously, the values H_k, $k = i+1, \dots, n$, are uniquely determined from the points t_k. We will show that for a convenient choice of the numbers t_k, the polygonal line $s(x)$ will approximate λ better than $s_{\lambda,p}$. Using the representation

$$s(x) = 1 - H_{k-1}(x - z_k)/(h t_k)$$

for $x_{k-1} \le x \le x_k$, and $H_k/H_{k-1} = -(1-t_k)/t_k$, we easily obtain

$$(2.2) \qquad \int_{x_i}^{x_n} |\lambda(x) - s(x)|^p \, dx = \frac{h}{p+1} \sum_{k=i}^{n-1} [|H_k|^p t_{k+1} + |H_{k+1}|^p (1 - t_{k+1})].$$

In the case $p > 1$, we choose $t_k = 1/(1+q)$, $k = i+1, \ldots, n$, where $1 > q > 0$ is arbitrary but fixed. Then (2.2) implies that

$$\int_{x_i}^{x_n} |\lambda(x) - s(x)|^p \, dx \le \frac{h|H_i|^p}{p+1} \left(\frac{1}{1+q} + \frac{q^p}{1-q^p} \right).$$

The right-hand side of the last inequality is less than $h|H_i|^p/(p+1)$ for q sufficiently small and positive. This, with (2.1), proves the lemma for $p > 1$.

For the case $p = 1$, we choose $t_k = 1/(1+q_k), k = i+1, \ldots, n$, where $1 > q_k > 0$. Then (2.2) implies that

$$\int_{x_i}^{x_n} |\lambda(x) - s(x)| \, dx$$

$$\le \frac{h|H_i|}{2} (1 - q_{i+1} q_{i+2} \cdots q_n + 2q_{i+1}^2 + 2 \sum_{k=i+1}^{n-1} q_{i+1} q_{i+2} \cdots q_k q_{k+1}^2).$$

It is clear that for suitable positive $q_{i+1}, \ldots, q_n$ the right-hand side of the last inequality can be made less than $h|H_i|/2$. This again contradicts the fact that $s_{\lambda,1}$ is a best approximation to λ. □

In what follows we shall use the notation $z_{k,p}^n = x_{k-1} + ht_{k,p}^{(n)}$, $k = 1, 2, \ldots, n$, or briefly, $z_k = x_{k-1} + ht_k$ $(0 < t_k < 1)$, for the points of interpolation on $[0, x_n]$ of the best L_p approximant, so that we have

$$s_{\lambda,p}(z_i) = \lambda(z_i), \quad i = 1, 2, \ldots, n.$$

Such points z_k do exist (in the case $p = 1$, for those best approximants $s_{\lambda,1}$ for which $s_{\lambda,1}(0) < 1$). Indeed, let $s_{\lambda,p}(x_i) = 1 + H_i$ and $s_{\lambda,p}(x_{i+1}) = 1 + H_{i+1}$. Lemma 2.1 implies that $H_i H_{i+1} \ne 0$. If we suppose that $H_i H_{i+1} > 0$, then the polygonal line s, such that $s(x_j) = 1 + \overline{H_j}$ where $\overline{H_j} = H_j$, $j = 0, \ldots, i$, and $\overline{H_j} = -H_j$, $j = i+1, \ldots, n$, gives better approximation of λ over $[x_0, x_n]$.

The points z_k completely characterize the best approximant (for $p = 1$ only in $[x_0, x_n]$). Furthermore, notice that on the interval $[x_0, x_1]$, $s_{\lambda,p}(x)$ overshoots λ by the amount $s_{\lambda,p}(x_1) - 1 = (1-t_1)/t_1$; on $[x_2, x_3]$ the overshoot is $(1-t_1)(1-t_2)(1-t_3)/t_1 t_2 t_3$; etc.

Lemma 2.2. *For $p > 1$ or for $p = 1$ and $s_{\lambda,1}(0) < 1$, the numbers $t_1, t_2, \ldots, t_n$ satisfy the system*

$$
\begin{aligned}
&\varphi_p(1 - t_n) = 0,\\
&\varphi_p(1 - t_{n-1}) + \varphi_p(t_n) = 0,\\
&\ldots\\
&\varphi_p(1 - t_1) + \varphi_p(t_2) = 0,
\end{aligned}
\tag{2.3}
$$

where

$$
\varphi_p(x) := \frac{1}{p+1}\left(\frac{1-x}{x}\right)^{p-1}(1-x)^2 + \frac{x^2}{p+1} - x, \ x \in (0,1).
\tag{2.4}
$$

Proof. From Lemma 2.1 it follows that $s_{\lambda,p}(x_i) \neq 1$, $i = 0, 1, \ldots, n$. Let H_i be the numbers for which $s_{\lambda,p}(x_i) = 1 + H_i$ ($H_i \neq 0$ and $\operatorname{sgn}(H_i) = -\operatorname{sgn}(H_{i+1})$). Then

$$
(1 - t_k)/t_k = -H_k/H_{k-1} \quad \text{and} \quad |s_{\lambda,p}(x) = H_{k-1}(z_k - x)/(t_k h) + 1
$$

for $x \in [x_{k-1}, x_k]$. The orthogonality property of the best approximant $s_{\lambda,p}$ yields

$$
\int_{-nh}^{nh} |\lambda(x) - s_{\lambda,p}(x)|^{p-1}\operatorname{sgn}(\lambda(x) - s_{\lambda,p}(x))B_k(x)\,dx = 0
\tag{2.5}
$$

for $k = -n, -n+1, \ldots, n$. It is easy to see that (2.3) follows from the equalities (2.5) for $k = 1, 2, \ldots, n$. In fact, for $0 < k < n$, since B_k vanishes outside the interval $[x_{k-1}, x_{k+1}]$, we have

$$
\begin{aligned}
0 &= \int_{x_{k-1}}^{x_{k+1}} |\lambda(x) - s_{\lambda,p}(x)|^{p-1}\operatorname{sgn}(\lambda(x) - s_{\lambda,p}(x))B_k(x)\,dx \\
&= \frac{\operatorname{sgn}(H_{k-1})}{h}\Bigg\{\left|\frac{H_{k-1}}{ht_k}\right|^{p-1}\Bigg[-\int_{x_{k-1}}^{z_k}(z_k - x)^{p-1}(x - x_{k-1})\,dx \\
&\qquad + \int_{z_k}^{x_k}(x - z_k)^{p-1}(x - x_{k-1})\,dx\Bigg] \\
&\qquad + \left|\frac{H_k}{ht_{k+1}}\right|^{p-1}\Bigg[\int_{x_k}^{z_{k+1}}(z_{k+1} - x)^{p-1}(x_{k+1} - x)\,dx \\
&\qquad - \int_{z_{k+1}}^{x_{k+1}}(x - z_{k+1})^{p-1}(x_{k+1} - x)\,dx\Bigg]\Bigg\} \\
&= h\operatorname{sgn}(H_{k-1})\Bigg[\frac{|H_{k-1}|^{p-1}}{t_k^{p-1}}\left(\frac{-t_k^{p+1}}{p(p+1)} + \frac{(1-t_k)^p}{p} - \frac{(1-t_k)^{p+1}}{p(p+1)}\right) \\
&\qquad + \frac{|H_k|^{p-1}}{t_{k+1}^{p-1}}\left(\frac{t_{k+1}^p}{p} - \frac{t_{k+1}^{p+1}}{p(p+1)} - \frac{(1-t_{k+1})^{p+1}}{p(p+1)}\right)\Bigg].
\end{aligned}
$$

Since $|H_k|/|H_{k-1}| = (1-t_k)/t_k$, this equality reduces to

$$\varphi_p(1-t_k) + \varphi_p(t_{k+1}) = 0.$$

In the same way, one can show that t_n satisfies the equation $\varphi_p(1-t_n) = 0$. □

Next we show that the system (2.3) has a unique solution.

Lemma 2.3. *Let $p \in [1,\infty)$. There exists a unique solution $t_{k,p}^{(n)}$, $k = 1,2,\ldots,n$, of the system (2.3) such that*

(i) $1/2 < t_{n,p}^{(n)} < t_{n-1,p}^{(n)} < \ldots < t_{1,p}^{(n)} < \theta_p < 1$, *where θ_p is the larger of the two solutions of the equation $\varphi_p(1-x) + \varphi_p(x) = 0$;*

(ii) $|\theta_p - t_{1,p}^{(n)}| < (p+1)^{-(n-1)}$;

(iii) *θ_p is a decreasing function of p for $p \in [1,\infty)$. Moreover, $\theta_p \to 1$ as $p \to 1$ and $\theta_p \to 1/2$ as $p \to \infty$.*

Proof. (i) From the identities (cf. (2.4))

$$\varphi_p(x) = \left(\frac{1-x}{x}\right)^{p-1} \frac{(1-x)^2}{p+1} + \frac{x^2}{p+1} - x,$$

$$\frac{d}{dx}\varphi_p(x) = -\frac{1}{p+1}\left(\frac{1-x}{x}\right)^{p}(p-1+2x) + \frac{2x}{p+1} - 1,$$

$$\frac{d^2}{dx^2}\varphi_p(x) = \frac{1}{(p+1)x^2}\left(\frac{1-x}{x}\right)^{p-1}(2x^2+2(p-1)x+p(p-1)) + \frac{2}{p+1},$$

it is clear that the function $\varphi_p(x)$, $x \in (0,1)$, is decreasing and concave upward. Furthermore, $\varphi_1(0) = 1/2$, $\varphi_p(x) \to \infty$ as $x \to 0$ for $p > 1$, and $\varphi_p(1/2) = -p/[2(p+1)]$, $\varphi_p(1) = -p/(p+1)$. Thus the function $\varphi_p(1-x)$ is increasing and has a unique zero $t_{n,p}^{(n)}$ with $t_{n,p}^{(n)} > 1/2$. Moreover, from the relations $\varphi_p(1-t_{n-1,p}^{(n)}) = -\varphi_p(t_{n,p}^{(n)}) > 0 = \varphi_p(1-t_{n,p}^{(n)})$, it follows that $t_{n-1,p}^{(n)} > t_{n,p}^{(n)}$ because the function $\varphi_p(1-x)$ is increasing. By induction, it is easy to verify that $t_{i-1,p}^{(n)} > t_{i,p}^{(n)}$ for $i = 1,2,\ldots,n$. In the same way, the inequality $t_{i,p}^{(n)} < \theta_p$ implies $t_{i-1,p}^{(n)} < \theta_p$, because

$$\varphi_p(1-t_{i-1,p}^{(n)}) = -\varphi_p(t_{i,p}^{(n)}) < -\varphi_p(\theta_p) = \varphi_p(1-\theta_p).$$

This proves assertion (i).

(ii) The concavity of the function $\varphi_p(1-x)$ implies that

$$\frac{\theta_p - t_{i-1,p}^{(n)}}{\theta_p - t_{i,p}^{(n)}} \le \frac{\varphi_p(1-\theta_p) - \varphi_p(1-t_{i-1,p}^{(n)})}{\varphi_p(1-\theta_p) - \varphi_p(1-t_{i,p}^{(n)})}.$$

The right-hand side of this inequality is equal to $\varphi_p'(\xi)/\varphi_p'(1-\xi)$ for some $\xi \in (t_{i,p}^{(n)}, \theta_p)$. Thus we have

$$|\theta_p - t_{i-1,p}^{(n)}| \le |\theta_p - t_{i,p}^{(n)}| \left|\frac{\varphi_p'(\xi)}{\varphi_p'(1-\xi)}\right|. \tag{2.6}$$

From the monotonicity of the functions $\varphi_p(x)$ and $\varphi_p(1-x)$ and the fact that $\xi > t_{n,p}^{(n)}$, we conclude

$$\left|\frac{\varphi_p'(\xi)}{\varphi_p'(1-\xi)}\right| \le \left|\frac{\varphi_p'(t_{n,p}^{(n)})}{\varphi_p'(1-t_{n,p}^{(n)})}\right|.$$

Since $\varphi_p(1-t_{n,p}^{(n)})=0$, i.e.,

$$\left(\frac{t_{n,p}^{(n)}}{1-t_{n,p}^{(n)}}\right)^p=\frac{p+t_{n,p}^{(n)}}{t_{n,p}^{(n)}},$$

we obtain

$$\left|\frac{\varphi_p'(t_{n,p}^{(n)})}{\varphi_p'(1-t_{n,p}^{(n)})}\right|=\frac{t_{n,p}^{(n)}}{p+t_{n,p}^{(n)}}<\frac{1}{p+1}.$$

The last inequality together with (2.6) proves assertion (ii).

(iii) Set $F(x,p):=\varphi_p(1-x)+\varphi_p(x)$. There are two solutions of the equation $F(x,p)=0$ in the interval $(0,1)$ for fixed p. We shall prove that the solution $\theta(p)=\theta_p$ of this equation that belongs to the interval $\left(\frac{1}{2},1\right)$ is a decreasing function of p by showing that

$$\frac{d\theta(p)}{dp}=-\frac{(\partial F/\partial p)(\theta(p),p)}{(\partial F/\partial x)(\theta(p),p)}<0$$

for $p>1$. For this purpose, we prove that the functions $(\partial F/\partial p)(\theta(p),p)$ and $(\partial F/\partial x)(\theta(p),p)$ are positive for $p>1$. We have

$$\begin{aligned}F(x,p) &= \left(\frac{1-x}{x}\right)^{p-1}\frac{(1-x)^2}{p+1}+\left(\frac{x}{1-x}\right)^{p-1}\frac{x^2}{p+1}\\ &\quad+\frac{(1-x)^2+x^2}{p+1}-1,\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial x}F(x,p) &= \frac{1}{p+1}\left[\left(\frac{x}{1-x}\right)^p(p+1-2x)\right.\\ &\quad\left.-\left(\frac{1-x}{x}\right)^p(p-1+2x)+4x-2\right],\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial x^2}F(x,p) &= \frac{1}{(p+1)x(1-x)}\left[\left(\frac{x}{1-x}\right)^p(2x^2-2x(p+1)+p(p+1))\right.\\ &\quad\left.+\left(\frac{1-x}{x}\right)^p(2x^2+2x(p-1)+p(p-1))+4x(1-x)\right].\end{aligned}$$

Clearly, $(\partial^2F/\partial x^2)(x,p)>0$ for $x\in\left(\frac{1}{2},1\right)$; hence $(\partial F/\partial x)(\theta(p),p)>0$ because $(\partial F/\partial x)(1/2,p)=0$.

To prove that $(\partial F/\partial p)(\theta(p),p)>0$ we first need to perform some operations. Using the equation $F(\theta(p),p)=0$ we determine

$$\frac{\partial}{\partial p}F(\theta,p)=\frac{1}{p+1}\ln\frac{\theta}{1-\theta}\left[\left(\frac{\theta}{1-\theta}\right)^{p-1}\theta^2-\left(\frac{1-\theta}{\theta}\right)^{p-1}(1-\theta)^2\right]-\frac{1}{p+1},$$

where $\theta = \theta(p)$. Thus we need to prove that the function

$$(2.7)\quad \begin{aligned} G(x,p) &= \ln\frac{x}{1-x}\left[\left(\frac{x}{1-x}\right)^{p-1}x^2\right. \\ &\quad \left. - \left(\frac{1-x}{x}\right)^{p-1}(1-x)^2\right] - 1, \quad x \in \left(\tfrac{1}{2}, 1\right) \end{aligned}$$

is positive for $x = \theta = \theta(p)$. The function $G(x,p)$ is increasing since $(\partial G/\partial x)(x,p) > 0$ for $x \in (1/2,1)$, but it changes sign. Setting $z = [(1-\theta)/\theta]^{p-1}$ we find from the equation $F(\theta,p) = 0$ that

$$z = \frac{2\theta^2}{2\theta(1-\theta) + p + \sqrt{p^2 + 4p\theta(1-\theta)}}.$$

Using the relations

$$\ln\frac{\theta}{1-\theta} = \frac{1}{p-1}\ln\frac{1}{z}, \quad \frac{1}{z} > p, \qquad \frac{\theta^2}{z} > p, \qquad z(1-\theta)^2 < \frac{1}{7+16p}$$

(the last inequality follows from

$$\frac{2t^2}{2t + p + \sqrt{p^2 + 4pt}} \le \frac{1}{7+16p},$$

where $t = \theta(1-\theta) \in (0, 1/4)$), we obtain

$$G(\theta,p) \ge \frac{\ln p}{p-1}\left(p - \frac{1}{7+16p}\right) - 1.$$

The right-hand side of this inequality is an increasing function of p for $p \ge 1 + e^{-2}$, and is positive for $p = 1 + e^{-2}$. This proves (iii) for $p \ge 1 + e^{-2}$.

The case $1 < p < 1 + e^{-2}$ is more complicated. We set

$$\lambda(p) := \frac{1}{p\ln p}\ln\frac{e}{(p-1)\ln[e/(p-1)]},$$

and $\varepsilon(p) := p^{-\lambda(p)}$. First, we will show that $\theta(p) \ge \dfrac{1}{1+\varepsilon(p)}$. To do this, it is enough to verify the inequality

$$f_p\left(\frac{1}{1+\varepsilon(p)}\right) \ge g_p\left(\frac{1}{1+\varepsilon(p)}\right),$$

where

$$f_p(x) := \left(\frac{1-x}{x}\right)^{p-1}$$

and

$$g_p(x) := \frac{2x^2}{2x(1-x)+p+\sqrt{p^2+4px(1-x)}}, \quad x \in \left(\tfrac{1}{2}, 1\right).$$

(Note that the function f_p is decreasing, the function g_p is increasing, and

$$z = f(\theta(p)) = g(\theta(p)), \qquad f_p\left(\frac{1}{2}\right) > g_p\left(\frac{1}{2}\right), \qquad f_p(1) < g_p(1)).$$

It is easy to see that

$$(2.8) \qquad f_p\left(\frac{1}{1+\varepsilon(p)}\right) - g_p\left(\frac{1}{1+\varepsilon(p)}\right) \geq \frac{p\varepsilon^{p-1} + (2p+1)\varepsilon^p - 1}{(2p+1)\varepsilon + p}.$$

Using the relations $\ln p < p - 1$ and $e^x > 1 + x$, we have

$$\varepsilon^{p-1} = \exp(-\lambda(p)(p-1)\ln p) \geq 1 - \frac{p-1}{p} \ln \frac{e}{(p-1)\ln[e/(p-1)]},$$

and of course,

$$\varepsilon^p = \exp(-p\lambda(p)\ln p) = \frac{p-1}{e} \ln \frac{e}{(p-1)}.$$

Thus, the right-hand side of (2.8) is greater than

$$\frac{p - 1 + (p-1)\left(\frac{2p+1}{e} - 1)\right)\ln[e/(p-1)] + (p-1)\ln\ln[e/(p-1)]}{(2p+1)\varepsilon + p},$$

which is positive for $1 < p < 1 + e^{-2}$. So, in this case $\theta(p) \geq \dfrac{1}{1+\varepsilon(p)}$, and therefore

$$\frac{\theta(p)}{1-\theta(p)} \geq \frac{1}{\varepsilon(p)},$$

or

$$\ln \frac{\theta(p)}{1-\theta(p)} \geq \ln \frac{1}{\varepsilon(p)} = \frac{1}{p} \ln \frac{e}{(p-1)\ln[e/(p-1)]}.$$

Using this inequality, and

$$\frac{\theta^2}{z} > p, \qquad z(1-\theta)^2 < \frac{1}{7+16p}$$

again, we have from (2.7)

$$\begin{aligned} G(\theta, p) &\geq \frac{1}{p} \ln \frac{e}{(p-1)\ln \frac{e}{p-1}} \left(p - \frac{1}{7+16p}\right) - 1 \\ &\geq \ln \frac{e}{(p-1)\ln \frac{e}{p-1}} \left(1 - \frac{1}{23}\right) - 1. \end{aligned}$$

Since the right-hand side of the last inequality is positive for $1 < p < 1+e^{-2}$, the monotonicity is proved.

Finally we observe that for each $\varepsilon > 0$ we have $\varphi_p(x) + \varphi_p(1-x) \to \infty$ as $p \to \infty$ uniformly for $x \in [\frac{1}{2}+\varepsilon, 1]$, and so $\theta_p \to 1/2$ as $p \to \infty$. Furthermore, $\varphi_p(x) + \varphi_p(1-x) \to 2x(x-1)$ as $p \to 1$ uniformly for $x \in [\frac{1}{2}, 1-\varepsilon]$ so that $\theta_p \to 1$, as $p \to 1$. □

Corollary. *For $p \in [1,\infty)$, let $\{t_k\}_1^n = \{t_{k,p}^{(n)}\}_1^n$ be as in Lemma 2.3 and set $z_k = x_{k-1} + ht_k$, $z_{-k} = -z_k$, for $k = 1,2,\dots,n$. For $p > 1$, the best L_p approximant $s_{\lambda,p}$ is the unique polygonal line s that interpolates λ in the points $\{z_k, z_{-k}\}_1^n$ and satisfies $s(0) = 0$. For $p = 1$, every best L_1 approximant $s_{\lambda,1}$ satisfies $s_{\lambda,1}(0) \in [-1,1]$. Furthermore, for each $\alpha \in [-1,1]$, there exists a unique best L_1 approximant satisfying $s_{\lambda,1}(0) = \alpha$ and this best approximant is the unique polygonal line s with $s(0) = \alpha$ that interpolates λ in the points $\{z_k, z_{-k}\}_1^n$.*

Proof. For $p > 1$, the assertion of the corollary is immediate from the preceding lemmas. For $p = 1$, it is easy to see that if the polygonal line s satisfies $|s(0)| > 1$, then it cannot be a best approximant of λ. Moreover, using elementary computations, it can be shown that, for any two polygonal lines $s^{(1)}, s^{(2)}$ with $-1 \le s^{(1)}(0) < s^{(2)}(0) \le 1$ that interpolate λ in the points $\{z_k, z_{-k}\}_1^n$, we have $||\lambda - s^{(1)}||_{L_1} = ||\lambda - s^{(2)}||_{L_1}$. If $\alpha \in (-1,1)$ and $s^{(1)}$ is the polygonal line such that $s^{(1)}(0) = \alpha$ and $s^{(1)}$ interpolates λ in the points $\{z_k, z_{-k}\}_1^n$, then

$$\int_{-nh}^{nh} \operatorname{sgn}(\lambda(x) - s^{(1)}(x))\, B_k(x)\, dx = 0$$

for $k = -n, -n+1, \dots, n$ (including $k = 0$). But then $s^{(1)}$ is a best approximant. If $\alpha = 1$ (or $\alpha = -1$) so that $s^{(2)} \equiv \lambda$ on $[0,1]$ (on $[-1,0]$) and $s^{(2)}$ interpolates λ at the points $\{z_{-k}\}_1^n$ ($\{z_k\}_1^n$), then $s^{(2)}$ is also a best approximant because $||\lambda - s^{(1)}||_{L_1} = ||\lambda - s^{(2)}||_{L_1}$. The uniqueness assertion follows from the uniqueness of the points t_k (and hence z_k) established in Lemma 2.3. □

We now summarize the results of this section.

Proposition 2.4. *If $p > 1$ and $s_{n,\lambda,p}(x)$ denotes the best L_p-polygonal line approximation to $\lambda(x) = \operatorname{sgn}(x)$ over $[-1,1]$ with knots at $\{i/n\}_{i=-n}^n$,* *then*

$$\begin{aligned}\lim_{n\to\infty}\max_{x\in(0,1)}[s_{n,\lambda,p}(x)-\lambda(x)] &= \lim_{n\to\infty}[s_{n,\lambda,p}(\tfrac{1}{n})-1]\\ &= \lim_{n\to\infty}\left[\frac{1}{t_{1,p}^{(n)}}-1\right] = \frac{1}{\theta_p}-1,\end{aligned}\tag{2.9}$$

where $t_{1,p}^{(n)}$ and θ_p are defined as in Lemma 2.3. Consequently, the limit in (2.9) is an increasing function of p that tends to zero as $p \to 1$ and tends to 1 as $p \to \infty$. Furthermore, for each $\varepsilon > 0$,

$$\lim_{n\to\infty}\max_{\varepsilon\le]x]\le 1}|s_{n,\lambda,p}(x)-\lambda(x)| = 0.$$

If $p = 1$, then for any best polygonal line $s_{n,\lambda,1}$, we have

$$\lim_{n\to\infty}\max_{x\in(0,1]}[s_{n,\lambda,1}(x)-\lambda(x)] = \lim_{n\to\infty}[s_{n,\lambda,1}(\tfrac{1}{n})-1] = 0$$

and

$$\lim_{n\to\infty}\min_{x\in[-1,0)}[s_{n,\lambda,1}(x)-\lambda(x)] = \lim_{n\to\infty}[s_{n,\lambda,1}(-\tfrac{1}{n})+1] = 0.$$

3. Strong uniqueness properties

In this section we establish some strong uniqueness properties of best L_p-polygonal line approximants that will enable us to generalize the results of the preceding section to a class of functions f having a jump discontinuity at $x = 0$.

Lemma 3.1. *There exist constants c_1, c_2 so that for any best L_1-approximant $s^*_{\lambda,1}$ to $\lambda(x)$ on $[-nh, nh]$ and any $s \in S_\Delta$ with $s(0) = s^*_{\lambda,1}(0)$, the following inequality holds*

$$\begin{aligned}\|s-s^*_{\lambda,1}\|_C &\le \frac{c_1}{h}\{\|s-\lambda\|_{L_1}-\|s^*_{\lambda,1}-\lambda\|_{L_1}\}\\ &+ c_2\left\{|s^*_{\lambda,1}(x_1)-\lambda(x_1)|+|s^*_{\lambda,1}(x_{-1})-\lambda(x_{-1})|\right\},\end{aligned}\tag{3.1}$$

where $\|\cdot\|_C$ denotes the uniform norm over $[-nh, nh]$.

Proof. Set $r := s^*_{\lambda,1}-s$ and $g := s^*_{\lambda,1}-\lambda$, and suppose that $\|r\|_C = |r(x_k)|$, $k > 0$. Suppose also that $|r(x_k)| > 4|s^*_{\lambda,1}(x_1)-\lambda(x_1)|$ (otherwise (3.1) clearly holds). We will use the following notations (cf. Section 2):

$$\begin{aligned}
s^*_{\lambda,1}(x) &= \sum_{i=-n}^{n} \alpha_i^* B_i(x), \\
s(x) &= \sum_{i=-n}^{n} \alpha_i B_i(x), \\
\Phi(\alpha_{-n}, \alpha_{-n+1}, \dots, \alpha_n) &= \int_{-nh}^{nh} \left| \sum_{i=-n}^{n} \alpha_i B_i(x) - \lambda(x) \right| dx, \\
\Psi(t) &= \Phi(\boldsymbol{\alpha}^* + t(\boldsymbol{\alpha} - \boldsymbol{\alpha}^*)),
\end{aligned}$$

where $\boldsymbol{\alpha} = (\alpha_{-n}, \alpha_{-n+1}, \dots, \alpha_n)$ and $\boldsymbol{\alpha}^* = (\alpha^*_{-n}, \alpha^*_{-n+1}, \dots, \alpha^*_n)$. Since

$$\frac{\partial}{\partial \alpha_k} \Phi(\boldsymbol{\alpha}) = \int_{-nh}^{nh} B_k(x) \operatorname{sgn}(s(x) - \lambda(x))\, dx,$$

for almost all t we have

$$\begin{aligned}
\Psi'(t) &= \sum_{k=-n}^{n} (\alpha_k - \alpha_k^*) \frac{\partial \Phi}{\partial \alpha_k} (\boldsymbol{\alpha}^* + t(\boldsymbol{\alpha} - \boldsymbol{\alpha}^*)) \\
&= \sum_{k=-n}^{n} (\alpha_k - \alpha_k^*) \int_{-x_n}^{x_n} B_k(x) \\
&\qquad \times \operatorname{sgn} \left(\sum_{j=-n}^{n} (\alpha_j^* + t(\alpha_j - \alpha_j^*)) B_j(x) - \lambda(x) \right) dx \\
&= \int_{-x_n}^{x_n} \left(\sum_{k=-n}^{n} (\alpha_k - \alpha_k^*) B_k(x) \right) \\
&\qquad \times \operatorname{sgn} \left(\sum_{j=-n}^{n} (\alpha_j^* + t(\alpha_j - \alpha_j^*)) B_j(x) - \lambda(x) \right) dx \\
&= \int_{-x_n}^{x_n} (s(x) - s^*_{\lambda,1}(x)) \operatorname{sgn}(s^*_{\lambda,1}(x) + t(s(x) - s^*_{\lambda,1}(x)) - \lambda(x))\, dx.
\end{aligned}$$

Letting $A := \|s - \lambda\|_{L_1} - \|s^*_{\lambda,1} - \lambda\|_{L_1}$, we get

$$\begin{aligned} A &= \Phi(\boldsymbol{\alpha}) - \Phi(\boldsymbol{\alpha}^*) = \Psi(1) - \Psi(0) \\ &= \int_0^1 dt \int_{-x_n}^{x_n} (s(x) - s^*_{\lambda,1}(x)) \operatorname{sgn}(s^*_{\lambda,1}(x) - \lambda(x) + t(s(x) - s^*_{\lambda,1}(x)))\, dx, \end{aligned}$$

or

$$A = \|g - r\|_{L_1} - \|g\|_{L_1} = \int_0^1 dt \int_{-x_n}^{x_n} -r(x) \operatorname{sgn}(g(x) - tr(x))\, dx.$$

The differentiation above is valid if the set where $s^*_{\lambda,1} - \lambda$ vanishes does not have positive Lebesgue measure. If $s^*_{\lambda,1} \equiv \lambda$ on the interval $[x_0, x_n]$, for example, we can make the same computations, but restrict the integration to those subintervals of $[x_0, x_n]$ where $s^*_{\lambda,1} \not\equiv s$. In spite of this, the last formulas for A will be true, because $r(x) = s^*_{\lambda,1}(x) - s(x)$.

Since $\dfrac{\partial}{\partial \alpha_k}\Phi(\boldsymbol{\alpha}^*) = 0$ for $k \neq 0$ and $\alpha_0^* = \alpha_0$, it follows that

$$\int_{-x_n}^{x_n} r(x) \operatorname{sgn}(g(x))\, dx = 0,$$

and hence

$$A = \int_0^1 dt \int_{-x_n}^{x_n} -r(x)[\operatorname{sgn}(g(x) - tr(x)) - \operatorname{sgn}(g(x))]\, dx.$$

(Note that $-r[\operatorname{sgn}(g - tr) - \operatorname{sgn}(g)] \geq 0$ for any reals r and g and $t \geq 0$.) If the function g vanishes identically in the interval $[x_{k-1}, x_{k+1}]$ ($[x_{k-1}, x_k]$ if $k = n$), then

$$A \geq \int_0^1 dt \int_{x_{k-1}}^{x_{k+1}} |r(x)|\, dx \geq \frac{h}{2}|r(x_k)|$$

and the lemma is proved.

Thus, we suppose that the function g is not identically zero in the interval $[x_{k-1}, x_{k+1}]$ ($g(x_k)) \neq 0$) and we consider two cases:

(i) $\operatorname{sgn}(r(x_k)) = \operatorname{sgn}(g(x_k))$ and (ii) $\operatorname{sgn}(r(x_k)) = -\operatorname{sgn}(g(x_k))$.

CASE (i). If $k = n$, then

$$A \geq \int_{1/2}^{1} \frac{dt}{t} \int_{z_n}^{x_n} -tr(x)[\operatorname{sgn}(g(x) - tr(x)) - \operatorname{sgn}(g(x))]\, dx,$$

where $z_n = x_{n-1} + h/\sqrt{2}$ is the zero of g in this interval (see Lemma 2.3). Since, by assumption,

$$|r(x_n)| > 4|s^*_{\lambda,1}(x_1) - \lambda(x_1)| = 4 \max_{x \in ([x-n, x_n]} |s^*_{\lambda,1}(x) - \lambda(x)|, \tag{3.2}$$

it follows that

$$|tr(x)| > |g(x)|, \quad t \geq \frac{1}{2},$$

and also

$$\operatorname{sgn}(tr(x)) = \operatorname{sgn}(g(x))$$

for $x \in [z_n, x_n]$. Thus

$$A \geq \int_{z_n}^{x_n} |r(x)|\, dx \geq \frac{1}{2}|r(x_n)|(x_n - z_n) = \frac{h}{2}|r(x_n)|(1 - \frac{1}{\sqrt{2}}),$$

which yields (3.1). If $k < n$ it is easy to see from (3.2) that

$$A \geq \int_{1/2}^{1} dt \int_{x_k}^{x_k + h/3} 2|r(x)|\, dx \geq \frac{h}{6}|r(x_k)|.$$

CASE (ii). In this case $A \geq \int_{1/2}^{1} dt \int_{u_k}^{z_k} |r(x)|\, dx$, where u_k is the intercept point of the graph of g and the line passing through the points $(x_{k-1}, r(x_k)/2)$ and $(x_k, r(x_k)/2)$. Since $z_k - x_{k-1} \geq h/\sqrt{2}$, we have

$$A \geq |r(z_k)|\frac{z_k - u_k}{2} = |r(z_k)|\frac{(z_k - u_k)^2}{2(x_k - u_k)} \geq h\left(\frac{\sqrt{2} - 2}{2}\right)^2 |r(z_k)|.$$

This proves Lemma 3.1. □

Lemma 3.2. *For any function $f \in L_p[x_{-n}, x_n]$, $p > 1$, and any polygonal line $s \in S_\Delta$ the following inequalities hold:*

(i) $\|s - s_{f,p}\|_{L_p}^2 \le c(p)\left(\|f - s\|_{L_p}^p - \|f - s_{f,p}\|_{L_p}^p\right)$ *for* $p \ge 2$;

(ii) $$\frac{\|s - s_{f,p}\|_{L_2}^2}{(\|f - s\|_C + \|s - s_{f,p}\|_C)^{2-p}} \le c(p)\left(\|f - s\|_{L_p}^p - \|f - s_{f,p}\|_{L_p}^p\right)$$ *for* $1 < p < 2$,

where $s_{f,p}$ is the best L_p-polygonal line approximant of f.

The proof is the same as that of Theorem 3.5 in [2], so we omit the details.

4. Gibbs phenomenon for a class of jump functions

Using basically the same argument as in the proof of Theorem 4.1 in [2] (with Lemma 3.1 in the case $p = 1$ and Lemma 3.2 when $p > 1$) we obtain the following extension of our results to a class of functions having a jump discontinuity at $x = 0$.

Theorem 4.1. *Suppose that the function*

$$f(x) - \frac{f(0^+) - f(0^-)}{2}\lambda(x)$$

is absolutely continuous for $x \in [-1, 1]$, where $\lambda(x) = \operatorname{sgn}(x)$ and let $s_{n,p,f}$ and $s_{n,p,\lambda}$, $p \ge 1$, denote the best L_p–polygonal line approximations with knots at $\{i/n\}_{i=-n}^{n}$ of the functions f and λ, respectively, over $[-1, 1]$. (In the case $p = 1$ we choose the polygonal line $s_{n,1,\lambda}$ so that $s_{n,1,\lambda}(0) = s_{n,1,f}(0)$.) Then we have

$$\lim_{n\to\infty} \|s_{n,p,f} - f - \frac{f(0^+) - f(0^-)}{2}(s_{n,p,\lambda} - \lambda)\|_C = 0,$$

where $\|.\|_C$ denotes the uniform norm on $[-1, 1]$.

Remark. Since in the case $p = 1$ there is no Gibbs phenomenon for the function λ, the same is true for the function f.

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