Numer. Math. (1997) 76: 489–513

Numerische Mathematik

Electronic Edition

Approximation of conformal mappings of annular regions*

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Received February 2, 1996

Summary. In this paper we examine the convergence rates in an adaptive version of an orthonormalization method for approximating the conformal mapping f of an annular region A onto a circular annulus. In particular, we consider the case where f has an analytic extension in compl$(\overline{A})$ and, for this case, we determine optimal ray sequences of approximants that give the best possible geometric rate of uniform convergence. We also estimate the rate of uniform convergence in the case where the annular region A has piecewise analytic boundary without cusps. In both cases we also give the corresponding rates for the approximations to the conformal module of A.

Mathematics Subject Classification (1991): 30C30, 65E05, 41A20

1 Introduction

Let A be an annular region of the complex plane bounded by two piecewise analytic Jordan curves Γ_{e} and Γ_{i}, where Γ_{i} is interior to Γ_{e}, and denote the exterior of Γ_{e} by Ω (so that $\infty \in \Omega$) and the interior of Γ_{i} by G. Also, assume that $0 \in G$, let z_0 be a fixed point on Γ_{i} and denote by f the conformal mapping of A onto a circular annulus $E := \{w : 1 < |w| < M\}$, so that $f(z_0) = 1$. Here the outer radius M of E is uniquely determined by A and is called the *conformal module* of A.

* Research supported, in part, by a University of Cyprus research grant
** Research done in partial fulfillment of Ph.D. degree at the University of South Florida
*** Research supported, in part, by NSF grant DMS 950-1130
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The work of this paper is concerned with certain aspects of the convergence theory of an orthonormalization method (ONM) which was studied by Papamichael et al [8], [9], [10], [11] and [12], for the purpose of approximating the conformal mapping f and the conformal module M. The details of the basic ONM are as follows:

Let $L_2(A)$ be the Hilbert space of all square integrable functions that are analytic in A, with inner product

$$(u,v) := \iint_A u(z)\overline{v(z)}d\sigma_z \tag{1.1}$$

(where $d\sigma_z$ denotes 2-dimensional Lebesgue measure) and corresponding norm $\|u\|_2 := (u,u)^{\frac{1}{2}}$. Also let $L_2^S(A)$ denote the set

$$L_2^S(A) := \{u : u \in L_2(A) \text{ and } u \text{ has a single-valued indefinite integral}\}.$$

The set $L_2^S(A)$ is, in fact, a closed subspace of $L_2(A)$ and, therefore, is itself a Hilbert space with inner product (1.1). Next, let the functions g and H be defined respectively by

$$g(z) := \log f(z) - \log z \tag{1.2}$$

and

$$H(z) := g'(z) = \frac{f'(z)}{f(z)} - \frac{1}{z}, \tag{1.3}$$

so that g is a branch analytic and single-valued in A, $H \in L_2^S(A)$, and

$$f(z) = z\,\mathrm{e}^{g(z)}. \tag{1.4}$$

With these notations, it is well known ([3, pp. 250–51], [10, p. 686] and [11, p. 101]) that the conformal module M of A is related to H by means of

$$M = \exp\left\{ \left(\frac{1}{i}\int_{\partial A} \frac{\log|z|}{z}dz - \|H\|_2^2 \right) / 2\pi \right\}. \tag{1.5}$$

Let $\mathscr{R}_{m,n}$ be the set of all rational functions of the form

$$R_{m,n}(z) := \sum_{\substack{k=-m \\ k\neq -1}}^{n} a_k z^k. \tag{1.6}$$

Then, the basic ONM consists of the following steps:

(i) Orthonormalize the set of monomials $\{z^k\}_{k=-m}^n$, $k \neq -1$, by means of the Gram-Schmidt process in order to obtain the orthonormal set $\{\eta_k\}_{k=-m}^n$, $k \neq -1$.

(ii) Approximate the function H by the Fourier sum

$$R_{m,n}^*(z) := \sum_{\substack{k=-m \\ k\neq -1}}^{n} (H,\eta_k)\eta_k = \sum_{\substack{k=-m \\ k\neq -1}}^{n} a_k^* z^k, \tag{1.7}$$

where the coefficients (H,η_k) and hence the coefficients a_k^* can be computed, without the explicit knowledge of H, with the help of the formula

$$(\eta_k, H) = \mathrm{i} \int_{\partial A} \eta_k \log |z| dz ; \tag{1.8}$$

see e.g. [3, pp. 249–50] and [11, p. 101]. (The Fourier sum $R^*_{m,n}$ is of course the best approximant to H in $L^S_2(A)$ out of the subspace $\mathscr{R}_{m,n}$; i.e.,

$$\|H - R^*_{m,n}\|_2 = \inf_{R_{m,n} \in \mathscr{R}_{m,n}} \|H - R_{m,n}\|_2 \ .) \tag{1.9}$$

(iii) Using (1.3)–(1.5), approximate the mapping function f and the conformal module M respectively by

$$f_{m,n}(z) := \frac{z}{z_0} \exp \left(\int_{z_0}^{z} R^*_{m,n}(\zeta) d\zeta \right) \tag{1.10}$$

and

$$M_{m,n} := \exp \left\{ \left(\frac{1}{\mathrm{i}} \int_{\partial A} \frac{\log |z|}{z} dz - \|R^*_{m,n}\|_2^2 \right) / 2\pi \right\} . \tag{1.11}$$

The results of many numerical experiments ([8], [9], [10], [11] and [12]) suggest that (at least in the diagonal case $m = n$) uniform convergence of $f_{m,n}$ to f holds on $\overline{A}$, under some mild assumptions on the geometry of A. On the other hand, there is no adequate theory available to justify the ONM except for the case when f is analytic on the boundary of A (see [3, p. 250] and [12, p. 654]). Even in the latter case, the available theory concerns "diagonal" approximants with "equal" number of positive and negative powers of z, although it is reasonable to expect that this diagonal selection of powers need not always be the best choice.

In this paper we seek to overcome some of the above shortcomings in the theory of the ONM. In particular, we consider the case where the function H can be extended analytically in compl($\overline{A}$) and, for this case, we derive optimal ray sequences of approximants to H. More specifically, we derive (in terms of the analytic continuation properties of H) the asymptotically optimal proportion of positive and negative powers of z in the best approximations $R^*_{m,n}$ to H. We also study the case where f is not analytic on ∂A and fill partially the gap in the theory of the ONM, by showing that uniform convergence on $\overline{A}$ holds if A is bounded by piecewise analytic curves without cusps. Finally, we consider the convergence of the ONM approximants to the conformal module M and present results that provide theoretical justification for certain experimental observations concerning the quality of these approximants.

The organization of the paper is as follows: In Sect. 2, we state (without proofs) our main results. In Sect. 3, we describe an adaptive version of the ONM, based on the choice of optimal ray sequences of approximants, and also present the results of some numerical experiments illustrating the application of this adaptive method. Finally, in Sect. 4 we present the proofs of the results of Sect. 2. Here, our analysis is based mainly on methods of proof used in earlier papers ([1], [5], [6] and [7]) in connection with the problem of approximating, by means of Bieberbach polynomials, the conformal mapping of a simply-connected domain onto a disk.

2 Main results

2.1 Analytic case – optimal ray sequences

Let ϕ be the conformal mapping of $G := \mathrm{Int}(\Gamma_{\rm i})$ onto the unit disk $D := \{w : |w| < 1\}$, normalized by $\phi(0) = 0$ and $\phi'(0) > 0$, and let Φ denote the conformal mapping of the unbounded domain $\Omega := \mathrm{Ext}(\Gamma_{\rm e})$ onto the exterior of the unit circle $D' := \{w : |w| > 1\}$, normalized by $\Phi(\infty) = \infty$ and $\lim_{z\to\infty} \Phi(z)/z > 0$. Next, assume that the function H of (1.3) is analytic on $\overline{A}$ and suppose that its "nearest" singularities in G and in Ω are situated on the level curves $L_{r_{\rm i}} := \{z : |\phi(z)| = r_{\rm i},\ 0 < r_{\rm i} < 1\}$ and $L_{r_{\rm e}} := \{z : |\Phi(z)| = r_{\rm e},\ 1 < r_{\rm e} < \infty\}$, respectively. In other words, we assume that the function H is analytic in the annular region

$$A_H := \mathrm{Ext}(L_{r_{\rm i}}) \cap \mathrm{Int}(L_{r_{\rm e}}), \tag{2.1}$$

bounded by the curves $L_{r_{\rm i}}$ and $L_{r_{\rm e}}$, and has singularities on each of these curves.

We now consider "ray sequences" $\{R_{m(j),n(j)}\}_{j=1}^{\infty}$ of rational functions of the form (1.6) such that

$$m(j) \to \infty,\ \ n(j) \to \infty,\ \ \text{as } j \to \infty, \tag{2.2}$$

and the following limit exists

$$\lim_{j\to\infty} \frac{m(j)}{m(j) + n(j)} =: \alpha, \quad 0 \le \alpha \le 1. \tag{2.3}$$

We also assume that there exists a constant $c > 0$ such that

$$|m(j + 1) - m(j)| < c \ \text{ and } \ |n(j + 1) - n(j)| < c, \tag{2.4}$$

for all $j = 1,\ 2,\ \ldots$.

The result of the following theorem shows that the optimal proportion of positive and negative powers of z in the best approximation $R^*_{m,n}$ to H can be described asymptotically in terms of the location of the nearest singularities of H.

Theorem 2.1. *Let H be analytic on $\overline{A}$ and let $R^*_{m,n}$ be the best $L_2^{\rm S}(A)$ approximation to H out of $\mathscr{R}_{m,n}$. Then, a ray sequence $\{R^*_{m(j),n(j)}\}_{j=1}^{\infty}$, satisfying* (2.2)–(2.4)*, is optimal in the sense of geometric convergence rate if and only if*

$$\lim_{j\to\infty} \frac{m}{m + n} = \frac{\log r_{\rm e}}{\log r_{\rm e} - \log r_{\rm i}} =: \alpha^*. \tag{2.5}$$

In this case,

$$\limsup_{j\to\infty} \|f - f_{m,n}\|_{\infty}^{1/(m+n)} = \limsup_{j\to\infty} \left\|H - R^*_{m,n}\right\|_{\infty}^{1/(m+n)} = r_{\rm e}^{\alpha^* - 1} = r_{\rm i}^{\alpha^*}. \tag{2.6}$$

Furthermore, an optimal ray sequence $\{R^*_{m,n}\}$ and the corresponding sequence of approximants $\{f_{m,n}\}$ to the mapping function f converge locally uniformly in the annular region A_H.

In the above and in what follows $\|\cdot\|_\infty$ denotes the uniform norm on $\overline{A}$, i.e. for any function h analytic in A and bounded on $\overline{A}$,

$$\|h\|_\infty := \sup_{z\in\overline{A}} |h(z)| \ .$$

Also, the statement that the rate of geometric convergence in (2.6) is optimal means that for any ray sequence $\{R_{m(j),n(j)}\}_{j=1}^\infty$ satisfying (2.2)–(2.4),

$$\limsup_{j\to\infty} \|H - R_{m,n}\|_\infty^{1/(m+n)} \geq r_i^{\alpha^*}.$$

The following result gives the corresponding estimate of the error in the ONM approximation to the conformal module M of A.

Theorem 2.2. *Let $\{M_{m(j),n(j)}\}_{j=1}^\infty$ be a sequence of approximants to M corresponding to an optimal ray sequence $\{R^*_{m(j),n(j)}\}_{j=1}^\infty$. Then, this sequence of approximants tends monotonically (with $m+n$) to M from above, and*

$$\limsup_{j\to\infty}\{M_{m,n} - M\}^{1/(m+n)} = r_e^{2(\alpha^*-1)} = r_i^{2\alpha^*}. \tag{2.7}$$

2.2 Piecewise analytic boundary

Here we assume that the boundary components Γ_i and Γ_e of A are piecewise analytic Jordan curves without cusps. This means that each of Γ_i and Γ_e consist of a finite number of analytic arcs, where two adjacent arcs meet at a corner and form there an interior (with respect to A) angle $\nu\pi$, with $0 < \nu < 2$. Thus, in each case, the exterior (with respect to A) angle is $\lambda\pi$ with $\nu + \lambda = 2$.

Our next result extends Theorem 1 of [7] (concerning the convergence of Bieberbach polynomials for approximating the conformal maps of simply-connected regions) to the mapping of annular regions.

Theorem 2.3. *Suppose that the boundary components Γ_i and Γ_e of A are piecewise analytic without cusps and let $\lambda_i\pi$ and $\lambda_e\pi$ $(0 < \lambda_i, \lambda_e < 2)$ be respectively the smallest exterior (with respect to A) angles among all joints on Γ_i and Γ_e. Suppose, in addition, that $m,\ n \in \mathbb{N}$ satisfy*

$$0 < c_1 \leq \frac{m}{n} \leq c_2 < \infty, \tag{2.8}$$

for some fixed numbers c_1 and c_2. Then, with $f_{m,n}$ defined as in (1.10), *there exists a constant K, depending only on c_1, c_2 and the geometry of A, such that*

$$\|f - f_{m,n}\|_\infty \leq K \frac{\log\max(m,n)}{\min(m^{\gamma_i}, n^{\gamma_e})}, \tag{2.9}$$

with $\gamma_i = \lambda_i/(2-\lambda_i)$ and $\gamma_e = \lambda_e/(2-\lambda_e)$.

Remark. Because of (2.8), the result of the theorem is equivalent to the estimate

(2.10) $$\|f - f_{m,n}\|_\infty = O\left(\frac{\log n}{n^\gamma}\right) = O\left(\frac{\log m}{m^\gamma}\right),$$

where $\gamma := \min(\gamma_i, \gamma_e)$. We note that the exponent γ cannot be increased in general. This is so because if $\gamma > \min(\gamma_i, \gamma_e)$, then the converse theorem argument of [14] would imply that $f \in \mathrm{Lip}\,\beta$ for some $\beta > (2 - \min(\lambda_i, \lambda_e))^{-1}$ and would contradict the known boundary behavior of f (see Sect. 4, Lemma 4.4). It follows that Theorem 2.3 is sharp up to the logarithmic factor in the numerator of (2.9).

Theorem 2.4. *Suppose that the annular region A satisfies the conditions of Theorem* 2.3*. Then there exists a constant K, depending only on the geometry of A, such that*

(2.11) $$0 \le M_{m,n} - M \le K \frac{\log \max(m,n)}{\min(m^{2\gamma_i}, n^{2\gamma_e})}, \quad m,\ n = 2,\ 3,\ \ldots\ ,$$

where $\gamma_i = \lambda_i/(2 - \lambda_i)$ *and* $\gamma_e = \lambda_e/(2 - \lambda_e)$.

We end this section by noting that the results of Theorems 2.1–2.4 confirm the experimental observation ([9, p. 47], [10, Sect. 5] and [11, Sect. 4]) that the ONM approximations to the conformal module M are more accurate than the corresponding approximations to the mapping function f.

3 Adaptive ONM – numerical examples

The results of Theorem 2.1 suggest a process for "balancing" the boundary errors

(3.1) $$\max_{z \in \Gamma_i} |f(z) - f_{m,n}(z)| \quad \text{and} \quad \max_{z \in \Gamma_e} |f(z) - f_{m,n}(z)|.$$

In this section we show how this idea can be used in order to develop an adaptive version of the classical ONM, in cases when the function $H(z) := \log f(z) - \log z$ can be extended analytically to an open set containing $\overline{A}$. We also present two numerical examples illustrating the application of the adaptive ONM and the convergence results of Sect. 2.1.

We let $\mathscr{E}_i^{(m,n)}$ and $\mathscr{E}_e^{(m,n)}$ denote estimates of the maximum error in modulus in the approximation $f_{m,n}$ to f on the boundary curves Γ_i and Γ_e respectively. These are obtained, as indicated in [10], [11] and [12], by means of

(3.2) $$\mathscr{E}_i^{(m,n)} := \max_j |1 - |f_{m,n}(z_j^i)|\ |$$

and

(3.3) $$\mathscr{E}_e^{(m,n)} := \max_j |M_{m,n} - |f_{m,n}(z_j^e)|\ |,$$

where $\{z_j^i\}$ and $\{z_j^e\}$ are two sets of "boundary test points" on Γ_i and Γ_e respectively. (We expect (3.3) to be a reasonable estimate of the actual error in modulus on Γ_e because, as we remarked at the end of Sect. 2, the approximations $\{M_{m,n}\}$ to M are more accurate than the corresponding approximations $\{f_{m,n}\}$ to f; see Theorems 2.1–2.4 and also Lemma 4.3 in Sect. 4.) Then, the *adaptive ONM algorithm* can be described as follows:

1. Set $m := m_{\min}$ and $n := n_{\min}$, where $m_{\min}$ and $n_{\min}$ denote respectively the minimum number of negative and positive powers of z to be used in the basis set.
2. Compute the ONM approximations $f_{m,n}$ to f and $M_{m,n}$ to M, using the basis set

$$\{z^k\}_{k=-m}^{n}, \; k \neq -1. \tag{3.4}$$

3. Compute the error estimates $\mathscr{E}_{\mathrm{i}}^{(m,n)}$ and $\mathscr{E}_{\mathrm{e}}^{(m,n)}$, by means of (3.2)–(3.3).
4. If $\mathscr{E}_{\mathrm{i}}^{(m,n)} < \mathscr{E}_{\mathrm{e}}^{(m,n)}$ then:
   – Introduce into the basis set (3.4) the next positive power of z, i.e. set $n := n + 1$.

   Otherwise:
   – Introduce into the basis set (3.4) the next negative power of z, i.e. set $m := m + 1$.
5. Check the termination criterion and either terminate the process or go to Step 2.

Regarding Step 5, the procedure is terminated when after a certain pair (m, n), due to the numerical instability of the Gram-Schmidt process, the maximum error $\max(\mathscr{E}_{\mathrm{i}}^{(m,n)}, \mathscr{E}_{\mathrm{e}}^{(m,n)})$ does not decrease any further.

In presenting the numerical results we make use of the following notations:

– ζ_{i} and ζ_{e}: These denote the locations of the nearest singularities of H in G and Ω respectively.
– r_{i} and r_{e}: As in Sect. 2, these denote the level indices of the nearest singularities of the function H, i.e. $r_{\mathrm{i}} := |\phi(\zeta_{\mathrm{i}})|$ and $r_{\mathrm{e}} := |\Phi(\zeta_{\mathrm{e}})|$. (The values of r_{i} and r_{e} are obtained by using either the exact mapping functions ϕ or Φ, when these are available, or are computed in approximate form by using the conformal mapping package BKMPACK of Warby [17].)
– α^*: This denotes the optimal ratio given by (2.5), i.e.

$$\alpha^* := \frac{\log r_{\mathrm{e}}}{\log r_{\mathrm{e}} - \log r_{\mathrm{i}}}. \tag{3.5}$$

– $\mathscr{E}_f^{(m,n)}$: This denotes the larger of the two error estimates (3.2) and (3.3), i.e.

$$\mathscr{E}_f^{(m,n)} = \max(\mathscr{E}_{\mathrm{i}}^{(m,n)}, \mathscr{E}_{\mathrm{e}}^{(m,n)}). \tag{3.6}$$

– $\mathscr{E}_M^{(m,n)}$: This provides an estimate of the error in the approximation $M_{m,n}$ to the conformal module M by means of

$$\mathscr{E}_M^{(m,n)} = M_{\widehat{m},\widehat{n}} - M_{m,n}, \tag{3.7}$$

where $\widehat{m} + \widehat{n}$ is the largest value of the sum of indices $m + n$ used in the adaptive ONM algorithm. (That is, this estimate is computed by using, instead of the exact value of M, the most accurate approximation to M that we have available.)

– δ_f and δ_M: These are respectively estimates of the orders of the errors (2.6) and (2.7). They are determined as follows: Let (m_1, n_1) and (m_2, n_2) be two pairs of indices such that

$$\frac{m_1}{m_1+n_1} \approx \frac{m_2}{m_2+n_2} \approx \alpha^*, \tag{3.8}$$

and let $\mathscr{E}_f^{(m_j,n_j)}$ and $\mathscr{E}_M^{(m_j,n_j)}$; $j = 1, 2$, denote the error estimates (3.6) and (3.7) corresponding to the pairs (m_j, n_j); $j = 1, 2$, respectively. Then, we assume that

$$\mathscr{E}_f^{(m,n)} \approx C_1\, r_{\mathrm{i}}^{\delta_f \alpha^*(m+n)} \quad \text{and} \quad \mathscr{E}_M^{(m,n)} \approx C_2\, r_{\mathrm{i}}^{\delta_M \alpha^*(m+n)}, \tag{3.9}$$

and seek to estimate δ_f and δ_M by means of the formulae

$$\delta_f = \frac{1}{\alpha^*(m_2+n_2-m_1-n_1)} \left[\frac{\log(\mathscr{E}_f^{(m_2,n_2)}/\mathscr{E}_f^{(m_1,n_1)})}{\log r_{\mathrm{i}}} \right] \tag{3.10}$$

and

$$\delta_M = \frac{1}{\alpha^*(m_2+n_2-m_1-n_1)} \left[\frac{\log(\mathscr{E}_M^{(m_2,n_2)}/\mathscr{E}_M^{(m_1,n_1)})}{\log r_{\mathrm{i}}} \right]; \tag{3.11}$$

see (2.6) and (2.7). (That is, from the theory we expect to obtain values $\delta_f \approx 1$ and $\delta_M \approx 2$.)

Example 3.1. Square with circular hole.

Let A be the annular region bounded internally by the circle

$$\Gamma_{\mathrm{i}} := \{z : |z| = 0.8\}$$

and externally by the square

$$\Gamma_{\mathrm{e}} := \{z : z = \pm 1 + \mathrm{i}y, |y| \le 1\} \cup \{z : z = x \pm \mathrm{i}, |x| \le 1\}.$$

Then, the nearest singularities of H occur at the points

$$\zeta_{\mathrm{i}} = 0.4 \quad \text{and} \quad \zeta_{\mathrm{e}} = 1.6;$$

see [11, p. 95]. Also, the mapping function ϕ is known trivially, i.e. $\phi(z) = 1.25z$. Therefore,

$$r_{\mathrm{i}} = 0.5.$$

Finally, the value of r_{e} is found in approximate form by using BKMPACK for the approximation of the mapping function Φ. In this way we find that (correct to six decimal places) r_{e} is given by

$$r_{\mathrm{e}} = 1.413\,129.$$

Hence, from (3.5), α^* is given (correct to two decimal places) by

$$\alpha^* = 0.33\,.$$

Example 3.2. Ellipse with circular hole.

Let A be the annular region bounded internally by the circle

$$\Gamma_{\rm i} := \{z : |z| = 2/3\}$$

and externally by the ellipse

$$\Gamma_{\rm e} := \{z : z = x + {\rm i}y,\ (x+1)^2/4 + y^2 = 1\}\ .$$

In this case, the nearest singularities of H occur at the points

$$\zeta_{\rm i} = 0.113\,004\,586 + {\rm i}\,0.279\,182\,239 \quad\text{and}\quad \zeta_{\rm e} = 4/(9\overline{\zeta}_{\rm i}),$$

where $\zeta_{\rm i}$ is correct to all figures quoted; see [11, p. 100]. Also, the mapping functions ϕ and Φ are known exactly, i.e.

$$\phi(z) = 1.5z \quad\text{and}\quad \Phi(z) = \frac{z+1}{3}\left\{1 + \left(1 - \frac{3}{(z+1)^2}\right)^{1/2}\right\};$$

see e.g. [2, p. 225]. In this way we find that (correct to six decimal places) $r_{\rm i}$ and $r_{\rm e}$ are given by

$$r_{\rm i} = 0.451\,778 \quad\text{and}\quad r_{\rm e} = 1.409\,707.$$

Hence, from (3.5), α^* is given (correct to two decimal places) by

$$\alpha^* = 0.30\,.$$

The results for Examples 3.1 and 3.2 are given in Tables 1, 2 and 3, 4, respectively. All these results were obtained by using the adaptive ONM algorithm with starting pairs of indices $(m_{\min}, n_{\min}) = (9, 7)$ for Example 3.1, and $(m_{\min}, n_{\min}) = (4, 4)$ for Example 3.2.

Tables 1 and 3 contain, in each case, the values of the pairs (m, n) that were chosen by the adaptive process at each step of the algorithm, the corresponding values of the ratio $m/(m+n)$ and the values of the boundary errors $\mathscr{E}_{\rm i}^{(m,n)}$ and $\mathscr{E}_{\rm e}^{(m,n)}$. These results illustrate how the algorithm seeks to balance the magnitude of the two errors (by introducing the "appropriate" power of z into the basis set) and show that, for the "optimal" values of (m, n) chosen by the algorithm, the ratios $m/(m+n)$ are (as predicted by Theorem 2.1) close to the value of α^*. The domain of Example 3.1 has 4-fold rotational symmetry about the origin. Because of this the corresponding approximations (1.7) take the following form:

$$R^*_{m,n}(z) := \sum_{\substack{k=-m(4)n \\ k\neq -1}} a_k^* z^k,$$

with

$$m := m(j) = 4j + 1\ ,\quad n := n(j) = 4j - 1;\ j = 1,\ 2,\ \ldots\ .$$

Table 1. Square with circular hole: Steps of the adaptive algorithm

(m,n)	$m/(m+n)$	$\mathscr{E}_{\mathrm{i}}^{(m,n)}$	$\mathscr{E}_{\mathrm{e}}^{(m,n)}$
(9, 7)	0.56	6.5e-04	3.7e-03
(9,11)	0.45	7.6e-05	6.1e-04
(9,15)	0.38	4.5e-05	1.5e-04
(9,19)	0.32	5.3e-05	6.6e-05
(9,23)	0.28	5.1e-05	4.3e-05
(13,23)	0.36	2.7e-06	7.1e-06
(13,27)	0.32	2.8e-06	3.2e-06
(13,31)	0.30	2.9e-06	2.2e-06
(17,31)	0.35	1.4e-07	3.4e-07
(17,35)	0.33	1.5e-07	1.6e-07
(17,39)	0.30	1.5e-07	1.1e-07
(21,39)	0.35	7.4e-09	1.7e-08
(21,43)	0.33	7.4e-09	8.1e-09
(21,47)	0.31	7.5e-09	5.7e-09
(25,47)	0.35	3.8e-10	9.3e-10
(25,51)	0.33	3.8e-10	4.3e-10
(25,55)	0.31	3.8e-10	3.0e-10
(29,55)	0.34	2.0e-11	5.1e-11
(29,59)	0.33	2.0e-11	2.4e-11
(29,63)	0.32	2.0e-11	1.7e-11
(33,63)	0.34	1.1e-12	3.6e-12

Tables 2 and 4 contain the ONM approximations $M_{m,n}$, for some of the optimal pairs (m,n) of Tables 1 and 3, together with the corresponding estimates δ_f and δ_M that measure respectively the orders of the errors in the approximations $f_{m,n}$ to f and $M_{m,n}$ to M. These estimates were computed from (3.10) and (3.11) by keeping the pair (m_1,n_1) fixed, as the initial pair in each table, and taking recursively (m_2,n_2) to be each subsequent pair shown in the table. As expected (see (2.6) and (2.7)), the values of δ_f and δ_M are, in all cases, close to 1 and 2, respectively.

Table 2. Square with circular hole: Orders of the errors $\mathscr{E}_f^{(m,n)}$ and $\mathscr{E}_M^{(m,n)}$

(m,n)	$M_{m,n}$	δ_f	δ_M
(9,19)	1.342 990 380 6369	–	–
(9,23)	1.342 990 374 7144	0.20	0.54
(13,23)	1.342 990 365 9443	1.20	2.05
(13,27)	1.342 990 365 6469	1.09	2.08
(13,31)	1.342 990 365 6320	0.85	1.66
(17,31)	1.342 990 365 6002	1.14	2.10
(17,35)	1.342 990 365 5994	1.09	2.08
(17,39)	1.342 990 365 5993	0.95	1.84
(21,39)	1.342 990 365 5992	1.12	2.10

Table 3. Ellipse with circular hole: Steps of the adaptive algorithm

(m,n)	$m/(m+n)$	$\mathscr{E}_{\mathrm{i}}^{(m,n)}$	$\mathscr{E}_{\mathrm{e}}^{(m,n)}$
(4, 4)	0.50	1.8e-02	4.5e-02
(4, 5)	0.44	1.0e-02	3.3e-02
(4, 6)	0.40	5.2e-03	1.5e-02
(4, 7)	0.36	5.7e-03	9.7e-03
(4, 8)	0.33	5.5e-03	9.5e-03
(4, 9)	0.31	5.4e-03	6.6e-03
(4,10)	0.29	5.6e-03	5.4e-03
(5,10)	0.33	6.5e-03	6.3e-03
(6,10)	0.38	1.5e-03	2.3e-03
(6,11)	0.35	1.6e-03	2.2e-03
(6,12)	0.33	1.5e-03	1.8e-03
(6,13)	0.32	1.5e-03	1.2e-03
(7,13)	0.35	6.7e-04	1.1e-03
(7,14)	0.33	6.7e-04	8.1e-04
(7,15)	0.32	6.7e-04	8.0e-04
(7,16)	0.30	6.7e-04	6.9e-04
(7,17)	0.29	6.8e-04	5.7e-04
(8,17)	0.32	3.9e-04	3.4e-04
(9,17)	0.35	4.6e-05	1.1e-04
(9,18)	0.33	4.8e-05	9.6e-05
(9,19)	0.32	4.6e-05	7.9e-05
(9,20)	0.31	4.6e-05	4.7e-05
(9,21)	0.30	4.6e-05	4.5e-05
(10,21)	0.32	5.9e-05	8.3e-05
(10,22)	0.31	5.9e-05	7.4e-05
(10,23)	0.30	5.9e-05	7.1e-05

4 Proofs

4.1 Proofs of Theorems 2.1 and 2.2

For the proofs of our main results we shall make use of several lemmas. The first of these is an analogue, for rational functions, of the well-known Bernstein-Walsh lemma for polynomials.

Lemma 4.1. *For any rational function of the form*

$$r_{m,n}(z) = \sum_{k=-m}^{n} a_k z^k \tag{4.1}$$

we have that

$$|r_{m,n}(z)| \le \|r_{m,n}\|_{\partial\Omega} \, |\Phi(z)|^n \,, \quad z \in \Omega, \tag{4.2}$$

and

$$|r_{m,n}(z)| \le \frac{\|r_{m,n}\|_{\partial G}}{|\phi(z)|^m}\,, \quad z \in G, \tag{4.3}$$

where $\|\cdot\|_{\partial\Omega}$ *and* $\|\cdot\|_{\partial G}$ *denote the uniform norms on* $\partial\Omega$ *and* ∂G *respectively.*

Table 4. Ellipse with circular hole: Orders of the errors $\mathcal{E}_f^{(m,n)}$ and $\mathcal{E}_M^{(m,n)}$

(m,n)	$M_{m,n}$	δ_f	δ_M
(4, 9)	1.419 811 519	–	–
(4,10)	1.419 765 672	.70	1.87
(6,12)	1.419 699 037	1.08	1.82
(6,13)	1.419 694 052	1.03	1.81
(7,14)	1.419 686 002	1.10	2.36
(7,15)	1.419 686 000	.98	2.10
(7,16)	1.419 685 652	.94	2.01
(7,17)	1.419 685 406	.87	1.93
(8,17)	1.419 685 112	.98	1.94
(9,18)	1.419 684 686	1.26	2.33
(9,19)	1.419 684 675	1.23	2.24
(9,20)	1.419 684 652	1.29	2.32
(9,21)	1.419 684 651	1.22	2.19
(10,21)	1.419 684 641	1.01	2.31
(10,22)	1.419 684 636	.99	2.75
(10,23)	1.419 684 635	–	–

Proof. The function $r_{m,n}(z)/[\Phi(z)]^n$ is analytic in Ω and continuous on $\overline{\Omega}$. Thus, by the maximum modulus principle,

$$\frac{|r_{m,n}(z)|}{|\Phi(z)|^n} \leq \left\| \frac{r_{m,n}}{\Phi^n} \right\|_{\partial\Omega} = \|r_{m,n}\|_{\partial\Omega}\,, \quad z \in \Omega,$$

from which (4.2) follows.

Similarly, the function $r_{m,n}(z)[\phi(z)]^m$ is analytic in G and continuous on $\overline{G}$. Hence,

$$|r_{m,n}(z)| \cdot |\phi(z)|^m \leq \|r_{m,n}\phi^m\|_{\partial G} = \|r_{m,n}\|_{\partial G}\,, \quad z \in G,$$

from which (4.3) follows. □

Lemma 4.2. *For any rational function of the form* (4.1) *there exists a constant* K*, depending only on the geometry of A, such that*

$$\|H - (r_{m,n} - a_{-1}/z)\|_2 \leq K\|H - r_{m,n}\|_2.$$

Proof. Let $\Gamma := \{z : |f(z)| = r\}$, where $r \in (1, M)$ is fixed. Then, since the indefinite integral of H is single-valued in A,

$$\int_\Gamma \{H(t) - r_{m,n}(t)\}dt = -2\pi \mathrm{i} a_{-1}.$$

On the other hand by [4, p. 4]

$$\begin{aligned} \left| \int_\Gamma \{H(t) - r_{m,n}(t)\}dt \right| &\leq |\Gamma| \max_{t\in\Gamma} |H(t) - r_{m,n}(t)| \\ &\leq \frac{|\Gamma|}{\sqrt{\pi}\mathrm{dist}(\Gamma, \partial A)} \|H - r_{m,n}\|_2. \end{aligned}$$

Thus,

$$\begin{aligned}\|H-(r_{m,n}-a_{-1}/z)\|_2 &\le \|H-r_{m,n}\|_2+|a_{-1}|\cdot\|1/z\|_2\\ &\le K\|H-r_{m,n}\|_2. \quad \square\end{aligned}$$

Proof of Theorem 2.1. Our first task is to show that

$$\limsup_{j\to\infty}\|H-R^*_{m,n}\|_\infty^{1/(m+n)} \ge \max\left(r_{\mathrm{e}}^{\alpha-1}, r_{\mathrm{i}}^{\alpha}\right) \tag{4.4}$$

for any ray sequence satisfying (2.2)–(2.4). We do this as follows:

Suppose that

$$\limsup_{j\to\infty}\|H-R^*_{m,n}\|_\infty^{1/(m+n)} < \max\left(r_{\mathrm{e}}^{\alpha-1}, r_{\mathrm{i}}^{\alpha}\right), \tag{4.5}$$

and assume that the max in (4.5) is equal to r_{i}^{α}, so that

$$\limsup_{j\to\infty}\|H-R^*_{m,n}\|_\infty^{1/m} < r_{\mathrm{i}}. \tag{4.6}$$

Next, suppose that the value of lim sup in (4.6) is equal to $q<r_{\mathrm{i}}$ and let $\varepsilon>0$ be such that $q+\varepsilon<r_{\mathrm{i}}$. Then, by making use of Lemma 4.1 (and Inequalities (2.4)), we find that for $z\in L_{q+\varepsilon}:=\{z:|\phi(z)|=q+\varepsilon\}$

$$\begin{aligned}&\sum_{j=1}^{\infty}|R^*_{m(j+1),n(j+1)}(z)-R^*_{m(j),n(j)}(z)|\\ &\le \sum_{j=1}^{\infty}\|R^*_{m(j+1),n(j+1)}-R^*_{m(j),n(j)}\|_\infty (q+\varepsilon)^{-\max(m(j),m(j+1))}\\ &\le \sum_{j=1}^{\infty}\left\{\|H-R^*_{m(j),n(j)}\|_\infty+\|H-R^*_{m(j+1),n(j+1)}\|_\infty\right\}(q+\varepsilon)^{-\max(m(j),m(j+1))}\\ &\le K_1\sum_{j=1}^{\infty}\left(q+\frac{\varepsilon}{2}\right)^{\min(m(j),m(j+1))}(q+\varepsilon)^{-\max(m(j),m(j+1))}\\ &\le K_2\sum_{j=1}^{\infty}\left(\frac{q+\frac{\varepsilon}{2}}{q+\varepsilon}\right)^{m(j)}<\infty.\end{aligned}$$

This shows that the series

$$\sum_{j=1}^{\infty}\left\{R^*_{m(j+1),n(j+1)}(z)-R^*_{m(j),n(j)}(z)\right\}+R^*_{m(1),n(1)}(z) \tag{4.7}$$

converges uniformly on $L_{q+\varepsilon}$. Since (by (4.6)) the series converges uniformly to H on $\overline{A}$, it follows that it converges in the region between Γ_{i} and $L_{q+\varepsilon}$. In other words, we have shown that (4.7) converges to an analytic continuation of H through the level curve $L_{r_{\mathrm{i}}}$, which is a contradiction. A similar argument can be used to establish a contradiction in the case when the max in (4.5) is equal to $r_\mathrm{e}^{\alpha-1}$. This proves (4.4). (Observe that the right hand side of (4.4) takes its minimal value when α is given by (2.5), i.e. when α satisfies the equation $r_\mathrm{e}^{\alpha-1} = r_\mathrm{i}^{\alpha}$).

We shall now show that (2.6) holds for a ray sequence defined by (2.5). For this, we first note that the function H can be expressed as

$$H(z) = H_\mathrm{e}(z) + H_\mathrm{i}(z), \quad z \in A_H,$$

where H_e is analytic inside L_{r_e} and H_i is analytic outside L_{r_i} (including the point at ∞) and $H_\mathrm{i}(\infty) = 0$. Also, by the results of Walsh [16, pp. 75–80], there exist two sequences of polynomials $\{P_n(z)\}_{n=1}^{\infty}$, where $\deg P_n \leq n$, and $\{Q_m(t)\}_{m=1}^{\infty}$, where $\deg Q_m \leq m$, such that

$$\limsup_{n\to\infty} \|H_\mathrm{e} - P_n\|_{\partial\Omega}^{1/n} = \frac{1}{r_\mathrm{e}}, \tag{4.8}$$

and

$$\limsup_{m\to\infty} \|H_\mathrm{i} - Q_m\left(\frac{1}{z}\right)\|_{\partial G}^{1/m} = r_\mathrm{i}. \tag{4.9}$$

Further, from Lemma 4.2,

$$\begin{aligned}
\limsup_{j\to\infty} \|H - R_{m,n}^{*}\|_2^{1/(m+n)} &\leq \limsup_{j\to\infty} \left\| H - \left(P_n + Q_m\left(\frac{1}{z}\right)\right)\right\|_2^{1/(m+n)} \\
&\leq \limsup_{j\to\infty} \left\{ \|H_\mathrm{e} - P_n\|_\infty + \left\| H_\mathrm{i} - Q_m\left(\frac{1}{z}\right)\right\|_\infty \right\}^{1/(m+n)} \\
&\leq r_\mathrm{e}^{\alpha^*-1} = r_\mathrm{i}^{\alpha^*},
\end{aligned} \tag{4.10}$$

where, for the last step, we made use of (4.8), (4.9) and (2.5). Next, by using Lemma 4.1 and the estimate (cf. [4, p. 4])

$$|H(z) - R_{m,n}^{*}(z)| \leq \frac{1}{\sqrt{\pi}\,\mathrm{dist}(z, \partial A)} \|H - R_{m,n}^{*}\|_2, \quad z \in A,$$

we can show (with the help of series (4.7)) that

$$\limsup_{j\to\infty} \|H - R_{m,n}^{*}\|_\infty^{1/(m+n)} \leq \limsup_{j\to\infty} \|H - R_{m,n}^{*}\|_2^{1/(m+n)}$$

which, in view of (4.4) and (4.10), gives that

$$\limsup_{j\to\infty} \|H - R_{m,n}^{*}\|_\infty^{1/(m+n)} = \limsup_{j\to\infty} \|H - R_{m,n}^{*}\|_2^{1/(m+n)} = r_\mathrm{e}^{\alpha^*-1} = r_\mathrm{i}^{\alpha^*}. \tag{4.11}$$

In order to show that an optimal ray sequence $\{R_{m,n}^{*}\}_{j=1}^{\infty}$ converges to H locally uniformly in A_H, we consider again the series (4.7) and repeat the argument of the first part of our proof for z lying respectively on two level curves $L_{r_\mathrm{i}+\varepsilon} := \{z : |\phi(z)| = r_\mathrm{i} + \varepsilon\}$ and $L_{r_\mathrm{e}-\varepsilon} := \{z : |\Phi(z)| = r_\mathrm{e} - \varepsilon\}$, where $\varepsilon > 0$ is sufficiently small. The local uniform convergence of $\{R_{m,n}^{*}\}_{j=1}^{\infty}$ to H in A_H is then established by letting $\varepsilon \to 0$ in the two resulting estimates.

To complete the proof we recall that H is analytic and single-valued in the annular region A_H and that

$$f(z) = \frac{z}{z_0} \exp\left(\int_{z_0}^{z} H(\zeta)d\zeta\right), \tag{4.12}$$

where the integration is carried over a rectifiable path γ_z lying in A_H (see (1.2), (1.3) and (1.4)). Since for z belonging to any compact subset of A_H there is an upper bound for the length of γ_z, it follows from (4.12) and (1.10) that

$$\begin{aligned} |f(z) - f_{m,n}(z)| &= \left|\frac{z}{z_0}\right| \left|\exp\left(\int_{z_0}^{z} H(\zeta)d\zeta\right) - \exp\left(\int_{z_0}^{z} R^*_{m,n}(\zeta)d\zeta\right)\right| \\ &\le C_1 \left|\frac{z}{z_0}\right| \left|\int_{z_0}^{z} \left\{H(\zeta) - R^*_{m,n}(\zeta)\right\} d\zeta\right| \\ &\le C_1 \left|\frac{z}{z_0}\right| |\gamma_z| \max_{\zeta\in\gamma_z} |H(\zeta) - R^*_{m,n}(\zeta)|. \end{aligned}$$

This shows that $\{f_{m,n}\}_{j=1}^{\infty}$ also converges locally uniformly to f in A_H. Thus, by choosing arbitrary $z \in \overline{A}$ and $\gamma_z \subset \overline{A}$, we obtain from the previous estimate and (4.11) that

$$\limsup_{j\to\infty} \|f - f_{m,n}\|_{\infty}^{1/(m+n)} \le r_{\mathrm{i}}^{\alpha^*}.$$

To prove that equality holds, we recall that

$$g(z) - g(z_0) = \int_{z_0}^{z} H(\zeta)d\zeta, \ z \in A_H,$$

let

$$g_{m,n}(z) := \int_{z_0}^{z} R^*_{m,n}(\zeta)d\zeta, \ z \in A_H, \quad m, \ n = 2, \ 3, \ \ldots ,$$

and observe that for $z \in A_H$

$$C_2 \left|\frac{z}{z_0}\right| \left|\int_{z_0}^{z} \left\{H(\zeta) - R^*_{m,n}(\zeta)\right\} d\zeta\right| \le |f(z) - f_{m,n}(z)|.$$

This gives

$$\limsup_{j\to\infty} \|g - g(z_0) - g_{m,n}\|_{\infty}^{1/(m+n)} \le \limsup_{j\to\infty} \|f - f_{m,n}\|_{\infty}^{1/(m+n)}.$$

Finally, if we assume that

$$\limsup_{j\to\infty} \|f - f_{m,n}\|_{\infty}^{1/(m+n)} < r_{\mathrm{i}}^{\alpha^*},$$

then by applying the same argument as in the first part of this proof to the sequence of rationals $\{g_{m,n}\}_{j=1}^{\infty}$ (instead of $\{R^*_{m,n}\}_{j=1}^{\infty}$) we can show that this sequence converges to $g(z)-g(z_0)$ in a region that contains A_H in its interior. That is we can show that g, and hence H, are analytic on $\overline{A}_H$ which is a contradiction. □

Lemma 4.3. *For the ONM approximation $M_{m,n}$ to the conformal module M we have that*

$$0 \le M_{m,n} - M \sim \|H - R^*_{m,n}\|_2^2, \tag{4.13}$$

where $a \sim b$ means that there exist two constants K_1 and K_2 (that depend only on the geometry of the annular region A) such that $bK_1 \le a \le bK_2$.

Proof. One can readily see from (1.5) and (1.11) that

$$M_{m,n} - M \sim \|H\|_2^2 - \|R^*_{m,n}\|_2^2.$$

The result follows because, since $R^*_{m,n}$ is the Fourier sum of H,

$$\|H\|_2^2 - \|R^*_{m,n}\|_2^2 = \|H - R^*_{m,n}\|_2^2. \qquad \square$$

Proof of Theorem 2.2. The sequence of approximations $\{M_{m,n}\}$ to M, given by (1.11), decrease monotonically with $m+n$. This is so because, from (1.7),

$$\|R^*_{m,n}\|_2^2 = \sum_{\substack{k=-m\\k\ne-1}}^{n} |(H, \eta_k)|^2.$$

Therefore, the result of Theorem 2.2 follows immediately from Lemma 4.3 and (4.11). $\square$

4.2 Proofs of Theorems 2.3 and 2.4

We shall follow essentially the methods of proof used by Andrievskii [1] and Gaier [6], in connection with the use of Bieberbach polynomials for the conformal mapping of simply-connected domains. In particular, the proof of Theorem 2.3 involves the following three steps: (i) estimating the L_2-error in the approximation $R^*_{m,n}$ to H, using the extremal property (1.9); (ii) relating the norms $\|r_{m,n}\|_\infty$ and $\|r'_{m,n}\|_2$, where $r_{m,n}$ is a rational approximant of the form (4.1); (iii) estimating the uniform error in the approximation $f_{m,n}$ to f, thus proving (2.9). First, however, we present a lemma concerning the boundary behavior of the mapping function f and its derivatives. This extends a well-known result of Warschawski [18] (for the conformal mapping of simply-connected domains) to the mapping of annular regions. The lemma follows immediately from Warschawski's result through the localization principle, i.e. by considering a small simply-connected neighborhood of a boundary point.

Lemma 4.4. *With the hypotheses of Theorem 2.3, suppose that two analytic arcs, of either $\Gamma_{\rm i}$ or $\Gamma_{\rm e}$, meet at a point ζ and form there a corner of interior (with respect to A) angle $\nu\pi$, $0 < \nu \le 2$. Then, all the limits*

$$\lim_{z\to\zeta} (f(z) - f(\zeta))/(z-\zeta)^{\frac{1}{\nu}} = \Lambda_0, \ \ \Lambda_0 \ne 0, \tag{4.14}$$

and

(4.15) $$\lim_{z\to\zeta}(z-\zeta)^{n-\frac{1}{\nu}} f^{(n)}(z) = \Lambda_n,\ n = 1, 2, \ldots, \ \Lambda_1 \neq 0,$$

exist for unrestricted approach $z \to \zeta$, $z \in A$. Similarly, for the inverse mapping $\tau := f^{-1}$, all the limits

(4.16) $$\lim_{w\to f(\zeta)}(\tau(w)-\zeta)/(w-f(\zeta))^{\nu} = \Lambda_0',\ \Lambda_0' \neq 0,$$

and

(4.17) $$\lim_{w\to f(\zeta)}(w-f(\zeta))^{n-\nu}\tau^{(n)}(w) = \Lambda_n',\ n = 1, 2, \ldots, \ \Lambda_1' \neq 0,$$

exist for unrestricted approach $w \to f(\zeta)$, $w \in E$.

(I) The L_2-error in the approximation to H

Lemma 4.5. *Under the hypotheses of Theorem* 2.3, *there exists a constant K, depending only on the geometry of A, such that*

(4.18) $$\|H - R^*_{m,n}\|_2 \leq K \frac{\{\log \max(m,n)\}^{1/2}}{\min(m^{\gamma_i}, n^{\gamma_e})},\ m,\ n = 2, 3, \ldots,$$

where $\gamma_i = \lambda_i/(2-\lambda_i)$ and $\gamma_e = \lambda_e/(2-\lambda_e)$.

Proof. The proof consists of the following three main steps: (i) reducing the problem of estimating the L_2-error of the approximation to H to that of estimating the uniform error of an approximation to a modified function; (ii) estimating this uniform error by making use of results from the theory of uniform approximations; (iii) establishing the result of the lemma by making use of the extremal property of $R^*_{m,n}$.

(i) Let h denote the function

(4.19) $$h(z) := (H(z) + 1/z)\omega(z),$$

where

$$\omega(z) := \prod_{j=1}^{l}(z - z_j),$$

and the product extends over all corner points $z_j \in \partial A$, $j = 1, 2, \ldots, l$. Also, let $\nu_j\pi$, $0 < \nu_j < 2$, $j = 1, 2, \ldots, l$, be respectively the inner angles at the points z_j. Then, (1.3) and (4.15) imply that for $z \in A$, near z_j, $j = 1, 2, \ldots, l$,

$$|h(z)| = O(|z - z_j|^{\frac{1}{\nu_j}}),$$

and this in turn implies that h is continuous on $\overline{A}$.

Consider the set of rational functions of the form

(4.20) $$r_{m,n}(z) = \sum_{k=-m}^{n} a_k z^k,$$

and let $r^*_{m,n}$ be the best uniform approximant to h on $\overline{A}$ out of this set. Next, for $n \geq l$, let p_{l-1} be the Lagrange polynomial interpolating $r^*_{m,n}$ at the points $\{z_j\}_{j=1}^{l}$, i.e.

$$p_{l-1}(z) := \sum_{j=1}^{l} \frac{\omega(z)}{\omega'(z_j)(z-z_j)} r^*_{m,n}(z_j),$$

and define the function $\widetilde{r}_{m,n-l}$ by means of the equation

$$r^*_{m,n}(z) - p_{l-1}(z) = \omega(z)\, \widetilde{r}_{m,n-l}(z). \tag{4.21}$$

Our next task is to consider using the function $\widetilde{r}_{m,n-l} - 1/z$ in order to approximate H in the $L_2(A)$ norm. For this we first note that, since $h(z_j) = 0$, $j = 1, 2, \ldots, l$, and the degree of p_{l-1} is fixed,

$$\begin{aligned} |p_{l-1}(z)| &\leq \max_{1\leq j\leq l} |r^*_{m,n}(z_j)| \sum_{j=1}^{l} \frac{|\omega(z)|}{|\omega'(z_j)||z-z_j|} \\ &\leq C(B)\|h - r^*_{m,n}\|_\infty,\ z \in B, \end{aligned} \tag{4.22}$$

where $B \subset \mathbb{C}$ is an arbitrary compact set. Next, we choose a positive number $\delta_{m,n} < 1$ small enough so that each disk $S_j = \{z : |z - z_j| \leq \delta_{m,n}\}$, $j = 1, 2, \ldots, l$, is contained in the annular region $A_{m,n}$ bounded by the level curves $L_{1+1/n} := \{z : |\Phi(z)| = 1 + 1/n\}$ and $L_{1-1/m} := \{z : |\phi(z)| = 1 - 1/m\}$. (For this it is sufficient to take

$$\delta_{m,n} = o(\min(m^{-2},\ n^{-2})),\ m,\ n = 1, 2, \ldots ;$$

see [15, p. 181].)

Since $r^*_{m,n}(z) - p_{l-1}(z)$ is uniformly bounded on $\overline{A}$, by a constant independent of m and n, it follows (from Lemma 4.1) that the same is true for $\overline{A}_{m,n}$. Thus, by the maximum modulus principle,

$$\left|\widetilde{r}_{m,n-l}(z) \frac{\omega(z)}{z - z_j}\right| \leq \frac{C_1}{\delta_{m,n}},\ z \in S_j,$$

and therefore,

$$|\widetilde{r}_{m,n-l}(z)| \leq \frac{C_2}{\delta_{m,n}},\ z \in \bigcup_{j=1}^{l} S_j. \tag{4.23}$$

Let $s_j = \{z : |z - z_j| \leq \delta^2_{m,n}\}$, $j = 1, 2, \ldots, l$, and note that

$$\begin{aligned} \|H(z) - (\widetilde{r}_{m,n-l} - 1/z)\|_2^2 &= \sum_{j=1}^{l} \iint_{A\cap s_j} |H(z) + 1/z - \widetilde{r}_{m,n-l}(z)|^2 d\sigma_z \\ &+ \iint_{A\setminus \cup_{j=1}^{l} s_j} |H(z) + 1/z - \widetilde{r}_{m,n-l}(z)|^2 d\sigma_z, \end{aligned} \tag{4.24}$$

where, as before, $d\sigma_z$ denotes 2-dimensional Lebesgue measure. Then, for the integrals in (4.24), the triangle inequality gives that

$$\left(\iint_{A\cap s_j} |H(z)+1/z-\widetilde{r}_{m,n-l}(z)|^2 d\sigma_z\right)^{1/2} \le \left(\iint_{A\cap s_j} |H(z)+1/z|^2 d\sigma_z\right)^{1/2} + \left(\iint_{A\cap s_j} |\widetilde{r}_{m,n-l}(z)|^2 d\sigma_z\right)^{1/2},$$

$j=1,\ 2,\ \dots,\ l$, where from (4.15) and (1.3),

$$\iint_{A\cap s_j} |H(z)+1/z|^2 d\sigma_z \le C \iint_{A\cap s_j} |z-z_j|^{\frac{2}{\nu_j}-2} d\sigma_z = O(\delta_{m,n}^2),$$

and from (4.23),

$$\iint_{A\cap s_j} |\widetilde{r}_{m,n-l}(z)|^2 d\sigma_z = O(\delta_{m,n}^2).$$

Also, from (4.21) and (4.22),

$$\iint_{A\setminus\cup_{j=1}^l s_j} |H(z)+1/z-\widetilde{r}_{m,n-l}(z)|^2 d\sigma_z = \iint_{A\setminus\cup_{j=1}^l s_j} \left|\frac{h(z)-\widetilde{r}_{m,n-l}(z)\omega(z)}{\omega(z)}\right|^2 d\sigma_z$$
$$\le C\|h-r_{m,n}^*\|_\infty^2 \iint_{A\setminus\cup_{j=1}^l s_j} \frac{d\sigma_z}{|\omega(z)|^2},$$

where it is easy to see that

$$\iint_{A\setminus\cup_{j=1}^l s_j} \frac{d\sigma_z}{|\omega(z)|^2} \le C\int_{\delta_{m,n}^2}^1 \frac{dr}{r} = O(|\log\delta_{m,n}|).$$

Hence, by combining all the above estimates we find that

$$\|H-(\widetilde{r}_{m,n-l}-1/z)\|_2^2 = O(|\log\delta_{m,n}|)\|h-r_{m,n}^*\|_\infty^2 + O(\delta_{m,n}^2),$$

i.e.

$$\|H-(\widetilde{r}_{m,n-l}-1/z)\|_2 = O(|\log\delta_{m,n}|^{1/2}\cdot\|h-r_{m,n}^*\|_\infty+\delta_{m,n}). \tag{4.25}$$

(ii) In order to investigate the rate of decrease of $\|h-r_{m,n}^*\|_\infty$ we need to consider the smoothness properties of h on ∂A, i.e. we need to determine the behavior of h in the neighborhood of every corner z_j, $j=1,\ 2,\ \dots,\ l$. For this we proceed as follows:

By differentiating (1.3) and applying induction we find, from (4.15), that for $z\in\partial A$, near z_j, $j=1,\ 2,\ \dots,\ l$,

$$|(H(z)+1/z)^{(k)}| = O(|z-z_j|^{\frac{1}{\nu_j}-k-1}),\ k=0,\ 1,\ 2,\ \dots\ .$$

Hence, by using (4.19), we can verify that for $z\in\partial A$, near z_j, $j=1,\ 2,\ \dots,\ l$,

$$|h^{(k)}(z)| = O(|z-z_j|^{\frac{1}{\nu_j}-k}),\ k=0,\ 1,\ 2,\ \dots\ ; \tag{4.26}$$

see the proof of Theorem 3 in [7]).

Next, with the help of Cauchy's formula we express h as

(4.27) $$h(z) = h_e(z) + h_i(z), \ z \in \overline{A},$$

where h_e is analytic inside and continuous on $\Gamma_e := \partial\Omega$ and h_i is analytic outside and continuous on $\Gamma_i := \partial G$. Moreover, since h_e is analytic on Γ_i, the function h_i has the same boundary properties on Γ_i as the function h. Similarly, the behavior of h on Γ_e is inherited by h_e. Thus, in particular, from (4.26) we have that

(4.28) $$|h_e^{(k)}(z)| = O(|z - z_j|^{\frac{1}{\nu_j} - k}), \ k = 0, \ 1, \ 2, \ \dots ,$$

for $z \in \Gamma_e$ near a corner point $z_j \in \Gamma_e$.

Let Ψ be the conformal mapping of $D' = \{w : |w| > 1\}$ onto the exterior of Γ_e normalized by $\Psi(\infty) = \infty$ and $\lim_{w\to\infty} \Psi(w)/w > 0$. Then, by following the proof of Theorem 3 of [7], we find that $(h_e \circ \Psi)^{(q_j - 1)} \in \text{Lip}\ \beta_j$ in a neighborhood of $w_j = \Psi^{-1}(z_j)$, $z_j \in \Gamma_e$, where

$$\frac{2}{\nu_j} = q_j + \beta_j, \ q_j \in \mathbb{N}, \ 0 < \beta_j \le 1.$$

Further, since $\lambda_e = 2 - \max_{z_j \in \Gamma_e} \nu_j$, there exists a polynomial P_n of degree at most n in z such that

(4.29) $$\|h_e - P_n\|_{\partial\Omega} = O(n^{-\gamma_e}), \ n = 1, \ 2, \ \dots ,$$

where $\gamma_e = \lambda_e/(2 - \lambda_e)$; see Equation (2.3) of [7]. Similarly, there exists a polynomial Q_m of degree at most m in $1/z$ such that

(4.30) $$\|h_i - Q_m(1/z)\|_{\partial G} = O(m^{-\gamma_i}), \ m = 1, \ 2, \ \dots ,$$

where $\gamma_i = \lambda_i/(2 - \lambda_i)$, $\lambda_i = 2 - \max_{z_j \in \Gamma_i} \nu_j$. Thus, since

$$\|h - r^*_{m,n}\|_\infty \le \|h_e - P_n\|_{\partial\Omega} + \|h_i - Q_m(1/z)\|_{\partial G},$$

(4.29) and (4.30) give that

$$\|h - r^*_{m,n}\|_\infty = O(\max(m^{-\gamma_i}, n^{-\gamma_e})), \ m, \ n = 1, \ 2, \ \dots .$$

Finally, we recall that the quantity $\delta_{m,n}$ in (4.25) can be made sufficiently small so that

$$\delta_{m,n} \le \max(m^{-\gamma_i}, \ n^{-\gamma_e}).$$

Therefore,

(4.31) $$\|H - (\widetilde{r}_{m,n-l} - 1/z)\|_2 = O\left(\frac{\{\log \max(m,n)\}^{1/2}}{\min(m^{\gamma_i}, n^{\gamma_e})}\right), \ m, \ n = 2, \ 3, \ \dots .$$

(iii) Let

$$\widetilde{r}_{m,n-l}(z) - 1/z = \sum_{k=-m}^{n-l} \widetilde{a}_k z^k.$$

Then, from Lemma 4.2 we have that

$$\|H - (\widetilde{r}_{m,n-l} - 1/z - \widetilde{a}_{-1}/z)\|_2 \le K\|H - (\widetilde{r}_{m,n-l} - 1/z)\|_2,$$

where K depends only on the geometry of A. Therefore, (4.18) follows from (4.31) and the extremal property (1.9) of $R^*_{m,n}$. □

(II) Relating the norms

The result of this subsection (i.e. of Lemma 4.6) holds under much weaker assumptions about the geometry of A than those of Theorem 2.3. In fact, for our purposes here we need only assume that the two boundary components Γ_{i} and Γ_{e} of A are arbitrary Jordan curves.

We note that Lemma 4.6 is the doubly-connected analogue of Andrievskii's lemma [1] for simply-connected domains with quasiconformal boundaries. We also note that a simpler proof and some extensions of Andrievskii's lemma were given by Gaier [5], that Gaier's approach served as the basis for generalizing the lemma to arbitrary Jordan domains in [13] and that our method of proving Lemma 4.6 is also based on Gaier's approach.

Lemma 4.6. *Let $E := \{w : 1 < |w| < M\}$ and assume that the mapping function $\tau := f^{-1} : E \to A$ satisfies a Hölder condition*

$$(4.32)\qquad |\tau(w_1) - \tau(w_2)| \le C|w_1 - w_2|^{\nu},\ w_1,\ w_2 \in \overline{E},$$

for some $\nu \in (0, 1]$. Also, let $r_{m,n}$ be a rational function of the form

$$(4.33)\qquad r_{m,n}(z) = \sum_{k=-m}^{n} a_k z^k, \quad m,\ n = 2,\ 3,\ \dots,$$

such that there exist a point $\zeta \in A$ and a constant $b > 0$ for which $|r_{m,n}(\zeta)| \le b$. Then, there exists a constant K, depending only on ζ, b and the geometry of A, such that

$$(4.34)\qquad \|r_{m,n}\|_\infty \le K\{\log\max(m, n)\}^{1/2}\|r'_{m,n}\|_2.$$

Proof. In the proof we shall use the letter K to denote constants (not necessarily the same), depending only on ζ, b and the geometry of A.

We assume, without loss of generality, that $\|r'_{m,n}\|_2 \le 1$ and note that the function $h(w) := r_{m,n}(\tau(w))$ is analytic in E. Therefore, h has a Laurent expansion

$$(4.35)\qquad h(w) = \sum_{k=-\infty}^{\infty} c_k w^k,\ w \in E,$$

and hence

$$(4.36)\qquad \iint_E |h'(w)|^2 d\sigma_w = \pi \sum_{k=-\infty}^{\infty} k|c_k|^2(M^{2k} - 1).$$

On the other hand,

$$(4.37)\qquad \iint_E |h'(w)|^2 d\sigma_w = \iint_A |r'_{m,n}(z)|^2 d\sigma_z \le 1.$$

Thus, for $|w| = \rho,\ 1 < \rho < M$, the use of Schwarz's inequality gives that

$$|h(w)| \le \sum_{k=-\infty}^{\infty} |c_k|\rho^k \le |c_0| + \frac{1}{\sqrt{\pi}} \left(\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{\rho^{2k}}{k(M^{2k}-1)} \right)^{1/2} . \tag{4.38}$$

Regarding the sum on the right of (4.38), this can be estimated by means of

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\rho^{2k}}{k(M^{2k}-1)} &\le \frac{M^2}{M^2-1} \sum_{k=1}^{\infty} \frac{\rho^{2k}}{kM^{2k}} \le \frac{M^2}{M^2-1} \log \frac{1}{1-\rho/M} \\ &= \frac{M^2}{M^2-1} \log \frac{M}{M-\rho} \end{aligned} \tag{4.39}$$

and

$$\begin{aligned} \sum_{k=-\infty}^{-1} \frac{\rho^{2k}}{k(M^{2k}-1)} &\le \frac{M^2}{M^2-1} \sum_{k=1}^{\infty} \frac{1}{k\rho^{2k}} \le \frac{M^2}{M^2-1} \log \frac{1}{1-1/\rho} \\ &\le \frac{M^2}{M^2-1} \log \frac{M}{\rho-1}. \end{aligned} \tag{4.40}$$

Also, for any fixed ρ_0, $1 < \rho_0 < M$, we have that

$$|c_0| \le \frac{1}{2\pi} \int_{|w|=\rho_0} \frac{|h(w)|}{|w|} |dw| \le \max_{|f(t)|=\rho_0} |r_{m,n}(t)|. \tag{4.41}$$

Further, it is easy to see that the point ζ can be connected to any other point $z \in \{t : |f(t)| = \rho_0\}$ by a path γ_z such that $|\gamma_z| < K_1$ and $\operatorname{dist}(\gamma_z, \partial A) > K_2$, where K_1 and K_2 are positive constants independent of z. Thus, with the help of

$$|r'_{m,n}(t)| \le \frac{1}{\sqrt{\pi}\operatorname{dist}(t,\partial A)} \|r'_{m,n}\|_2 \le \frac{1}{\sqrt{\pi}\operatorname{dist}(t,\partial A)},\ t \in A,$$

we find that

$$|r_{m,n}(z)| \le |r_{m,n}(\zeta)| + \left| \int_{\zeta}^{z} r'_{m,n}(t)dt \right| \le b + \frac{K_1}{\sqrt{\pi}K_2}, \tag{4.42}$$

for $z \in \{t : |f(t)| = \rho_0\}$. Hence, from (4.38)–(4.42),

$$|h(w)| \le b + \frac{K_1}{\sqrt{\pi}K_2} + \frac{M}{\sqrt{\pi(M^2-1)}} \left(\log \frac{M^2}{(M-\rho)(\rho-1)} \right)^{1/2}, \tag{4.43}$$

for $|w| = \rho,\ 1 < \rho < M$.

Consider now an annular region $A_{\rho_i,\rho_e} := \{z : \rho_i < |f(z)| < \rho_e\}$, where $1 < \rho_i < \rho_e < M$, and denote by L_{ρ_e} and L_{ρ_i} its outer and inner boundary components. Also, let Ω_{ρ_e} and G_{ρ_i} denote respectively the domains exterior to L_{ρ_e} and interior to L_{ρ_i}, and let Φ_{ρ_e} and ϕ_{ρ_i} denote the conformal mappings $\Phi_{\rho_e} : \Omega_{\rho_e} \to D' = \{w : |w| > 1\}$, with $\Phi_{\rho_e}(\infty) = \infty$ and $\lim_{z\to\infty} \Phi_{\rho_e}(z)/z > 0$, and $\phi_{\rho_i} : G_{\rho_i} \to D = \{w : |w| < 1\}$, with $\phi_{\rho_i}(0) = 0$ and $\phi'_{\rho_i}(0) > 0$. Next, denote by $A_{m,n}$ the annular region bounded by the level curves $L_{\rho_e,n} := \{z :$

$|\Phi_{\rho_e}(z)| = 1 + 1/n\}$ and $L_{\rho_i,m} := \{z : |\phi_{\rho_i}(z)| = 1 - 1/m\}$, and note that ρ_i and ρ_e can be chosen so that the region A lies entirely within $A_{m,n}$. The latter can be shown as follows:

From [15, p. 181] we have that

$$\mathrm{dist}(L_{\rho_e,n}, L_{\rho_e}) \geq \frac{\mathrm{diam} L_{\rho_e}}{4n^2} \quad \text{and} \quad \mathrm{dist}(L_{\rho_i,m}, L_{\rho_i}) \geq \frac{\mathrm{dist}(0, L_{\rho_i})}{4m^2}.$$

On the other hand,

$$|\tau(M\mathrm{e}^{\mathrm{i}\theta}) - \tau(\rho_e \mathrm{e}^{\mathrm{i}\theta})| \leq C(M - \rho_e)^\nu$$

and

$$|\tau(\mathrm{e}^{\mathrm{i}\theta}) - \tau(\rho_i \mathrm{e}^{\mathrm{i}\theta})| \leq C(\rho_i - 1)^\nu,$$

where $\theta \in [0, 2\pi)$. Therefore, by choosing ρ_i and ρ_e so that

$$\frac{\mathrm{diam} L_{\rho_e}}{4n^2} > C(M - \rho_e)^\nu \quad \text{and} \quad \frac{\mathrm{dist}(0, L_{\rho_i})}{4m^2} > C(\rho_i - 1)^\nu,$$

we can ensure that $\overline{A} \subset A_{m,n}$.

Finally, from Lemma 4.1 and (4.43) we find that

$$\begin{aligned} \|r_{m,n}\|_\infty &\leq \max_{z \in A_{m,n}} |r_{m,n}(z)| \\ &\leq K\{\log \max(m,n)\}^{1/2} \max\left(\left(1 + 1/n\right)^n, \left(1 - 1/m\right)^{-m}\right) \\ &\leq K\{\log \max(m,n)\}^{1/2}. \quad \square \end{aligned}$$

We remark that the result of Lemma 4.6 is sharp in the sense that the factor $\{\log \max(m,n)\}^{1/2}$ cannot be improved even for $E = \{w : 1 < |w| < M\}$. This can be seen by considering the functions

$$r_{0,n}(z) = \sum_{k=1}^{n} \frac{z^k}{kM^k}$$

and

$$r_{m,0}(z) = \sum_{k=-m}^{-1} \frac{z^k}{k}.$$

(III) Estimating the errors $||f - f_{m.n}||_\infty$ *and* $|M - M_{m,n}|$

Proof of Theorem 2.3. We now use the letter C to denote constants, not necessarily the same, that depend only on the geometry of A.

We first integrate along some rectifiable path in A, from z_0 to z, and obtain the single-valued analytic functions

$$g(z) - g(z_0) = \int_{z_0}^{z} H(\zeta)d\zeta, \ z \in A, \tag{4.44}$$

and

$$g_{m,n}(z) = \int_{z_0}^{z} R^*_{m,n}(\zeta)d\zeta, \ z \in A, \ m, \ n = 2, \ 3, \ \ldots \ . \tag{4.45}$$

We then seek to estimate the error $\|g-g(z_0)-g_{m,n}\|_\infty$, by following the technique of [1] and [6]. For this we proceed as follows:

For any $m, \ n \in \mathbb{N}$ satisfying (2.8) we choose $k(m)$ and $k(n)$ so that $2^{k(m)} \le m < 2^{k(m)+1}$ and $2^{k(n)} \le n < 2^{k(n)+1}$. Then, it follows from Lemma 4.5 that

$$\|g'_{2^{k(m)+1},2^{k(n)+1}} - g'_{m,n}\|_2 \le C \frac{\{\log\max(m,n)\}^{1/2}}{\min(m^{\gamma_i}, n^{\gamma_e})}. \tag{4.46}$$

Next, we note that the function $g_{2^{k(m)+1},2^{k(n)+1}} - g_{m,n}$ satisfies all the conditions of Lemma 4.6. (This follows from Lemma 4.4 and the fact that $R^*_{m,n}$ converges to H locally uniformly in A.) Therefore,

$$\|g_{2^{k(m)+1},2^{k(n)+1}} - g_{m,n}\|_\infty \le C \frac{\log\max(m,n)}{\min(m^{\gamma_i}, n^{\gamma_e})}. \tag{4.47}$$

Similarly,

$$\|g_{2^{j+1},2^{k+1}} - g_{2^j,2^k}\|_\infty \le C \frac{\max(j+1,k+1)}{\min(2^{j\gamma_i}, 2^{k\gamma_e})}, \ j, \ k = 1, \ 2, \ \ldots \ . \tag{4.48}$$

Since, for any $z \in A$,

$$\begin{aligned} g(z) - g(z_0) - g_{m,n}(z) &= \{g_{2^{k(m)+1},2^{k(n)+1}}(z) - g_{m,n}(z)\} \\ &+ \sum_{j=k(m)+1}^{\infty} \{g_{2^{j+1},2^{k(n)-k(m)+j+1}}(z) - g_{2^j,2^{k(n)-k(m)+j}}(z)\}, \end{aligned}$$

it follows, from (4.47) and (4.48), that

$$\begin{aligned} \|g - g(z_0) - g_{m,n}\|_\infty &\le C \frac{\log\max(m,n)}{\min(m^{\gamma_i}, n^{\gamma_e})} \\ &+ C \sum_{j=k(m)+1}^{\infty} \frac{\max(j+1, k(n)-k(m)+j+1)}{\min(2^{j\gamma_i}, 2^{(k(n)-k(m)+j)\gamma_e})}. \end{aligned}$$

Therefore, in view of (2.8),

(4.49) $$\|g - g(z_0) - g_{m,n}\|_\infty \le K \frac{\log \max(m,n)}{\min(m^{\gamma_i}, n^{\gamma_e})},$$

where K depends on the geometry of A and the numbers c_1 and c_2 in (2.8).

Finally, by recalling

$$f(z) = z\,e^{g(z)} = \frac{z}{z_0} e^{g(z)-g(z_0)},\ z \in \overline{A}$$

and taking into account (1.10), we obtain

$$\begin{aligned} \|f - f_{m,n}\|_\infty &\le \|f\|_\infty \|1 - f_{m,n}/f\|_\infty \le M\|1 - \exp\{g_{m,n} - (g - g(z_0))\}\|_\infty \\ &\le C\|g_{m,n} - (g - g(z_0))\|_\infty . \end{aligned}$$

Thus, (2.9) follows from (4.49). □

Proof of Theorem 2.4. This follows at once from Lemmas 4.3 and 4.5. □

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