COEFFICIENT AND INTEGRAL MEAN ESTIMATES FOR ALGEBRAIC AND TRIGONOMETRIC POLYNOMIALS WITH RESTRICTED ZEROS

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1. *Introduction*

The results of this paper concern the following two conjectures which have appeared in the literature.

CONJECTURE 1. (P. Erdös [1]) *Let $T_n(\theta)$ be a trigonometric polynomial of degree n all of whose zeros are real,* i.e., *$T_n(\theta)$ has 2n zeros in* $[0, 2\pi)$, *and let* $M = \max_{\forall\theta} |T_n(\theta)|$ *Then*

$$\int_0^{2\pi} |T_n(\theta)| \leqslant 4M. \tag{1}$$

CONJECTURE 2. (W. K. Hayman [3]) *Let $P_n(z) = \sum_0^n a_k z^k$ be a polynomial of degree n having all its zeros on the unit circle* $|z| = 1$, *and let* $M = \max_{|z|=1} |P_n(z)|$. *Then*

$$|a_k| \leqslant M/2, \quad \textit{for} \quad k = 0, 1, 2, \ldots, n. \tag{2}$$

Conjecture 1 has remained an open problem for over 30 years. In a recent paper [5] H. Kuhn states two related conjectures which imply Conjecture 1.

Conjecture 2, which actually appears in misprinted form in Hayman's problem book, has been verified for polynomials of degree $n = 1, 2,$ and 3; see [9]. Recently Suffridge [10] has shown that the study of polynomials having all their zeros on the unit circle has application to the theory of univalent functions.

The purpose of the present paper is to give a proof of Conjecture 1 and to give a proof of Conjecture 2 except in the case of a middle coefficient $a_{n/2}$ of a polynomial of even degree n. We also present related results as well as some open problems arising from our investigation.

2. *Main results*

It will be convenient to have for reference the following theorem first proved by P. Lax [6].

THEOREM A. *If $p(z)$ is a polynomial of degree n all of whose zeros lie on or exterior to the unit circle, then*

$$\max_{|z|=1} |p'(z)| \leqslant \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{3}$$

We first establish

Received 26 March 1973

* The research of this author was supported in part by the Air Force Office of Scientific Research, Grant AFOSR-73-2503.

THEOREM 1. *Let* $P(z) = \sum_0^n a_k z^k$ *be a polynomial of degree n having all its zeros on the unit circle, and let* $M = \max_{|z|=1} |P(z)|$. *Then for each* $q > 0$ *we have*

$$\int_0^{2\pi} |P(e^{i\theta})|^q \, d\theta \leqslant A_q(M/2)^q, \tag{4}$$

where

$$A_q = \int_0^{2\pi} |1+e^{i\theta}|^q \, d\theta = 2^{q+1} \sqrt{\pi}\, \Gamma(\tfrac{1}{2}q+\tfrac{1}{2})/\Gamma(\tfrac{1}{2}q+1).$$

Futhermore, equality holds in (4) *if and only if* $P(z) = M(\lambda z^n + \mu)/2$, *where* $|\lambda| = |\mu| = 1$.

Proof. Since all the zeros of $P(z)$ lie on $|z| = 1$, there exists a constant u, $|u| = 1$, such that the coefficients satisfy

$$a_k = u\bar{a}_{n-k}, \quad \text{for} \quad k = 0, 1, \ldots, n. \tag{5}$$

Using the relations (5) it is easy to verify that

$$P(z) = \big(zP'(z) + uQ(z)\big)/n, \tag{6}$$

where $Q(z) = z^{n-1}\overline{P'(1/\bar{z})}$. Now set $w(z) = zP'(z)/(uQ(z))$. Then we can write (6) in the form

$$P(z) = \frac{uQ(z)}{n}(1+w(z)),$$

and since $|Q(z)| = |P'(z)|$ for $|z| = 1$, it follows from Theorem A that

$$|P(e^{i\theta})| = \frac{|P'(e^{i\theta})|}{n} |1+w(e^{i\theta})| \leqslant \frac{M}{2}|1+w(e^{i\theta})|, \quad \forall\theta. \tag{7}$$

By the Gauss–Lucas theorem all the zeros of $P'(z)$ lie in $|z| \leqslant 1$, and hence the Blaschke product $w(z)$ is analytic on $|z| \leqslant 1$. Furthermore, $w(0) = 0$ and $|w(z)| = 1$ for $|z| = 1$. Thus the function $1+w(z)$ is subordinate to the function $1+z$ in $|z| < 1$, and so by applying a well-known property of subordination [2] we deduce from (7) that

$$\int_0^{2\pi} |P(e^{i\theta})|^q \, d\theta \leqslant (M/2)^q \int_0^{2\pi} |1+w(e^{i\theta})|^q \, d\theta \leqslant (M/2)^q A_q.$$

Now suppose that equality holds in (4). Then from (7) we must have $|P'(e^{i\theta})| = Mn/2$ for all θ. Since $P'(z)$ is a polynomial of degree $n-1$, this implies that $P'(z) = \lambda Mnz^{n-1}/2$, where $|\lambda| = 1$. But $P(z)$ must have all its zeros on $|z| = 1$, and so $P(z) = M(\lambda z^n + \mu)/2$, where $|\mu| = 1$. Finally, note that for every polynomial $P(z)$ of this form equality holds in (4). This completes the proof of Theorem 1.

An easy consequence of Theorem 1 is

THEOREM 2. *Let* $T_n(\theta)$ *be a trigonometric polynomial of degree n all of whose zeros are real, and let* $M = \max_{\forall\theta} |T_n(\theta)|$. *Then for each* $q > 0$ *we have*

$$\int_0^{2\pi} |T_n(\theta)|^q \, d\theta \leqslant A_q(M/2)^q, \tag{8}$$

and equality holds in (8) *if and only if*

$$T_n(\theta) = Me^{i\phi}\cos(n\theta+\tau), \tag{9}$$

where ϕ and τ are real constants.

Consequently, Conjecture 1 *is true and the extremal polynomials are given by* (9).

Proof. The trigonometric polynomial $T_n(\theta)$ can be written in the form $T_n(\theta) = e^{-in\theta}P_{2n}(e^{i\theta})$, where $P_{2n}(z)$ is an algebraic polynomial of degree $2n$ having all its zeros on $|z| = 1$. Thus (8) follows from (4). By Theorem 1, equality holds in (8) if and only if $P_{2n}(z) = M(\lambda z^{2n}+\mu)/2$, i.e., if and only if $T_n(\theta)$ is of the form (9).

To establish Conjecture 1 we simply take $q = 1$ in (8), in which case $A_q = 8$. This proves Theorem 2.

Concerning Conjecture 2 we prove

THEOREM 3. *Let the polynomial $P(z) = \sum_0^n a_k z^k$ be as in Theorem* 1. *Then*

$$|a_k| \leqslant M/2, \quad \text{for} \quad k = 0, 1, \ldots, n, \quad k \neq n/2. \tag{10}$$

Furthermore, excluding the case $k = n/2$, equality in (10) *can hold only for $k = 0$ or $k = n$ in which case $P(z)$ must be of the form $M(\lambda z^n+\mu)/2$, where $|\lambda| = |\mu| = 1$.*

Proof. Taking $q = 2$ in (4), we have $A_q = 4\pi$, and so

$$2\pi\sum_0^n |a_k|^2 = \int_0^{2\pi} |P(e^{i\theta})|^2\, d\theta \leqslant \pi M^2. \tag{11}$$

For $k \neq n/2$, i.e., $k \neq n-k$, we deduce from (11) and the relations (5) that

$$4\pi|a_k|^2 = 2\pi(|a_k|^2+|a_{n-k}|^2) \leqslant \pi M^2,$$

which implies (10).

If $|a_k| = M/2$, $k \neq n/2$, then equality must hold in (11) and consequently $P(z) = M(\lambda z^n+\mu)/2$, where $|\lambda| = |\mu| = 1$. Thus $k = 0$ or $k = n$. This proves Theorem 3.

3. *Related results and conjectures*

If $P(z)$ is of even degree $n = 2m$, then inequality (4) with $q = 1$ implies that $|a_m| \leqslant 2M/\pi$. We obtain a sharper estimate for the middle coefficient of a polynomial of even degree in

THEOREM 4. *Let $P(z) = \sum_0^n a_k z^k$, with $n = 2m$, be as in Theorem* 1. *Then*

$$|a_m| \leqslant M/\sqrt{3}. \tag{12}$$

Proof. With the same notation as in the proof of Theorem 1, we have $uQ(z) = \sum_{k=0}^{2m-1}(2m-k)\,a_k z^k$, and

$$zP'(z) = uQ(z)w(z). \tag{13}$$

Writing $w(z) = \sum_{1}^{\infty} \omega_\nu z^\nu$ and equating coefficients of z^m in (13) we obtain

$$ma_m = \sum_{k=0}^{m-1} (2m-k)\, a_k\, \omega_{m-k},$$

and hence

$$m^2|a_m|^2 \leqslant 4m^2 \sum_{k=0}^{m-1} |a_k|^2 \sum_{k=0}^{m-1} |\omega_{m-k}|^2 \leqslant 4m^2 \sum_{k=0}^{m-1} |a_k|^2, \tag{14}$$

where the last inequality follows from the fact that $|w(z)| \leqslant 1$ for $|z| \leqslant 1$. We now observe that inequality (11) implies that

$$2 \sum_{0}^{m-1} |a_k|^2 \leqslant M^2/2 - |a_m|^2,$$

and so from (14) there follows

$$m^2|a_m|^2 \leqslant 4m^2(M^2/4 - |a_m|^2/2),$$

which yields (12).

It is of interest to note that the above proof can be modified to show that if the polynomial $P(z)$ of Theorem 4 satisfies

$$\int_0^{2\pi} |P(e^{i\theta})|^2\, d\theta \leqslant 3\pi M^2/4, \quad \text{then } |a_m| \leqslant M/2.$$

For the special case when $n = 4$ we can obtain a sharp estimate for the middle coefficient.

THEOREM 5. *If* $P(z) = \sum_{0}^{4} a_k z^k$ *has all its zeros on* $|z| = 1$, *and* $M = \max_{|z|=1} |P(z)|$,

then

$$|a_2| \leqslant M/2. \tag{15}$$

Equality holds in (15) *for* $P(z) = M(\lambda z^2 + \mu)^2/4$, $|\lambda| = |\mu| = 1$.

Proof. Because of its technical nature we present only an outline of the proof.

It suffices to assume that a_0 and a_2 are real and positive so that $a_4 = a_0$ and $a_1 = \bar{a}_3$. We then show that if $a_0 \geqslant M/4$ or $|a_1| \geqslant M/4$, then (15) holds. The case $a_0 \geqslant M/4$ follows from the inequality $|P(1)+P(-1)| \leqslant 2M$; the case $|a_1| \geqslant M/4$ follows by applying Theorem A twice, first to $P(z)$ and then to $z^3\,\overline{P'(1/\bar{z})}$. Now let $\phi \equiv \arg a_3$ and write

$$e^{-2i\theta}P(e^{i\theta}) = a_2 + 2a_0 \cos 2\theta + 2|a_3| \cos(\theta+\phi),$$

so that

$$a_2 = -2a_0 \cos 2\theta_k - 2|a_3| \cos(\theta_k + \phi), \quad k = 1, 2, 3, 4, \tag{16}$$

where $z_k = e^{i\theta_k}$ are the zeros of $P(z)$. By the first part of the proof, inequality (15) need only be verified in the case where $a_0 < M/4$, $|a_1| < M/4$, $\cos 2\theta_k < 0$, and $\cos(\theta_k+\phi) < 0$ for all k. The next to last inequality together with the coefficient normalization imply that there are two θ_k's, say θ_1 and θ_2, which satisfy

$$\pi/4 < \theta_1 < 3\pi/4 \pmod{2\pi}, \quad 5\pi/4 < \theta_2 < 7\pi/4 \pmod{2\pi}.$$

Since $\cos(\theta_k+\phi) < 0$ for all k, it follows that

$$-\pi/4 < \phi < \pi/4 \pmod{2\pi} \quad \text{or} \quad 3\pi/4 < \phi < 5\pi/4 \pmod{2\pi}.$$

By considering the 4 separate cases $0 \leqslant \phi < \pi/4$, $-\pi/4 < \phi < 0$, $3\pi/4 < \phi \leqslant \pi$, and $\pi < \phi < 5\pi/4$, it can be shown that

$$a_2 \leqslant 2a_0 \cos 2\phi + 2|a_3| \cos 2\phi. \tag{17}$$

For example, in the case $0 \leqslant \phi < \pi/4$, inequality (17) follows from (16) with $k = 2$. Finally, since

$$e^{-2i\phi} P(e^{i\phi}) = a_2 + 2a_0 \cos 2\phi + 2|a_3| \cos 2\phi \leqslant M,$$

we deduce from (17) that (15) holds. This proves Theorem 5.

Based upon our study of the middle coefficient problem it seems likely that the following two results hold.

CONJECTURE 3. *Let $p(z) = \sum_0^n b_k z^k$ be a polynomial of degree n all of whose zeros lie on or exterior (interior) to $|z| = 1$, and let $M = \max_{|z|=1} |p(z)|$. Then*

$$|b_k| \leqslant M/2 \quad \text{for} \quad n/2 \leqslant k \leqslant n \quad (n/2 \geqslant k \geqslant 0).$$

CONJECTURE 4. *Let $T_n(\theta)$ be a real trigonometric polynomial of degree n all of whose zeros are real. Let $M = \max_{\forall\theta} |T_n(\theta)|$. Then*

$$\int_0^{2\pi} T_n(\theta)\, d\theta \leqslant \pi M,$$

with equality if and only if $T_n(\theta) = (M/2)(1+\cos(n\theta+\tau))$.

Another open problem of interest concerns coefficient estimates for polynomials of large degree. More precisely, let $\Pi_{n,M}$, be the class of all polynomials $P(z) = \sum_0^n a_k z^k$ of degree n which have all their zeros on $|z| = 1$ and satisfy $M = \max_{|z|=1} |P(z)|$. Let

$$\mu_{k,n} = \sup_{P \in \Pi_{n,M}} |a_k|.$$

Then what can be said about the sequence $\mu_{k,n}$ as $n \to \infty$? Of course Theorem 3 implies that $\overline{\lim_{n\to\infty}}\, \mu_{k,n} \leqslant M/2$, and this is best possible for $k = 0$. However for $k \geqslant 1$ it seems likely that a sharper estimate is possible as is suggested by

THEOREM 6. *For $k = 1$, $\overline{\lim_{n\to\infty}}\, \mu_{k,n} = M/e$.*

The proof requires two lemmas. The first appears in [9].

LEMMA 1. *Let $\{P_{n_j}(z)\}_{j=1}^{\infty}$ be a sequence of polynomials of respective degrees n_j ($n_1 < n_2 < \dots$) which have all of their zeros on $|z| = 1$. Suppose that $\lim_{j\to\infty} P_{n_j}(z) = F(z)$ for $|z| < 1$. Then*

$$\sup_j \left[\max_{|z|=1} |P_{n_j}(z)|\right] \geqslant 2 \sup_{|z|<1} |F(z)|.$$

The next lemma is a known [4] consequence of the coefficient theorem for functions of positive real part.

LEMMA 2. *Let $f(z)$ be analytic and zero-free in $|z| < 1$. Suppose that*

$$\sup_{|z|<1} |f(z)| = K < \infty.$$

Then $|f'(0)| \leqslant 2K/e$.

Proof of Theorem 6. Set $\mu_1 = \overline{\lim_{n\to\infty}}\, \mu_{1,n}$ and let

$$p_{n_j}(z) = \sum_0^{n_j} a_k^{(j)} z^j \in \Pi_{n_j, M}, \quad n_1 < n_2 < \dots,$$

be a sequence of polynomials such that $|a_1^{(j)}| \to \mu_1$ as $j \to \infty$. Since the $p_{n_j}(z)$ are uniformly bounded by M in $|z| < 1$, there exists a subsequence of the $p_{n_j}(z)$, which we continue to denote by $p_{n_j}(z)$, such that $\lim_{j\to\infty} p_{n_j}(z) = f(z)$ uniformly on closed subsets of $|z| < 1$. It is easy to see that $\mu_1 > 0$, and hence $f(z)$ is analytic and zero-free in $|z| < 1$. Furthermore, Lemma 1 implies that $\sup_{|z|<1} |f(z)| \leqslant M/2$. Hence from Lemma 2 we obtain

$$\mu_1 = \lim_{j\to\infty} |a_1^{(j)}| = |f'(0)| \leqslant M/e. \tag{18}$$

Now consider the function $g(z) = Me^{(z-1)/(z+1)}/2$. It is shown in [9] that there exists a sequence of polynomials $q_n(z) \in \Pi_{n, M}$ such that $\lim_{n\to\infty} q_n(z) = g(z)$ uniformly on closed subsets of $|z| < 1$. Since $g'(0) = M/e$, there follows $\mu_1 \geqslant M/e$. This last inequality together with (18) implies Theorem 6.

We conclude the paper with a result concerning self-inversive polynomials [8]. A polynomial $p(z) = \sum_0^n c_k z^k$ is said to be *self-inversive* if there exists a constant u, $|u| = 1$, such that

$$c_k = u\bar{c}_{n-k}, \quad \text{for} \quad k = 0, 1, \dots, n.$$

Self-inversive polynomials have the property that their zeros are symmetric with respect to the unit circle, and hence this class of polynomials includes those studied in Theorem 1.

Recently, Malik [7] gave a new proof of Theorem A. In this paper he showed that for *any* polynomial $p(z)$ of degree n with $\max_{|z|=1} |p(z)| = M$, there holds

$$|p'(z)| + |q'(z)| \leqslant Mn, \quad |z| = 1, \tag{19}$$

where $q(z) = z^n\overline{p(1/\bar{z})}$. In particular, if $p(z)$ is a self-inversive polynomial, then $p(z) = uq(z), |u| = 1$, and so (19) implies that

$$2|p'(z)| \leqslant Mn, \quad |z| = 1. \tag{20}$$

Using this last inequality we can prove

THEOREM 7. *If $p(z)$ is a self-inversive polynomial of degree n and $M = \max_{|z|=1} |p(z)|$, then*

$$\max_{|z|=1} |p'(z)| = Mn/2.$$

Proof. By inequality (20) we need only show that $\max_{|z|=1} |p'(z)| \geqslant Mn/2$. Let z_0, $|z_0| = 1$, be such that $|p(z_0)| = M$, and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the zeros of $p(z)$. Since the α's are symmetric with respect to the unit circle, it is easy to verify that

$$\operatorname{Re}\left[\frac{zp'(z)}{p(z)}\right] = \sum_{k=1}^{n} \operatorname{Re}\left[\frac{z}{z-\alpha_k}\right] = \frac{n}{2}, \quad \text{for} \quad |z| = 1.$$

Hence

$$\frac{|p'(z_0)|}{M} \geqslant \operatorname{Re}\left[\frac{z_0 p'(z_0)}{p(z_0)}\right] = \frac{n}{2},$$

so that $|p'(z_0)| \geqslant Mn/2$. This proves Theorem 7.

The authors wish to thank Professor J. Clunie for his helpful suggestions.

References

1. P. Erdös, " Note on some elementary properties of polynomials ", *Bull. Amer. Math. Soc.*, 46 (1940), 954–158.
2. G. M. Goluzin, *Geometric theory of functions of a complex variable*, Translations of Math. Monographs, vol. 26, *Amer. Math. Soc.* (Providence, R.I., 1969).
3. W. K. Hayman, *Research problems in function theory* (Athlone Press, London, 1967).
4. ~~J. Hummel and L. Zalcman, oral communication~~. AMER. MATH MONTHLY, 60 (1953), 131-132.
5. H. Kuhn, " Über eine Vermutung von P. Erdös ", *Arch. Math.* (Basel), 21 (1970), 185–191.
6. P. Lax, " Proof of a conjecture of P. Erdös on the derivative of a polynomial ", *Bull. Amer. Math. Soc.*, 50 (1944), 509–513.
7. M. A. Malik, " On the derivative of a polynomial ", ~~to appear.~~ JOURN. LONDON MATH SOC. (2) 1 (1969), 57-
8. M. Marden, " Geometry of polynomials ", *Amer. Math. Soc. Math. Surveys*, 3 (1966).
9. Z. Rubinstein and E. B. Saff, " Bounded approximation by polynomials whose zeros lie on a circle ", *Proc. Amer. Math. Soc.*, 29 (1971), 482–486.
10. T. J. Suffridge, " Extreme points in a class of polynomials having univalent sequential limits " *Trans. Amer. Math. Soc.*, 163 (1972), 225–237.

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