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Asymptotic Zero Distribution of Laurent-Type Rational Functions*

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We study convergence and asymptotic zero distribution of sequences of rational functions with fixed location of poles that approximate an analytic function in a multiply connected domain. Although the study of zero distributions of polynomials has a long history, analogous results for truncations of Laurent series have been obtained only recently by Edrei (*Michigan Math. J.* **29** (1982), 43–57). We obtain extensions of Edrei's results for more general sequences of Laurent-type rational functions. It turns out that the limiting measure describing zero distributions is a linear convex combination of the harmonic measures at the poles of rational functions, which arises as the solution to a minimum weighted energy problem for a special weight. Applications of these results include the asymptotic zero distribution of the best approximants to analytic functions in multiply connected domains, Faber–Laurent polynomials, Laurent–Padé approximants, trigonometric polynomials, etc. © 1997 Academic Press

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1. INTRODUCTION

The limiting behavior of zeros of sequences of polynomials is a classical subject that continues to receive much attention (see, e.g. [8, 11, 16]) because of its applications in function theory, numerical analysis and approximation theory. Two of the fundamental results of the subject are the theorems of Jentzsch [9] and Szegő [17] on the zero distribution of partial sums of a power series. It is rather surprising that although the study of zero distributions of power series sections has a long history, analogous results for truncations of Laurent series has been investigated only relatively recently by Edrei [5] who, in particular, proved the following.

THEOREM A. *Let $A=\{z: r<|z|<R\}$ be the exact annulus of convergence for the Laurent series*

$$f(z)=\sum_{k=-\infty}^{\infty} a_k z^k,$$

where $0<r<1<R<\infty$. Let $\Lambda_1=\{m_i\}_{i=1}^{\infty}$ and $\Lambda_2=\{n_i\}_{i=1}^{\infty}$ be two sequences of positive integers tending to ∞, such that

$$\lim_{i\to\infty} |a_{-m_i}|^{1/m_i}=r \tag{1.1}$$

and

$$\lim_{i\to\infty} |a_{n_i}|^{1/n_i}=\frac{1}{R}. \tag{1.2}$$

Consider all the zeros of the truncation

$$T_{n_i, m_i}(z)=\sum_{k=-m_i}^{n_i} a_k z^k$$

that lie in the angle $\theta_1\leqslant \arg z<\theta_2$ ($\theta_1<\theta_2\leqslant\theta_1+2\pi$). Then, as $i\to\infty$, there are

$$(1+o(1))\,\frac{\theta_2-\theta_1}{2\pi}\,m_i$$

of those zeros that have modulus <1 and

$$(1+o(1))\,\frac{\theta_2-\theta_1}{2\pi}\,n_i$$

of them have modulus >1.

Moreover, for any given $\varepsilon > 0$, *there are outside the annulus* $Re^{-\varepsilon} \leqslant |z| \leqslant Re^{\varepsilon}$ *at most* $o(n_i)$ *zeros of* $T_{n_i, m_i}(z)$ *of modulus* > 1 *and there are at most* $o(m_i)$ *zeros of modulus* < 1 *outside* $re^{-\varepsilon} \leqslant |z| \leqslant re^{\varepsilon}$.

This theorem is completely analogous to Szegő's result for power series [17]. It says that all but a negligible proportion of the zeros of the $T_{n_i, m_i}(z)$'s accumulate on the circles $|z| = r$ and $|z| = R$, and that the arguments of these zeros that are close to one of the circles, are equidistributed in the sense of Weyl.

In this paper we present various generalizations of Edrei's result to the zero distribution of certain sequences of rational functions having fixed location of poles and converging locally uniformly in finitely-connected domains to an analytic function $f(\not\equiv 0)$. As applications we describe the limiting zero distribution of Laurent-type approximants. The main tools of our investigation are the theories of weighted potentials and weighted polynomial zero distributions developed in [15] and [11].

The paper is organized as follows. In Section 2 we state and discuss our main results. Section 3 is devoted to applications such as zero distributions of trigonometric approximants and Laurent–Padé approximants. In Section 4 we discuss the theory associated with a weighted potential problem and, finally, in Section 5 we make use of this theory in order to prove the theorems of Sections 2 and 3.

2. MAIN RESULTS

Let $\mathscr{K}$ be a bounded continuum (not a single point) whose complement consists of a finite number of domains. We denote by $\overline{\mathbf{C}}$ the extended complex plane, by $\{G_l\}_{l=1}^n$ the set of bounded components of $\overline{\mathbf{C}} \backslash \mathscr{K}$ and by Ω the unbounded component. (It is clear that the G_l and Ω are simply connected domains and that $\overline{\mathbf{C}} \backslash \mathscr{K} = (\bigcup_{l=1}^n G_l) \cup \Omega$.) Finally, for each $l = 1, 2, ..., n$ we associate an arbitrary but fixed point $a_l \in G_l$.

By the Riemann mapping theorem there exists a unique conformal mapping $\phi_l \colon G_l \to D$ of G_l onto the open unit disk D, normalized by the conditions $\phi_l(a_l) = 0$ and $\phi_l'(a_l) > 0$. The quantity $R_l := 1/\phi_l'(a_l)$ is called the *interior conformal radius* of G_l with respect to a_l. Similarly, there exists a conformal mapping $\Phi \colon \Omega \to D'$ of the unbounded component Ω onto the exterior of the unit circle $D' = \{z \colon |z| > 1\}$ normalized by $\Phi(\infty) = \infty$ and $\lim_{z \to \infty} \Phi(z)/z = 1/C$, where $C := \operatorname{cap} \mathscr{K}$ is the logarithmic capacity (transfinite diameter) of $\mathscr{K}$ (cf. [19]).

We shall keep the same notation $\phi_l(z)$ for the extension of the conformal mapping $\phi_l \colon G_l \to D$ onto the boundary ∂G_l in the sense of Carathéodory's theory of prime ends [6]. Thus, for each $l = 1, 2, ..., n$, the mapping ϕ_l

is defined on the closure $\overline{G_l}$, i.e. $\phi_l\colon \overline{G_l} \to \bar{D}$. Similarly, for the exterior mapping we take $\Phi\colon \bar{\Omega} \to \overline{D'}$.

For our study of limiting distributions we shall utilize the measures

$$\mu_e(B) := \omega(\infty, B, \Omega) \tag{2.1}$$

and

$$\mu_l(B) := \omega(a_l, B, G_l), \qquad l = 1, ..., n, \tag{2.2}$$

for any Borel set $B \subset \mathbf{C}$, where $\omega(\infty, B, \Omega)$ is the harmonic measure of the set B at the point ∞ with respect to Ω, and $\omega(a_l, B, G_l)$ is the harmonic measure of B at the point a_l with respect to the domain G_l (cf. [13, 19]). We remark that μ_e is the same as the equilibrium measure for $\mathscr{K}$ in the sense of logarithmic potential theory. Another convenient way to describe the above harmonic measures is to interpret them as preimages of the normalized arclength measure on the unit circle $\{z\colon |z| = 1\}$ under the corresponding conformal mappings. That is,

$$\omega(\infty, B, \Omega) = m(\Phi(B \cap \partial\Omega)) \tag{2.3}$$

and

$$\omega(a_l, B, G_l) = m(\phi_l(B \cap \partial G_l)), \qquad l = 1, ..., n, \tag{2.4}$$

where $dm = d\theta/2\pi$ on $\{z\colon |z| = 1\}$.

We recall that μ_e and μ_l, $l = 1, ..., n$, are compactly supported unit Borel measures, i.e.

$$\|\mu_e\| = \|\mu_l\| = 1, \qquad l = 1, ..., n,$$

and $\operatorname{supp} \mu_e = \partial\Omega$, $\operatorname{supp} \mu_l = \partial G_l$ [13].

The main goal of this paper is to study the limiting zero distribution of sequences of rational functions that are identified by a multi-index $N = (k, m_1, m_2, ..., m_n)$ and have the form:

$$R_N(z) = \sum_{j=0}^{k} t_j^N z^j + \sum_{l=1}^{n} \sum_{j=1}^{m_l} s_{l,j}^N (z - a_l)^{-j}. \tag{2.5}$$

In other words, $R_N(z)$ is a *Laurent-type* rational function whose poles are located at the fixed points $a_l \in G_l$, $l = 1, ..., n$, and at $\infty \in \Omega$. If t_k^N and s_{l,m_l}^N, $l = 1, ..., n$, are nonzero, then the total number of poles of $R_N(z)$ in the extended complex plane $\bar{\mathbf{C}}$ is equal to

$$|N| = k + \sum_{l=1}^{n} m_l, \tag{2.6}$$

which is the norm of the multi-index N. We also note that, in this case,

$$R_N(z) = \frac{t_k^N P_N(z)}{\prod_{l=1}^{n} (z-a_l)^{m_l}}, \tag{2.7}$$

where $P_N(z)$ is a monic polynomial of degree $|N|$ whose zeros coincide with those of $R_N(z)$. Next we introduce the *normalized counting measure* in the zeros of $R_N(z)$:

$$\nu_N := \frac{1}{|N|} \sum_{P_N(z_j)=0} \delta_{z_j}, \tag{2.8}$$

where δ_z is the unit point mass at z and where all zeros are counted according to their multiplicities.

The results of this paper on zero distributions are all stated in terms of the weak* convergence of measures. We say that a sequence of Borel measures $\{\mu_n\}_{n=1}^{\infty}$ converges to the measure μ, as $n \to \infty$, in the *weak* topology* if

$$\lim_{n \to \infty} \int f \, d\mu_n = \int f \, d\mu$$

for any continuous function f on $\mathbf{C}$ having compact support.

Throughout the paper we assume that $k = k(i)$, $m_1 = m_1(i)$, ..., $m_n = m_n(i)$, $N = N(i)$, for some increasing sequence Λ of integers i, and that $k(i) \to \infty$, $m_l(i) \to \infty$, $l = 1, ..., n$, as $i \to \infty$, $i \in \Lambda$. Furthermore, we assume that the following limits exist:

$$\lim_{|N| \to \infty} \frac{m_l}{|N|} = \lim_{\substack{i \to \infty \\ i \in \Lambda}} \frac{m_l(i)}{|N(i)|} =: \alpha_l \qquad l = 1, ..., n. \tag{2.9}$$

This normalization means that we deal with so-called "ray sequences" of rational functions. Clearly,

$$\alpha_l \geqslant 0, \qquad l = 1, ..., n,$$

$$\lim_{\substack{i \to \infty \\ i \in \Lambda}} \frac{k(i)}{|N(i)|} = 1 - \sum_{l=1}^{n} \alpha_l,$$

and

$$\sum_{l=1}^{n} \alpha_l \leqslant 1.$$

Before stating our results in their full generality, it is convenient to consider the special case where $\mathscr{K}$ is the closure of an annular region A bounded by two Jordan curves, with one curve interior to the other, so that $\overline{\mathbf{C}}\setminus\mathscr{K}$ has only one bounded component which we denote by G_1. Assume that $a_1 = 0 \in G_1$ and consider the weak* limit of the normalized counting measures $\nu_{(k,m)}$ in the zeros of Laurent-type rational functions of the form

$$R_{(k,m)}(z) = \sum_{j=0}^{k} t_j^{k,m} z^j + \sum_{j=1}^{m} s_j^{k,m} z^{-j}, \tag{2.10}$$

where $k = k(i)$, $m = m(i)$. Also, in accordance with (2.9), we assume the existence of the limit

$$\lim_{k+m\to\infty} \frac{m}{k+m} = \lim_{\substack{i\to\infty\\ i\in\Lambda}} \frac{m(i)}{k(i)+m(i)} =: \alpha \tag{2.11}$$

and that $k(i)\to\infty$, $m(i)\to\infty$.

The following theorem is a special case of Theorem 2.2 below.

THEOREM 2.1. *Suppose that the sequence* $\{R_{(k,m)}(z)\}_{i\in\Lambda}$ *defined by* (2.10) *converges locally uniformly in a bounded annular region* A *to an analytic function that is not identically zero. Assume further that* (2.11) *holds and hat*

$$\lim_{\substack{i\to\infty\\ i\in\Lambda}} |t_k^{k,m}|^{1/k} = \frac{1}{C}, \tag{2.12}$$

$$\lim_{\substack{i\to\infty\\ i\in\Lambda}} |s_m^{k,m}|^{1/m} = R_1, \tag{2.13}$$

where $C := \operatorname{cap} \bar{A}$ *and* R_1 *is the inner conformal radius of* G_1 *with respect to the origin. Then*

$$\nu_{(k,m)} \xrightarrow{*} \mu_w := (1-\alpha)\,\mu_e + \alpha\mu_1, \qquad \text{as} \quad i\to\infty, \quad i\in\Lambda, \tag{2.14}$$

where, from (2.1), (2.2),

$$\mu_e = \omega(\infty, \cdot, \Omega) \qquad \text{and} \qquad \mu_1 = \omega(0, \cdot, G_1).$$

We remark that the limit measure μ_w defined in (2.14) is, in fact, the equilibrium measure for a weighted potential problem corresponding to a special weight w (see Section 4).

The "not identically zero function" assumption is essential in Theorem 2.1. To show this, we consider the sequence

$$R_{(n,\, n)} = z^n - \frac{1}{2^n z^n}, \qquad n = 1, 2, ...,$$

convergent to $f \equiv 0$ locally uniformly in $A := \{z\colon 1/2 < |z| < 1\}$ as $n \to \infty$. This sequence satisfies conditions (2.12) and (2.13) as well as (2.11) with $\alpha = 1/2$. However, $R_{(n,\, n)}$ has all zeros on $\{z\colon |z| = 1/\sqrt{2}\}$. In contrast, if we consider the modified sequence $\{R_{(n,\, n)} + 1\}_{n=1}^{\infty}$, then all the conditions of Theorem 2.1 hold and this sequence of rationals has one half of its zeros accumulating on $|z| = 1$ and the other half of its zeros accumulating on $|z| = 1/2$ in agreement with Theorem 2.1.

Some examples of Laurent-type rational approximants satisfying the conditions of Theorem 2.1 are given in Section 3. We also mention that it is possible to prove a result on the zero distribution of the partial sums of Faber-Laurent series to a function analytic in a doubly connected domain [18] via an application of Theorem 2.1. For the details see [14].

As an immediate corollary of Theorem 2.1 we obtain Edrei's result of Theorem A. Indeed, if $A = \{z\colon 0 < r < |z| < R < \infty\}$ is the exact annulus of convergence for the Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k,$$

then the truncations

$$T_{n_i,\, m_i}(z) = \sum_{k=-m_i}^{n_i} a_k z^k$$

converge locally uniformly in A to $f(z) \not\equiv 0$, as $i \to \infty$. Also, we assume without loss of generality that the limit

$$\alpha := \lim_{i \to \infty} \frac{m_i}{n_i + m_i}$$

exists (otherwise we can apply our results to a subsequence, for which this limit does exist). Then conditions (2.12) and (2.13) of Theorem 2.1 are satisfied in view of (1.1) and (1.2). Thus, Theorem 2.1 gives

$$\nu_N \xrightarrow{*} (1-\alpha)\, \mu_e + \alpha \mu_1 \tag{2.15}$$

in the weak* sense as $|N| = n_i + m_i \to \infty$, or as $i \to \infty$. Since in this case μ_e coincides with the normalized arclength measure on $\{z\colon |z| = R\}$ and μ_1 coincides with the normalized arclength measure on $\{z\colon |z| = r\}$, the result

(2.15) gives in a more compact manner (using weak* convergence) the conclusions of Theorem 1.1.

Theorem 2.1 is a special case of the following more general result.

THEOREM 2.2. *Let $\mathscr{K}$ be the closure of a multiply connected Jordan domain, i.e. $\partial\Omega$ and ∂G_l, $l=1, ..., n$, are Jordan curves without common points. Suppose that the sequence $\{R_N(z)\}_{i\in\Lambda}$ (cf. (2.5)) converges locally uniformly in the interior $\mathscr{K}^\circ$ of $\mathscr{K}$ to $f(z)(\not\equiv 0)$ and (2.9) holds.*

If

(i)
$$\lim_{\substack{i\to\infty\\ i\in\Lambda}} |t_k^N|^{1/k} = \frac{1}{C}$$

and

(ii)
$$\lim_{\substack{i\to\infty\\ i\in\Lambda}} |s_{l,m_l}^N|^{1/m_l} = R_l, \qquad l=1, ..., n,$$

then the normalized zero counting measures ν_N for R_N satisfy

(iii) $\nu_N \xrightarrow{*} \mu_w$ *in the weak* sense as $i\to\infty$, $i\in\Lambda$, where*

$$\mu_w := \left(1-\sum_{l=1}^{n}\alpha_l\right)\mu_e + \sum_{l=1}^{n}\alpha_l\mu_l. \tag{2.16}$$

Conversely, suppose that $\alpha_l>0$, $l=1, ..., n$, with $\sum_{l=1}^{n}\alpha_l\neq 1$. If each a_l has some neighborhood free of zeros of $\{R_N(z)\}_{i\in\Lambda}$, then (iii) *implies* (i) *and* (ii).

Our next result is a "one-sided" version of Theorem 2.2. To state it we recall that the logarithmic potential of a compactly supported Borel measure μ is defined by

$$U^\mu(z) := \int \log\frac{1}{|z-t|}\,d\mu(t).$$

Suppose that G is an open bounded set and $\operatorname{supp}\mu\subset G$. Then a measure $\hat\mu$ supported on ∂G is called the *balayage* of μ to the boundary of G if $\|\hat\mu\| = \|\mu\|$, the potential $U^{\hat\mu}$ is bounded on ∂G and

$$U^{\hat\mu}(z) = U^\mu(z) \qquad \text{q.e. on } \partial G$$

(see Chapter IV of [10]). By q.e. (quasi-everywhere) we mean that the above equality holds for all $z\in\partial G$ with the possible exception of a set of zero logarithmic capacity.

THEOREM 2.3. *If in Theorem 2.2 we assume that condition* (i) *holds* (*but not necessarily* (ii)), *then for any weak* limit* ν *of the measures* ν_N, *as* $i \to \infty$, $i \in \Lambda$, *we have*

$$U^{\nu}(z) = U^{\mu_w}(z), \qquad z \notin \bar{G} := \bigcup_{l=1}^{n} \overline{G_l} \tag{2.17}$$

and

$$\nu\,|_{\mathbf{C}\setminus\bar{G}} = \mu_w\,|_{\mathbf{C}\setminus\bar{G}} = \left(1 - \sum_{l=1}^{n} \alpha_l\right)\mu_e, \tag{2.18}$$

where μ_w *is given by* (2.16). *Furthermore,*

$$\hat{\nu}_N \xrightarrow{*} \mu_w, \qquad as \quad i \to \infty, \quad i \in \Lambda, \tag{2.19}$$

where $\hat{\nu}_N$ *denotes the measure obtained by balayage of the part of* ν_N *supported in* $G := \bigcup_{l=1}^{n} G_l$ *to* $\partial G = \bigcup_{l=1}^{n} \partial G_l$. (*Here the part of* ν_N *supported outside of* G *is kept fixed*).

Remark 2.4. Theorem 1 in [5] follows from Theorem 2.2 applied to the partial sums of Laurent series.

Remark 2.5. In Theorem 2.3 we actually require only that $k(i) \to \infty$ as $i \to \infty$, $i \in \Lambda$, so that $m_l(i)$, $l = 1, \ldots, n$, may be bounded. Thus, if we consider a sequence of polynomials $\{P_N(z)\}$ that converges locally uniformly in $\mathcal{K}^\circ$ to a nonzero analytic function such that (i) is satisfied, then we have $\alpha_l = 0$, $l = 1, \ldots, n$, and $\nu_N(B) \to 0$, as $i \to \infty$, for any compact $B \subset \overline{\mathbf{C}}\setminus\bar{\Omega}$. The last statement follows from Hurwitz's theorem and the maximum modulus principle, according to which $\{P_N(z)\}$ converges locally uniformly in $\mathbf{C}\setminus\bar{\Omega}$. Thus, we obtain from (2.19) of Theorem 2.3 that

$$\nu_N \xrightarrow{*} \mu_e \qquad \text{as} \quad i \to \infty, \quad i \in \Lambda.$$

It follows that in this case Theorem 2.3 reduces to certain results of Blatt, Saff and Simkani [2] on the zero distribution of polynomials.

It is of some interest to consider the analog of Theorem 2.2 for a set $\mathcal{K}$ with empty interior. In this case the condition of convergence for the sequence of rational functions can be replaced by a weaker one. Namely, we have the following.

THEOREM 2.6. *Let* $\mathcal{K}$ *be an arbitrary bounded continuum* (*not a single point*) *with empty interior and suppose that* $\overline{\mathbf{C}}\setminus\mathcal{K}$ *consists of a finite number* *of components Ω, $G_1, ..., G_n$. Assume that for the rational functions R_N of* (2.5) *we have* (2.9) *and*

$$\lim_{\substack{i \to \infty \\ i \in \Lambda}} \|R_N\|_{\mathscr{K}}^{1/|N|} = 1. \tag{2.20}$$

Then conditions (i) *and* (ii) *of Theorem* 2.2 *imply* (iii).

By the norm in (2.20) we mean the uniform norm, i.e.,

$$\|f\|_{\mathscr{K}} := \sup_{z \in \mathscr{K}} |f(z)|.$$

The last theorem can be applied, for example, in case when $\mathscr{K}$ is a single Jordan curve. This provides an extension of Edrei's result for Laurent series to the situation when the annulus of convergence A degenerates to the unit circle, i.e. with $\mathscr{K} = \mathbf{T} = \{z: |z| = 1\}$. We present this result in Theorem 3.1

We remark that it is possible to further relax the geometric conditions imposed on the set $\mathscr{K}$ in the theorems of this section. For example, one could allow every component of $\overline{\mathbf{C}} \backslash \mathscr{K}$ to be a finitely connected domain, where we assume, as before, that the number of components is finite. The analogues of our result in this case are straightforward. Another generalization is to allow finitely many fixed poles in every component of $\overline{\mathbf{C}} \backslash \mathscr{K}$ instead of just one.

3. APPLICATIONS

3.1. *Zeros of Fourier Sections*

Consider the Fourier expansion

$$f(z) = \sum_{k=-\infty}^{\infty} a_k e^{ik\theta} = \sum_{k=-\infty}^{\infty} a_k z^k \tag{3.1}$$

for a function $f \in L^2(\mathbf{T})$, where $\mathbf{T} = \{z: |z| = 1\}$. Equality in (3.1) is with respect to L^2 norm.

THEOREM 3.1. *Let $f \in L^2(\mathbf{T})$. Suppose that for the representation* (3.1), *the functions*

$$g(z) = \sum_{k=0}^{\infty} a_k z^k \tag{3.2}$$

(analytic in $|z|<1$*) and*

$$h(z)=\sum_{k=-\infty}^{-1} a_k z^k \tag{3.3}$$

(analytic in $|z|>1$*) cannot be analytically continued to an open set containing the unit circle. Then there exist two sequences of positive integers* $\{n_i\}_{i=1}^{\infty}$ *and* $\{m_i\}_{i=1}^{\infty}$ *such that for the normalized counting measure* $\nu_{n_i+m_i}$ *in the zeros of the truncation*

$$T_{n_i,m_i}(z)=\sum_{k=-m_i}^{n_i} a_k z^k, \tag{3.4}$$

we have

$$\nu_{n_i+m_i} \xrightarrow{*} \omega \qquad \text{as} \quad i\to\infty,$$

where ω *is the normalized arclength measure on* $\mathbf{T}$*, i.e.* $d\omega=d\theta/2\pi$.

We give a sketch of proof here. Since both functions $g(z)$ and $h(z)$ have singularities on $\mathbf{T}$, there exist two subsequences of positive integers $\{n_i\}_{i=1}^{\infty}$ and $\{m_i\}_{i=1}^{\infty}$ such that

$$\lim_{i\to\infty} |a_{n_i}|^{1/n_i}=1 \qquad \text{and} \qquad \lim_{i\to\infty} |a_{-m_i}|^{1/m_i}=1. \tag{3.5}$$

From the Hölder inequality we obtain

$$\|T_{n_i,m_i}\|_2 \leqslant \sqrt{2\pi}\,\|T_{n_i,m_i}\|_\infty \leqslant 2\pi\sqrt{n+m}\,\|T_{n_i,m_i}\|_2, \tag{3.6}$$

where $\|T_{n_i,m_i}\|_\infty=\max_{z\in\mathbf{T}}|T_{n_i,m_i}(z)|$. It follows that

$$\lim_{i\to\infty}\|T_{n_i,m_i}\|_\infty^{1/(n_i+m_i)}=\lim_{i\to\infty}\|T_{n_i,m_i}\|_2^{1/(n_i+m_i)}=1, \tag{3.7}$$

because $\lim_{i\to\infty}\|T_{n_i,m_i}\|_2=\|f\|_2\neq 0$.

Theorem 3.1 is now a consequence of Theorem 2.6 because the measures μ_e and μ_1 of Theorem 2.6 are given in this case by $\mu_e=\mu_1=\omega$; thus for any $\{n_i\}_{i=1}^{\infty}$ and $\{m_i\}_{i=1}^{\infty}$, satisfying (3.5) such that

$$\lim_{i\to\infty}\frac{m_i}{n_i+m_i}=\alpha,$$

we obtain

$$\mu_w=(1-\alpha)\,\mu_e+\alpha\mu_1=(1-\alpha)\,\omega+\alpha\omega=\omega.$$

3.2. *Zeros of Laurent–Padé Approximants with the Fixed Denominator Degree*

For the function

$$f(z)=\sum_{k=-\infty}^{\infty} c_k z^k$$

analytic in $A=\{z: r<|z|<R\}$ we consider its additive splitting

$$f(z)=f^+(z)+f^-(z), \qquad z\in A, \tag{3.8}$$

where

$$f^+(z)=\frac{c_0}{2}+\sum_{k=1}^{\infty} c_k z^k$$

is analytic in $\{z: |z|<R\}$ and

$$f^-(z)=\frac{c_0}{2}+\sum_{k=-\infty}^{-1} c_k z^k$$

is analytic in $\{z: |z|>r\}$.

Following [7], we introduce Laurent–Padé approximants of order (m, n). It is enough to consider the case $m\geqslant n$ for our purposes. This simplifies the definition of Laurent–Padé approximant of type (m, n) to the sum of classical Padé approximants of type (m, n) to $f^+(z)$ and $f^-(z)$ about 0 and ∞ respectively.

Let

$$r^+_{m,n}(z)=\frac{p^+_{m,n}(z)}{q^+_{m,n}(z)}$$

be the classical Padé approximant of type (m, n) to the function $f^+(z)$ about 0, i.e.

$$f^+(z)\, q^+_{m,n}(z)-p^+_{m,n}(z)=O(z^{m+n+1}).$$

We use the notations

$$p^+_{m,n}(z)=\sum_{j=0}^{m} a^{+,(j)}_{m,n}\, z^j$$

for the Padé numerator of type (m, n) and

$$q^+_{m,n}(z)=\sum_{j=0}^{n} b^{+,(j)}_{m,n}\, z^j$$

for the (normalized) Padé denominator with $b^{+,(0)}_{m,n}=1$.

We consider next the Padé approximant of type (m, n) to $f^{-}(z)$ about ∞, i.e.

$$r^{-}_{m,n}(z)=\frac{p^{-}_{m,n}(1/z)}{q^{-}_{m,n}(1/z)},$$

where

$$p^{-}_{m,n}\left(\frac{1}{z}\right)=\sum_{j=0}^{m} a^{-,(j)}_{m,n}\left(\frac{1}{z}\right)^{j}$$

and

$$q^{-}_{m,n}\left(\frac{1}{z}\right)=\sum_{j=0}^{n} b^{-,(j)}_{m,n}\left(\frac{1}{z}\right)^{j}$$

with $b^{-,(0)}_{m,n}=1$. For $p^{-}_{m,n}(1/z)$ and $q^{-}_{m,n}(1/z)$ we have

$$f^{-}(z)\, q^{-}_{m,n}\left(\frac{1}{z}\right)-p^{-}_{m,n}\left(\frac{1}{z}\right)=O\left(\frac{1}{z^{m+n+1}}\right).$$

The Laurent–Padé approximant of type (m, n) to $f(z)$ is then defined by

$$r_{m,n}(z):=r^{+}_{m,n}(z)+r^{-}_{m,n}(z)=\frac{p_{m,n}(z)}{q_{m,n}(z)},$$

where

$$p_{m,n}(z):=p^{+}_{m,n}(z)\, q^{-}_{m,n}(1/z)+p^{-}_{m,n}(1/z)\, q^{+}_{m,n}(z) \tag{3.9}$$

is a Laurent polynomial of degree at most m and

$$q_{m,n}(z):=q^{+}_{m,n}(z)\, q^{-}_{m,n}(1/z)$$

is a Laurent polynomial of degree at most n. We note that

$$p_{m,n}(z)=a^{+,(m)}_{m,n} b^{-,(0)}_{m,n} z^{m}+\cdots+a^{-,(m)}_{m,n} b^{+,(0)}_{m,n}\frac{1}{z^{m}}$$

or, since $b^{+,(0)}_{m,n}=b^{-,(0)}_{m,n}=1$,

$$p_{m,n}(z)=a^{+,(m)}_{m,n} z^{m}+\cdots+a^{-,(m)}_{m,n}\frac{1}{z^{m}}. \tag{3.10}$$

Since the zeros of $r_{m,n}(z)$ coincide with those of the numerator $p_{m,n}(z)$, as given by (3.10), it suffices to study the limiting behavior of zeros of this Laurent polynomial or Laurent-type rational function. Theorem 2.1 gives the necessary tools for such a study.

Suppose that $f(z)$ has a meromorphic continuation to the annulus $A_n := \{z: r_n < |z| < R_n\}$ such that this continuation (which we still denote by $f(z)$) has exactly n poles, counted according to multiplicities, in $\{z: r_n < |z| \leqslant r\}$ and n poles in $\{z: R \leqslant |z| < R_n\}$. Thus, the total number of poles of $f(z)$ in A_n is $2n$, where n is a fixed positive integer. We assume that A_n is the largest annulus with the above properties and that the conditions $R_n < \infty$, $r_n > 0$ hold.

THEOREM 3.2. *Under the above assumptions of $f(z)$, there exist two subsequences Λ_1 and Λ_2 of indices such that the normalized counting measure in zeros of Laurent–Padé approximants of type (m, n) to $f(z)$, i.e.,*

$$\nu_m = \frac{1}{2m} \sum_{r_{m,n}(z_j)=0} \delta_{z_j},$$

has weak limits satisfying*

$$\nu_m \big|_{|z| \geqslant (R_n + r_n)/2} \xrightarrow{*} \mu_1 \qquad as \quad m \to \infty, \quad m \in \Lambda_1, \tag{3.11}$$

where

$$d\mu_1 = \frac{1}{2} \frac{d\theta}{2\pi} \qquad on \quad |z| = R_n,$$

and

$$\nu_m \big|_{|z| \leqslant (R_n + r_n)/2} \xrightarrow{*} \mu_2 \qquad as \quad m \to \infty, \quad m \in \Lambda_2, \tag{3.12}$$

where

$$d\mu_2 = \frac{1}{2} \frac{d\theta}{2\pi} \qquad on \quad |z| = r_n.$$

This means that the weak* limit of restrictions of ν_m, $m \in \Lambda_1$, on $\{z: |z| \geqslant (R_n + r_n)/2\}$, as $m \to \infty$, coincides with one half of the normalized Lebesgue measure on $|z| = R_n$. On the other hand, when $m \to \infty$, $m \in \Lambda_2$, the weak* limit of the restrictions of ν_m on $\{z: |z| \leqslant (R_n + r_n)/2\}$ is equal to one half of the normalized Lebesgue measure on $|z| = r_n$.

The results of Theorem 3.2 are analogues of results in [4] for the zero distribution of classical Padé approximants with fixed denominator degree.

4. POTENTIAL THEORETIC BACKGROUND

In this section we discuss the theory associated with some weighted potential problems which plays a crucial role in the derivation of our results. In particular, we show that the measure μ_w defined in (2.16) arises in a natural way as the extremal measure for a weighted energy problem.

Let $\mathcal{K} \subset \mathbf{C}$ be an arbitrary compact set of positive capacity. For a Borel measure ν with compact supports $S_\nu \subset \mathbf{C} \setminus \mathcal{K}$ such that $\|\nu\| = \alpha$, $0 \leqslant \alpha \leqslant 1$, we consider weight function

$$w(z) := \begin{cases} e^{U^\nu(z)}, & z \in \mathcal{K}, \\ 0, & z \notin \mathcal{K}, \end{cases} \tag{4.1}$$

where U^ν is the logarithmic potential of ν. Note, that $w(z)$ is continuous as a function defined on $\mathcal{K}$ and

$$w(z) = e^{-Q(z)}, \qquad z \in \mathcal{K}, \tag{4.2}$$

where

$$Q(z) := -U^\nu(z), \qquad z \in \mathcal{K}. \tag{4.3}$$

Let $\mathcal{M}(\mathcal{K})$ denote the class of all Borel measures μ with total mass $\|\mu\| = 1$ supported on $\mathcal{K}$, and consider the following weighted energy problem:

"For the weighted energy integral

$$I_w(\mu) := \iint \log \frac{1}{|z-t|\, w(z)\, w(t)}\, d\mu(z)\, d\mu(t), \tag{4.4}$$

find

$$V_w := \inf_{\mu \in \mathcal{M}(\mathcal{K})} I_w(\mu) \tag{4.5}$$

and identify the extremal measure $\mu_w \in \mathcal{M}(\mathcal{K})$ for which the inf in (4.5) is attained."

We note that $w(z)$ is a continuous admissible weight on $\mathcal{K}$ in the sense of [12]. Therefore the following is known (cf. Theorem 3.1 of [12] and Section I.1 of [15]):

THEOREM 4.1. *For w as defined in* (4.1):

(a) *V_w is finite;*

(b) *$\exists$ a unique element $\mu_w \in \mathcal{M}(\mathcal{K})$ such that*

$$I_w(\mu_w) = V_w$$

and the logarithmic energy of μ_w is finite, i.e.

$$-\infty < \iint \log \frac{1}{|z-t|}\, d\mu_w(z)\, d\mu_w(t) < +\infty;$$

(c) *$U^{\mu_w}(z) + Q(z) \geqslant F_w$, for q.e. $z \in \mathcal{K}$, where*

$$U^{\mu_w}(z) := \int \log \frac{1}{|z-t|}\, d\mu_w(t)$$

and

$$F_w := V_w - \int Q(t)\, d\mu_w(t);$$

(d) *$U^{\mu_w}(z) + Q(z) = F_w$, for q.e. $z \in \operatorname{supp} \mu_w$.*

In fact, it is proved in [15] that properties (c) and (d) characterize the extremal measure: If a compactly supported measure $\sigma \in \mathcal{M}(\mathcal{K})$ has finite logarithmic energy and

$$U^{\sigma}(z) + Q(z) = F \qquad \text{q.e. on supp } \sigma, \tag{4.6}$$

$$U^{\sigma}(z) + Q(z) \geqslant F \qquad \text{q.e. on } \mathcal{K}, \tag{4.7}$$

then $\sigma = \mu_w$ and $F = F_w$. Using this result we can explicitly find the extremal measure for the weight $w(z)$ given by (4.1).

THEOREM 4.2. *Let $\mathcal{K} \subset \mathbf{C}$ be an arbitrary compact set of positive capacity and let ν be a Borel measure with compact support $S_\nu \subset \mathbf{C} \backslash \mathcal{K}$ such that $\|\nu\| = \alpha$, $0 \leqslant \alpha \leqslant 1$. Then the solution of the weighted energy problem for $w(z)$ as defined by* (4.1), *is given by*

$$\mu_w = (1-\alpha)\, \mu_{\mathcal{K}} + \hat{\nu}, \tag{4.8}$$

where $\mu_{\mathcal{K}}$ is the classical equilibrium distribution for $\mathcal{K}$ and $\hat{\nu}$ is the balayage of ν to $\mathcal{K}$.

Furthermore,

$$F_w = (1-\alpha)\log\frac{1}{C} + \int_\Omega g_\Omega(t, \infty)\, d\nu(t), \tag{4.9}$$

where $C := \operatorname{cap} \mathscr{K}$ *and* $g_\Omega(t, \infty)$ *is the Green function of the unbounded component* Ω *of* $\overline{\mathbf{C}} \backslash \mathscr{K}$ *with pole at* ∞.

Remark. For the notions of the equilibrium distribution, balayage and Green function see Chapter II and IV of [10].

Proof. Recall, that for the weight $w(z)$ given by (4.1)

$$Q(z) = -U^\nu(z), \qquad z \in \mathscr{K}.$$

Section IV.1 of [10] yields the existence of the balayage $\hat{\nu}$ which is the unique measure with $\operatorname{supp} \hat{\nu} \subset \partial\mathscr{K}$ and $\|\hat{\nu}\| = \|\nu\| = \alpha$, such that

$$U^{\hat{\nu}}(z) = U^\nu(z) + \int g_\Omega(t, \infty)\, d\nu(t) \qquad \text{for q.e.} \quad z \in \mathscr{K}, \tag{4.10}$$

and

$$U^{\hat{\nu}}(z) \leqslant U^\nu(z) + \int g_\Omega(t, \infty)\, d\nu(t) \qquad \forall z \in \mathbf{C}. \tag{4.11}$$

Since $U^\nu(z)$ is uniformly bounded on $\operatorname{supp} \hat{\nu} \subset \partial\mathscr{K}$, the last inequality implies by integration with respect to $\hat{\nu}$ that $\hat{\nu}$ has finite logarithmic energy. This means that $\mu := (1-\alpha)\,\mu_{\mathscr{K}} + \hat{\nu}$ also has finite logarithmic energy. Next, we recall from Frostman's theorem ([19, Section III.3]),

$$U^{\mu_{\mathscr{K}}}(z) = \log\frac{1}{C} \qquad \text{q.e. on } \mathscr{K}.$$

Then, it follows from (4.10) that q.e. on $\mathscr{K}$

$$\begin{aligned} U^\mu(z) + Q(z) &= (1-\alpha)\, U^{\mu_{\mathscr{K}}}(z) + U^{\hat{\nu}}(z) - U^\nu(z) \\ &= (1-\alpha)\log\frac{1}{C} + \int g_\Omega(t, \infty)\, d\nu(t). \end{aligned}$$

It is clear that $\operatorname{supp} \mu \subset \partial\mathscr{K} \subset \mathscr{K}$ and $\|\mu\| = (1-\alpha)\,\|\mu_{\mathscr{K}}\| + \|\hat{\nu}\| = 1 - \alpha + \alpha = 1$. Thus, by (4.6) and (4.7), μ must be the weighted equilibrium distribution with

$$F_w = (1-\alpha)\log\frac{1}{C} + \int g_\Omega(t, \infty)\, d\nu(t),$$

i.e. $\mu = \mu_w$. ∎

Let $\mathcal{K}$ be a bounded continuum whose complement consists of a finite number of domains and, with the notation introduced in Section 2, consider the weight function

$$w(z)=\begin{cases}\prod_{j=1}^{n}|z-a_j|^{-\alpha_j}, & z\in\mathcal{K},\\ 0, & z\in\mathbf{C}\backslash\mathcal{K},\end{cases}\tag{4.12}$$

where, as before, $\alpha_j\geqslant 0$, $j=1, ..., n$, $\sum_{j=1}^{n}\alpha_j\leqslant 1$, and for each $j=1, 2, ..., n$, a_j is a fixed point in the bounded component G_j of $\mathbf{C}\backslash\mathcal{K}$. Then $w(z)$ is continuous as a function defined on $\mathcal{K}$ and

$$w(z)=e^{-Q(z)},\qquad z\in\mathcal{K},\tag{4.13}$$

where

$$Q(z):=\sum_{j=1}^{n}\alpha_j\log|z-a_j|.\tag{4.14}$$

COROLLARY 4.3. *The solution of the minimal weighted energy problem for*

$$w(z)=\prod_{l=1}^{n}|z-a_l|^{-\alpha_l},\qquad z\in\mathcal{K},$$

is given by the measure

$$\mu_w=\left(1-\sum_{l=1}^{n}\alpha_l\right)\mu_e+\sum_{l=1}^{n}\alpha_l\mu_l.\tag{4.15}$$

Furthermore,

$$F_w=\left(1-\sum_{l=1}^{n}\alpha_l\right)\log\frac{1}{C},\tag{4.16}$$

where C is the logarithmic capacity of $\mathcal{K}$.

Proof. For the weight $w(z)$ defined by (4.12) we have that

$$Q(z)=-U^{\nu}(z),\qquad z\in\mathcal{K},$$

for the measure

$$\nu=\sum_{l=1}^{n}\alpha_l\delta_{a_l},$$

where δ_t is the Dirac δ-measure at the point t and

$$\alpha := \|\nu\| = \sum_{l=1}^{n} \alpha_l.$$

It is well known (cf. [10, p. 222]) that the balayage of the unit point mass δ_{a_l} supported in a_l to $\partial\mathscr{K}$ is just $\mu_l = \omega(a_l, \cdot, G_l)$, $l = 1, ..., n$. Therefore,

$$\hat{\nu} = \sum_{l=1}^{n} \alpha_l \mu_l$$

and so by Theorem 4.2,

$$\mu_w = \left(1 - \sum_{l=1}^{n} \alpha_l\right) \mu_e + \sum_{l=1}^{n} \alpha_l \mu_l.$$

Since in this case $\int g_\Omega(t, \infty)\, d\nu(t) = 0$, formula (4.9) yields

$$F_w = \left(1 - \sum_{l=1}^{n} \alpha_l\right) \log \frac{1}{C},$$

which completes the proof. ∎

5. PROOFS

5.1. *Lemmas*

Before we proceed with the proofs of our main results we need several important lemmas. The first of these is an analogue of the well-known Bernstein–Walsh lemma for rational functions $R_N(z)$.

Assume that E is a bounded continuum whose complement consists of a finite number of domains. Let us denote the bounded components of $\overline{\mathbf{C}} \setminus E$ by $\{\tilde{G}_l\}_{l=1}^{n}$ and the unbounded component by $\tilde{\Omega}$. Consider the conformal mappings $\tilde{\phi}_l \colon \tilde{G}_l \to D = \{w \colon |w| < 1\}$ and $\tilde{\Phi} \colon \tilde{\Omega} \to D' = \{w \colon |w| > 1\}$ with normalizations respectively $\tilde{\phi}_l(a_l) = 0$, $\tilde{\phi}_l'(a_l) > 0$, where $a_l \in \tilde{G}$, $l = 1, ..., n$, and $\tilde{\Phi}(\infty) = \infty$, $\tilde{\Phi}'(\infty) > 0$.

LEMMA 5.1. *For the rational function $R_N(z)$ defined by* (2.5) *we have that*

$$|R_N(z)| \leqslant \|R_N\|_{\partial\tilde{\Omega}}\, |\tilde{\Phi}(z)|^k, \qquad z \in \tilde{\Omega} \tag{5.1}$$

and

$$|R_N(z)| \leqslant \frac{\|R_N\|_{\partial \tilde{G}_l}}{|\tilde{\phi}_l(z)|^{m_l}}, \qquad z \in \tilde{G}_l, \quad l = 1, ..., n, \tag{5.2}$$

where the norms are Chebyshev norms.

Proof. The proof is standard: we consider the function $h(z) = R_N(z)/[\tilde{\Phi}(z)]^k$ and observe that it is analytic everywhere in $\tilde{\Omega}$. Thus, by the maximum modulus principle, we have that

$$\frac{|R_N(z)|}{|\tilde{\Phi}(z)|^k} \leqslant \left\| \frac{R_N}{\tilde{\Phi}^k} \right\|_{\partial \tilde{\Omega}} = \|R_N\|_{\partial \tilde{\Omega}}, \qquad z \in \tilde{\Omega}.$$

This gives (5.1).

Similarly, by considering the function $h(z) = R_N(z)[\tilde{\phi}_l(z)]^{m_l}$, which is analytic in $\tilde{G}_l$, we obtain in the same way that

$$|R_N(z)| \cdot |\tilde{\phi}_l(z)|^{m_l} \leqslant \|R_N \tilde{\phi}_l^{m_l}\|_{\partial \tilde{G}_l} = \|R_N\|_{\partial \tilde{G}_l}, \qquad z \in \tilde{G}_l, \quad l = 1, ..., n,$$

and (5.2) follows. ∎

As a consequence, we have the following:

LEMMA 5.2. *Let the continuum $\mathscr{K}$ satisfy the assumptions of Theorem 2.2, and assume that the sequence $\{R_N(z)\}_{i \in \Lambda}$ converges locally uniformly in $\mathscr{K}^\circ$ to $f(z) (\not\equiv 0)$. Then*

$$\lim_{\substack{i \to \infty \\ i \in \Lambda}} \|R_N\|_{\partial \Omega}^{1/k} = 1 \tag{5.3}$$

and

$$\lim_{\substack{i \to \infty \\ i \in \Lambda}} \|R_N\|_{\partial G_l}^{1/m_l} = 1, \qquad l = 1, ..., n. \tag{5.4}$$

Proof. Since $\mathscr{K}$ is the closure of a finitely connected Jordan domain $\mathscr{K}^\circ$, an argument similar to that of [20, p. 96], shows that for any $R > 1$ we can find a continuum $E \subset \mathscr{K}^\circ$ such that $\partial \Omega \subset E_R$, where E_R denotes the continuum bounded by the level curve $|\Phi(z)| = R$ and E. (It is enough to take E to be a Jordan curve sufficiently close to $\partial \Omega$.) Then, by the locally uniform convergence of $R_N(z)$ to $f(z) \not\equiv 0$ in $\mathscr{K}^\circ$, we have that

$$\|R_N\|_E \leqslant \|f - R_N\|_E + \|f\|_E \leqslant 2 \, \|f\|_E$$

for i sufficiently large. Thus, from (5.1),

$$\|R_N\|_{|\tilde{\Phi}(z)|=R} \leqslant 2\, \|f\|_E\, R^k,$$

and, by the maximum modulus principle, this yields

$$\limsup_{i\to\infty} \|R_N\|_{\partial\Omega}^{1/k} \leqslant \limsup_{i\to\infty} \|R_N\|_{E_R}^{1/k} \leqslant R.$$

Hence, on letting $R \to 1$ we get that

$$\limsup_{i\to\infty} \|R_N\|_{\partial\Omega}^{1/k} \leqslant 1. \tag{5.5}$$

Similarly, by taking E to be a Jordan curve close to ∂G_l, and using (5.2) we find that

$$\|R_N\|_{|\tilde{\phi}_l(z)|=1/R} \leqslant 2\, \|f\|_E\, R^{m_l}, \qquad l=1, ..., n.$$

Hence,

$$\limsup_{i\to\infty} \|R_N\|_{\partial G_l}^{1/m_l} \leqslant 1, \qquad l=1, ..., n. \tag{5.6}$$

If we assume that $\liminf_{i\to\infty} \|R_N\|_{\partial\Omega}^{1/k} = q < 1$, then we can conclude that there exists a subsequence $\{R_N\}_{i\in\Lambda'\subset\Lambda}$ uniformly convergent to zero on $\partial\Omega$. It then follows from (5.1), with $E=\partial\Omega$, that this convergence takes place in the strip between $\partial\Omega$ and the level curve defined by $|\Phi(z)|=R$ for some $R>1$, i.e.

$$\begin{aligned}\lim_{\substack{i\to\infty\\ i\in\Lambda'}} |R_N(z)|^{1/k} &\leqslant \lim_{\substack{i\to\infty\\ i\in\Lambda'}} \|R_N\|_{\partial\Omega}^{1/k}\cdot R\\ &\leqslant qR,\ z\in\{z\in t\colon 1\leqslant|\Phi(t)|\leqslant R\}.\end{aligned}$$

Thus, we only need to take R such that $qR<1$. But this implies the existence of analytic continuation of $f(z)$ through $\partial\Omega$, which vanishes identically in the strip. Thus, $f(z)$ must vanish everywhere, contradicting our assumption that $f(z)\not\equiv 0$. This proves (5.3).

The same argument can be applied in the case of ∂G_l, $l=1, ..., n$, to prove that

$$\liminf_{i\to\infty} \|R_N\|_{\partial G_l}^{1/m_l} < 1$$

is impossible for any $l=1, ..., n$, thus yielding (5.4). ∎

LEMMA 5.3. *For the leading coefficients of $R_N(z)$ defined by* (2.5) *we have that*

$$|t_k^N| \leqslant \frac{1}{C^k} \|R_N\|_{\partial\Omega}, \qquad C = \operatorname{cap} \mathscr{K}, \tag{5.7}$$

and

$$|s_{l,m_l}^N| \leqslant R_l^{m_l} \|R_N\|_{\partial G_l}, \tag{5.8}$$

$l = 1, ..., n$.

Proof. Following the proof of Lemma 5.1, and the maximum modulus principle we have

$$\frac{|R_N(z)|}{|\Phi(z)|^k} \leqslant \|R_N\|_{\partial\Omega}, \qquad z \in \Omega.$$

This gives (5.7), if we pass to the limit as $z \to \infty$. Similarly, by passing to the limit with $z \to a_l$ in

$$|R_N(z)| \cdot |\phi_l(z)|^{m_l} \leqslant \|R_N\|_{\partial G_l}, \qquad z \in G_l,$$

we obtain (5.8). ∎

COROLLARY 5.4. *Let $\mathscr{K}$ and $\{R_N\}_{i \in \Lambda}$ be the same as in Lemma* 5.2. *Then*

$$\limsup_{i \to \infty} |t_k^N|^{1/k} \leqslant \frac{1}{C} \tag{5.9}$$

and

$$\limsup_{i \to \infty} |s_{l,m_l}^N|^{1/m_l} \leqslant R_l, \qquad l = 1, ..., n. \tag{5.10}$$

Proof. Inequalities (5.9) and (5.10) follow immediately from (5.3), (5.4), (5.7) and (5.8). ∎

LEMMA 5.5. *With the assumptions of Theorem* 2.2, *the monic polynomial $P_N(z)$, in* (2.7), *is asymptotically extremal on $\mathscr{K}$ with respect to the weight $w(z)$ defined by* (4.12), *i.e.*

$$\lim_{\substack{i \to \infty \\ i \in \Lambda}} \|w^{|N|} P_N\|_{\mathscr{K}}^{1/|N|} = e^{-F_w}, \tag{5.11}$$

where F_w is given by (4.16).

Proof. Since $\mathscr{K}$ is a continuum and $a_l \notin \mathscr{K}$, $l=1, ..., n$, there exist two constants d_1 and d_2 such that

$$d_1 \leqslant |z-a_l| \leqslant d_2, \qquad \forall z \in \mathscr{K}, \quad l=1, ..., n.$$

Consider

$$\begin{aligned}
&\limsup_{i\to\infty} \|w^{|N|}P_N\|^{1/|N|} \\
&= \limsup_{i\to\infty} \left\| \left(\prod_{l=1}^{n} |z-a_l|^{-\alpha_l} \right)^{|N|} P_N(z) \right\|_{\mathscr{K}}^{1/|N|} \\
&= \limsup_{i\to\infty} \left\| \left(\prod_{l=1}^{n} |z-a_l|^{-\alpha_l} \right)^{|N|} \frac{\prod_{l=1}^{n} (z-a_l)^{m_l}}{t_k^N} R_N(z) \right\|_{\mathscr{K}}^{1/|N|} \\
&\leqslant \limsup_{i\to\infty} \left[\frac{1}{|t_k^N|^{1/|N|}} \left\| \prod_{l=1}^{n} (z-a_l)^{(m_l/|N|)-\alpha_l} \right\|_{\mathscr{K}} \cdot \|R_N\|_{\mathscr{K}}^{1/|N|} \right] \\
&= C^{1-\sum_{l=1}^{n}\alpha_l} = e^{-F_w}.
\end{aligned}$$

(For the above we made use of Theorem 2.2(i), (2.9), (5.3) and (5.4) in order to deduce that

$$\lim_{\substack{i\to\infty\\ i\in\Lambda}} \left\| \prod_{l=1}^{n} (z-a_l)^{(m_l/|N|)-\alpha_l} \right\|_{\mathscr{K}} = 1,$$

$$\lim_{\substack{i\to\infty\\ i\in\Lambda}} \|R_N\|_{\mathscr{K}}^{1/|N|} = 1,$$

and

$$\left. \lim_{\substack{i\to\infty\\ i\in\Lambda}} \frac{1}{|t_k^N|^{1/|N|}} = C^{1-\sum_{l=1}^{n}\alpha_l}. \right)$$

By Corollary 4.5 of [12] and Corollary 4.3 we have

$$\liminf_{i\to\infty} \|w^{|N|}P_N\|_{\mathscr{K}}^{1/|N|} \geqslant e^{-F_w}, \tag{5.12}$$

from which the lemma follows immediately. ∎

LEMMA 5.6. *Let ν_N be the measure defined by* (2.8). *Then, with the assumptions of Theorem 2.2 we have that*

$$\nu_N(B) \to 0, \qquad \text{as} \quad i\to\infty, \quad i\in\Lambda, \tag{5.13}$$

for any closed subset $B \subset (\bigcup_{l=1}^{n} G_l) \cup \Omega$.

Proof. If $B \subset \Omega$, then Lemma 5.6 is implied by Lemma 5.5 and Theorem 2.3(a) of [11]. In case of $B \subset G_l$, $1 \leqslant l \leqslant n$, the proof of this lemma is similar to the proof of Lemma 4.1 of [11] or can be directly reduced to it by the transformation $u = 1/(z - a_l)$. ∎

5.2. *Proofs of Theorems* 2.2, 2.3, *and* 2.6

Proof of Theorem 2.2. Let σ be any weak* limit of ν_N, i.e. $\nu_N \xrightarrow{*} \sigma$ for some subsequence $\Lambda' \subset \Lambda$. By the locally uniform convergence of $\{R_N(z)\}_{i \in \Lambda}$ to $f(z) \not\equiv 0$ we obtain from the Hurwitz theorem that

$$\sigma(B) = 0 \tag{5.14}$$

for any closed set $B \subset \mathscr{K}^\circ$. Then, Lemma 5.6 and (5.14) imply that $\operatorname{supp} \sigma \subset \partial\mathscr{K}$. Clearly, $\sigma \in \mathscr{M}(\mathscr{K})$. We know from Corollary 4.3 that $\operatorname{supp} \mu_w = \partial\mathscr{K}$. By the property (d) of Theorem 4.1 and (5.11) we obtain

$$\int \log |z - t| \, d\nu_N(t) - \int \log |z - t| \, d\mu_w(t) \leqslant \varepsilon_N \tag{5.15}$$

uniformly on $\partial\mathscr{K}$, where $\varepsilon_N \to 0$ as $i \to \infty$, $i \in \Lambda$. Using the principle of domination (Theorem II.3.2 in [15] and Second Maximum Principle in [10], p. 111) we conclude that (5.15) holds for all $z \in \mathbf{C}$.

If we let $\delta > 0$ and consider $\tilde{\nu}_N$, the normalized counting measure in zeros of $P_N(z)$ that are closer than δ to $\partial\mathscr{K}$, then by (5.15)

$$\int \log |z - t| \, d\tilde{\nu}_N(t) - \int \log |z - t| \, d\mu_w(t) \leqslant \varepsilon_N + o(1)$$

for every z such that $\operatorname{dist}(z, \partial\mathscr{K}) > \delta$. But $\tilde{\nu}_N \xrightarrow{*} \sigma$, as $i \to \infty$, $i \in \Lambda'$, therefore we have for every $z \notin \partial\mathscr{K}$

$$-\int \log \frac{1}{|z - t|} \, d\sigma(t) + \int \log \frac{1}{|z - t|} \, d\mu_w(t) \leqslant 0. \tag{5.16}$$

The function $u(z) := U^{\mu_w}(z) - U^{\sigma}(z)$ is harmonic in $\overline{\mathbf{C}} \setminus \partial\mathscr{K}$. If we show that $u(z)$ has a zero in some component of $\overline{\mathbf{C}} \setminus \partial\mathscr{K}$, then $u(z)$ vanishes identically there by (5.16) and the maximum principle [19].

First, we consider Ω, which is the unbounded component of $\overline{\mathbf{C}} \setminus \partial\mathscr{K}$. It is not difficult to see that

$$\lim_{z \to \infty} u(z) = \lim_{z \to \infty} \int \log \frac{1}{|z - t|} \, d(\mu_w - \sigma)(t) = 0 = u(\infty).$$

Consequently,

$$u(z) \equiv 0 \qquad \text{in } \Omega. \tag{5.17}$$

Using the assumption that the sequence $\{R_N(z)\}_{i \in \Lambda}$ converges locally uniformly to $f(z) \not\equiv 0$, we can choose a point $z_0 \in \mathscr{K}^\circ$ such that

$$\lim_{\substack{i \to \infty \\ i \in \Lambda}} R_N(z_0) = f(z_0) \neq 0.$$

This implies the existence of some neighborhood of z_0 that doesn't contain zeros of $R_N(z)$ by the Hurwitz theorem, for $i \in \Lambda$ large enough. Thus, we have by the weak* convergence

$$\begin{aligned} U^\sigma(z_0) &= \lim_{\substack{i \to \infty \\ i \in \Lambda'}} U^{\nu_N}(z_0) \\ &= \lim_{\substack{i \to \infty \\ i \in \Lambda'}} \frac{1}{|N|} \log \frac{1}{|P_N(z_0)|} \\ &= \lim_{\substack{i \to \infty \\ i \in \Lambda'}} \log \frac{|t_k^N|^{1/|N|} \prod_{l=1}^n |z_0 - a_l|^{-m_l/|N|}}{|R_N(z_0)|^{1/|N|}} \\ &= \lim_{\substack{i \to \infty \\ i \in \Lambda'}} \left(\frac{k}{|N|} \log \, |t_k^N|^{1/k} + \sum_{l=1}^n \frac{m_l}{|N|} \log \frac{1}{|z_0 - a_l|} \right) \\ &= \left(1 - \sum_{l=1}^n \alpha_l \right) \log \frac{1}{C} + \sum_{l=1}^n \alpha_l \log \frac{1}{|z_0 - a_l|}. \end{aligned} \tag{5.18}$$

For the potential of the weighted equilibrium distribution we obtain

$$\begin{aligned} U^{\mu_w}(z_0) &= \left(1 - \sum_{l=1}^n \alpha_l \right) \int \log \frac{1}{|z_0 - t|} \, d\mu_e(t) + \sum_{l=1}^n \alpha_l \int \log \frac{1}{|z_0 - t|} \, d\mu_l(t) \\ &= \left(1 - \sum_{l=1}^n \alpha_l \right) \log \frac{1}{C} + \sum_{l=1}^n \alpha_l \log \frac{1}{|z_0 - a_l|}. \end{aligned} \tag{5.19}$$

Comparing (5.18) and (5.19) we conclude that

$$U^{\mu_w}(z) = U^\sigma(z), \qquad \forall z \in \mathscr{K}^\circ. \tag{5.20}$$

Since both potentials are continuous in the fine topology (see Section V.3 of [10]) and the boundary of Ω in the fine topology is the same as the Euclidean boundary [15], then we obtain by (5.17) that $U^{\mu_w}(z) = U^\sigma(z)$, $\forall z \in \partial\Omega$. Thus, we have

$$U^{\mu_w}(z) = U^\sigma(z), \qquad \forall z \in \mathbf{C} \setminus \overline{G}, \tag{5.21}$$

where $G=\bigcup_{l=1}^{n} G_l$. By the unicity theorem [15] we get

$$\sigma|_{\partial\Omega}=\mu_w|_{\partial\Omega}.$$

Now, we apply the fine topology argument to the domain $\overline{\mathbf{C}}\backslash\overline{G}$ to conclude that

$$U^{\mu_w}(z)=U^{\sigma}(z) \qquad \text{on} \quad \partial(\overline{\mathbf{C}}\backslash\overline{G})=\bigcup_{l=1}^{n}\partial G_l.$$

Since $U^{\mu_w}(z)$ and $U^{\sigma}(z)$ coincide on $\partial\mathscr{K}$, which contains the supports of both measures, we obtain

$$U^{\mu_w}(z)=U^{\sigma}(z), \qquad \forall z\in\mathbf{C},$$

by the maximum principle for harmonic functions. Another application of the unicity theorem yields

$$\sigma\equiv\mu_w.$$

Let us turn to the proof of the converse part in Theorem 2.2. We have that $\nu_N \xrightarrow{*} \sigma$ as $i\to\infty$, $i\in\Lambda$, where $\sigma=\mu_w$. This implies

$$U^{\sigma}(z)=U^{\mu_w}(z), \qquad \forall z\in\mathbf{C}. \tag{5.22}$$

By the same argument as in (5.18) we get for $z_0\in\mathscr{K}^{\circ}$ such that $f(z_0)\neq 0$,

$$U^{\mu_w}(z_0)=\left(1-\sum_{l=1}^{n}\alpha_l\right)\log \lim_{\substack{i\to\infty\\ i\in\Lambda}}|t_k^N|^{1/k}+\sum_{l=1}^{n}\alpha_l\log\frac{1}{|z_0-a_l|}.$$

The last equality and (5.19) immediately gives condition (i) of Theorem 2.2.

In order to prove (ii) we use the assumption that every $a_l\in G_l$ has some zero free neighborhood, so that by the weak* convergence

$$\begin{aligned}
&U^{\mu_w}(a_l)\\
&\quad=\lim_{\substack{i\to\infty\\ i\in\Lambda}}\frac{1}{|N|}\log\frac{1}{|P_N(a_l)|}=\lim_{\substack{i\to\infty\\ i\in\Lambda}}\frac{1}{|N|}\log\left|\frac{t_k^N}{s_{l,m_l}^N\prod_{j\neq l}|a_l-a_j|^{m_j}}\right|\\
&\quad=\lim_{\substack{i\to\infty\\ i\in\Lambda}}\left(\frac{k}{|N|}\log|t_k^N|^{1/k}+\frac{m_l}{|N|}\log\frac{1}{|s_{l,m_l}^N|^{1/m_l}}+\log\prod_{j\neq l}|a_l-a_j|^{-(m_j/|N|)}\right)\\
&\quad=\left(1-\sum_{j=1}^{n}\alpha_j\right)\log\frac{1}{C}+\alpha_l\log\frac{1}{\lim_{m_l\to\infty}|s_{l,m_l}^N|^{1/m_l}}\\
&\qquad+\sum_{j\neq l}\alpha_j\log\frac{1}{|a_l-a_j|}.
\end{aligned}\tag{5.23}$$

We observe that the function $g(t)=(\phi_l(t)-\phi_l(a_l))/(t-a_l)$ is analytic in G_l and continuous on $\overline{G_l}$. Also, $g(t)\neq 0$, $\forall t\in \overline{G}_l$. This implies that $\log|g(t)|$ is harmonic in G_l and continuous on $\overline{G_l}$ with

$$\log|g(a_l)|=\log|\phi'(a_l)|=\log\frac{1}{R_l}.$$

Now we calculate

$$\begin{aligned}
U^{\mu_w}(a_l)&=\left(1-\sum_{j=1}^{n}\alpha_j\right)\log\frac{1}{C}+\sum_{j=1}^{n}\alpha_j\int\log\frac{1}{|z-a_l|}\,d\mu_j(z)\\
&=\left(1-\sum_{j=1}^{n}\alpha_j\right)\log\frac{1}{C}+\sum_{j\neq l}\alpha_j\log\frac{1}{|a_j-a_l|}\\
&\quad+\alpha_l\int\log\left|\frac{\phi_l(z)-\phi_l(a_l)}{z-a_l}\right|\,d\mu_l(z)\\
&=\left(1-\sum_{j=1}^{n}\alpha_j\right)\log\frac{1}{C}+\sum_{j\neq l}\alpha_j\log\frac{1}{|a_j-a_l|}\\
&\quad+\alpha_l\log|\phi_l'(a_l)|.
\end{aligned}\tag{5.24}$$

By (5.22) condition (ii) of Theorem 2.2 now follows from (5.23) and (5.24), $l=1, ..., n$. ∎

Proof of Theorem 2.3. The proofs of (2.17) and (2.18) are the same as in the proof of Theorem 2.2 so we need only to present the proof of (2.19). By (2.17) and (2.18) we have

$$U^{\nu|_{\bar{G}}}(z)=U^{\mu_w|_{\bar{G}}}(z),\qquad \forall z\notin \overline{G}.\tag{5.25}$$

Let σ be a weak* limit of the balayages of $\nu_N|_{\bar{G}}$ to ∂G. Then, by the lower envelope theorem [15] we have

$$\liminf_{\substack{i\to\infty\\ i\in\Lambda'}} U^{\widehat{\nu_N|_{\bar{G}}}}(z)=U^{\sigma}(z)\tag{5.26}$$

and

$$\liminf_{\substack{i\to\infty\\ i\in\Lambda'}} U^{\nu_N|_{\bar{G}}}(z)=U^{\nu|_{\bar{G}}}(z),\tag{5.27}$$

where both equalities hold q.e. in $\mathbf{C}\backslash\overline{G}$ for a subsequence $\Lambda'\subset\Lambda$. Since

$$U^{\widehat{\nu_N|_{\bar{G}}}}(z)=U^{\nu_N|_{\bar{G}}}(z),\qquad \forall z\in\mathbf{C}\backslash\overline{G},$$

by the definition of balayage, we obtain from (5.26) and (5.27)

$$U^{\sigma}(z) = U^{\nu|_{\bar{G}}}(z) \qquad \text{q.e. in } \mathbf{C}\backslash\bar{G}.$$

The last equality together with (5.25) yields

$$U^{\sigma}(z) = U^{\mu_w|_{\bar{G}}}(z) \qquad \text{q.e. in } \mathbf{C}\backslash\bar{G}.$$

But $\operatorname{supp} \sigma \subset \partial G$ and $\operatorname{supp} \mu_w|_{\bar{G}} \subset \partial G$, therefore both potentials are continuous in $\mathbf{C}\backslash\bar{G}$ and

$$U^{\sigma}(z) = U^{\mu_w|_{\bar{G}}}(z), \qquad \forall z \in \mathbf{C}\backslash\bar{G}.$$

Consequently, the measures must be identical by the Carleson unicity theorem (see [3] and [15]):

$$\sigma = \mu_w|_{\bar{G}}. \tag{5.28}$$

Thus, by (2.18) and (5.28) we have

$$\hat{\nu}_N \xrightarrow{*} \nu|_{\mathbf{C}\backslash\bar{\mathbf{G}}} + \sigma = \mu_w|_{\mathbf{C}\backslash\bar{\mathbf{G}}} + \mu_w|_{\bar{G}} = \mu_w,$$

as $i \to \infty$, $i \in \Lambda$. ∎

Proof of Theorem 2.6. This proof follows that of Theorem 2.2 and therefore is omitted.

5.3. *Proof of Theorem* 3.2

First, we show that $p_{m,n}(z)$ converges to some analytic function g locally uniformly in A_n, with $g \not\equiv 0$. To this end we recall that by the additive splitting (3.8) the function $f^+(z)$ must have a meromorphic continuation with precisely n poles in $\{z: R \leqslant |z| < R_n\}$ because $f^-(z)$ is analytic in $\{z: |z| > r\}$. Similarly, $f^-(z)$ has a meromorphic continuation with precisely n poles in $\{z: r_n < |z| < r\}$.

Suppose that $\{z_j^+\}_{j=1}^{k^+}$ are the poles of $f^+(z)$ with the corresponding multiplicities $\{l_j^+\}_{j=1}^{k^+}$ such that

$$\sum_{j=1}^{k^+} l_j^+ = n.$$

From the classical de Montessus de Ballore's theorem [1] we have

$$\lim_{m \to \infty} r_{m,n}^+(z) = f^+(z),$$

where the convergence is locally uniform in $\{z\colon |z|<R_n\}\backslash\{z_j^+\}_{j=1}^{k^+}$, and

$$\lim_{m\to\infty} q_{m,n}^+(z)=Q_n^+(z), \tag{5.29}$$

where

$$Q_n^+(z):=\prod_{j=1}^{k^+}\left(1-\frac{z}{z_j^+}\right)^{l_j^+}.$$

So,

$$p_{m,n}^+(z)\to f^+(z)\,Q_n^+(z), \qquad \text{as} \quad m\to\infty, \tag{5.30}$$

locally uniformly in $\{z\colon |z|<R_n\}$.

Similarly, we have

$$p_{m,n}^-\left(\frac{1}{z}\right)\to f^-(z)\,Q_n^-\left(\frac{1}{z}\right), \qquad \text{as} \quad m\to\infty, \tag{5.31}$$

locally uniformly in $\{z\colon |z|>r_n\}$, and

$$q_{m,n}^-\left(\frac{1}{z}\right)\to Q_n^-\left(\frac{1}{z}\right):=\prod_{j=1}^{k^-}\left(1-\frac{z_j^-}{z}\right)^{l_j^-}, \qquad \text{as} \quad m\to\infty, \tag{5.32}$$

locally uniformly in $\mathbf{C}$, where $\{z_j^-\}_{j=1}^{k^-}$ are the poles of $f^-(z)$ in $\{z\colon r_n<|z|\leqslant r\}$ with multiplicities $\{l_j^-\}_{j=1}^{n}$ such that

$$\sum_{j=1}^{k^-} l_j^- = n.$$

Taking into account (5.29), (5.30), (5.31) and (5.32) we obtain from (3.9)

$$p_{m,n}(z)\to(f^+Q_n^+\,Q_n^- + f^-Q_n^-\,Q_n^+)(z), \qquad \text{as} \quad m\to\infty,$$

locally uniformly in A_n. But

$$\begin{aligned} g(z) &:= (f^+Q_n^+\,Q_n^- + f^-Q_n^-\,Q_n^+)(z) \\ &= Q_n^+(z)\,Q_n^-(z)(f^+(z)+f^-(z)) \\ &= f(z)\,Q_n^+(z)\,Q_n^-(z) \end{aligned}$$

for any $z\in A_n$. Thus,

$$p_{m,n}(z)\to g(z), \qquad \text{as} \quad m\to\infty, \tag{5.33}$$

locally uniformly in A_n, where $g(z)$ is analytic in A_n doesn't vanish identically.

To finish the proof we need to find the mth root behavior for the leading coefficients of $p_{m,n}(z)$. Applying the result of [4, p. 263] to $f^+(z)$ and $p^+_{m,n}(z)$ we obtain that $\forall \Lambda_1 \subset \mathbf{N}$ such that

$$\lim_{\substack{m \to \infty \\ m \in \Lambda_1}} |a^{+,(m_k)}_{m_k,n}|^{1/m_k} = \frac{1}{R_n}. \tag{5.34}$$

Then (3.11) follows from (5.33), (5.34) and Theorem 2.3 (cf. (2.18)).

Using the transformation $w = 1/z$ we proceed in the same manner for $f^-(z)$ and $p^-_{m,n}(z)$ to deduce that $\exists \Lambda_2 \subset \mathbf{N}$ such that

$$\lim_{\substack{m \to \infty \\ m \in \Lambda_2}} |a^{-,(m_l)}_{m_l,n}|^{1/m_l} = r_n.$$

Another application of Theorem 2.3 (2.18) yields (3.12). ∎

REFERENCES

1. G. A. Baker, Jr., "Essentials of Padé Approximants," Academic Press, New York, 1975.
2. H. P. Blatt, E. B. Saff, and M. Simkani, Jentzsch–Szegő type theorems for the zeros of best approximants, *J. London Math. Soc.* **38** (1988), 307–316.
3. L. Carleson, Mergelyan's theorem on uniform polynomial approximation, *Math. Scand.* **15** (1964), 167–175.
4. A. Edrei, Angular distribution of the zeros of Padé polynomials, *J. Approx. Theory* **24** (1978), 251–265.
5. A. Edrei, Zeros of partial sums of Laurent series, *Michigan Math. J.* **29** (1982), 43–57.
6. G. M. Goluzin, "Geometric Theory of Functions of a Complex Variable," Translations of Mathematical Monographs, Vol. 26, American Math. Society, Providence, RI, 1969.
7. W. B. Gragg and G. D. Johnson, "The Laurent–Padé Table," Information Processing, Vol. 74, "Proc. 74th IEIP Congress, Amsterdam," pp. 632–637, North-Holland, Amsterdam, 1974.
8. R. Grothmann, Ostrowski gaps, overconvergence, and zeros of polynomials, *in* "Approximation Theory VI (New York)" (L. L. Schumaker, C. K. Chui, and J. D. Ward, Eds.), pp. 1–4, Academic Press, New York, 1989.
9. R. Jentzsch, "Untersuchungen zur Theorie Analytischer Funktionen," Ph.D. thesis, Berlin, 1914.
10. N. S. Landkof, "Foundations of Modern Potential Theory," Springer-Verlag, Berlin, 1972.
11. H. N. Mhaskar and E. B. Saff, The distribution of zeros of asymptotically extremal polynomials, *J. Approx. Theory* **65** (1991), 279–300.
12. H. N. Mhaskar and E. B. Saff, Weighted analogues of capacity, transfinite diameter and Chebyshev constant, *Constr. Approx.* **8** (1992), 105–124.
13. R. Nevanlinna, "Analytic Functions," Springer-Verlag, New York, 1970.
14. I. E. Pritsker, "Convergence and Zero Distribution of Laurent-Type Rational Functions," Ph.D. thesis, University of South Florida, 1995.

15. E. B. Saff and V. Totik, "Logarithmic Potentials with External Fields," Springer-Verlag, New York/Berlin, 1997.
16. H. Stahl and V. Totik, "General Orthogonal Polynomials," Encyclopedia of Mathematics, Vol. 43, Cambridge Univ. Press, New York, 1992.
17. G. Szegő, Uber die Nullstellen von Polynomen, die in einem Kreis gleichmässig konvergieren, *Sitzungsber. Ber. Math. Ges.* **21** (1922), 59–64.
18. H. Tietz, Faber series and the Laurent decomposition, *Michigan Math. J.* **4** (1957), 157–179.
19. M. Tsuji, "Potential Theory in Modern Function Theory," Maruzen, Tokyo, 1959.
20. J. L. Walsh, "Interpolation and Approximation by Rational Functions in the Complex Domain," Colloquium Publications, Vol. 20, American Math. Society, Providence, RI, 1969.