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CONSTRUCTIVE APPROXIMATION

Rational Approximation with Varying Weights I

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Abstract. We investigate two problems concerning uniform approximation by weighted rationals $\{w^n r_n\}_{n=1}^{\infty}$, where $r_n = p_n/q_n$ is a rational function of order n. Namely, for $w(x) := e^x$ we prove that uniform convergence to 1 of $w^n r_n$ is not possible on any interval $[0, a]$ with $a > 2\pi$. For $w(x) := x^\theta$, $\theta > 1$, we show that uniform convergence to 1 of $w^n r_n$ is not possible on any interval $[b, 1]$ with $b < \tan^4(\pi(\theta - 1)/4\theta)$. (The latter result can be interpreted as a rational analogue of results concerning "incomplete polynomials.") More generally, for $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta > 0$, we investigate for $w(x) = e^x$ and $w(x) = x^\theta$, the possibility of approximation by $\{w^n p_n/q_n\}_{n=1}^{\infty}$, where $\deg p_n \leq \alpha n$, $\deg q_n \leq \beta n$. The analysis utilizes potential theoretic methods. These are essentially sharp results though this will not be established in this paper.

1. Introduction and Main Results

1.1. For a positive, continuous function $f(x)$ on $\mathbf{R}_+ = [0, +\infty)$ we define

$$\delta_n(f; R) := \inf_{r \in \mathcal{R}_n} \left\| \frac{r(x)}{f(x)} - 1 \right\|_{[0,R]}, \tag{1.1}$$

where $\mathcal{R}_n$ is the set of all real rational functions of order $\leq n$ and $\|\cdot\|_{[a,b]}$ denotes the sup norm over the interval $[a, b]$. That is, we consider the best relative rational approximations to f on $[0, R]$. It is clear that $\delta_n(f; R) \to 0$ as $n \to \infty$ for any fixed $R \in (0, +\infty)$ and, moreover, it is always possible to find an increasing sequence $R_n \to \infty$ satisfying the condition

$$\delta_n(f, R_n) \to 0 \qquad \text{as} \quad n \to \infty. \tag{1.2}$$

On the other hand, if $f(x)$, say, decreases as $x \to \infty$, then R_n satisfying (1.2) cannot increase arbitrarily fast, which raises the question about the maximum possible rate of increase of R_n.

Here we consider this question for the "model" function $f(x) = e^{-x}$.

Theorem 1. *If (1.2) is true for $f(x) = e^{-x}$, then*

$$R_n \leq (1 + \varepsilon)2\pi n$$

for any $\varepsilon > 0$ and $n \geq n(\varepsilon)$.

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Remark A. We do not know whether the inequality $R_n \le 2\pi n$ is true for large enough n. However, Theorem 1 is sharp in the sense that the constant 2π cannot be replaced by any smaller constant. The proof of the last assertion will appear in [RSS].

Remark B. A weaker version of Theorem 1.1 with a constant of 8 replacing 2π follows easily from a result in [B] which says that, for a nonzero rational function of order $\le n$,

$$m\left\{x \in \mathbf{R}\colon \frac{r_n'(x)}{r_n(x)} \ge n\right\} \le 8.$$

Here m denotes the Lebesgue measure. It is reasonable to hypothesize from the results of this paper and its sequel that the constant in the above inequality should be 2π.

In Section 1.3 below we present a generalization of Theorem 1 concerning approximation by ray sequences of rational functions.

We note that the corresponding question about relative *polynomial* approximation is important for the investigation of strong asymptotics for orthogonal polynomials on $\mathbf{R}$ and $\mathbf{R}_+$. Such results dealing with relative polynomial approximation were obtained in [LS], [LR], [ST], and [To].

1.2. Another problem considered in this paper is the approximation of the sequence $x^{n\theta}$ on subintervals of $[0, 1]$. For $\theta > 0$, we set

$$\Delta_n(\theta, b) := \inf_{r \in \mathcal{R}_n} \|x^{n\theta} r(x) - 1\|_{[b,1]}, \qquad b \in (0, 1). \tag{1.3}$$

Theorem 2. *If $\Delta_n(\theta, b) \to 0$ as $n \to \infty$ and $\theta > 1$, then*

$$b \ge \tan^4\left(\frac{\pi}{4}\frac{\theta - 1}{\theta}\right). \tag{1.4}$$

Remark C. It is clear that for $\theta \le 1$ approximation is possible over $[0, 1]$. Furthermore, if $\theta > 1$, the right-hand side of (1.4) cannot be replaced by any larger constant (again this fact will appear in [RSS]).

In Section 1.4 we present a more general result dealing with approximation by ray sequences of rationals.

We note that Theorem 2 is closely related to the completeness of the system of "incomplete rational functions"

$$\left\{x^{n\theta}\frac{p_n(x)}{q_n(x)}\colon \deg p_n, \deg q_n \le n\right\}$$

in $C[b, 1]$. If $n\theta$ is an integer, then $x^{n\theta} p_n(x)/q_n(x)$ may be interpreted as a rational function of order $n(1+\theta)$ with $n\theta$ poles fixed at ∞ and $n\theta$ zeros fixed at 0. Corresponding questions for incomplete polynomials are considered in [Lo] and [SV]. Related questions for incomplete rationals in the complex plane are treated in [BC].

1.3. For fixed $\alpha, \beta \geq 0, \alpha + \beta > 0$, set

$$\delta_n(a; \alpha, \beta) := \inf_{p,q} \left\| e^{nx} \frac{p(x)}{q(x)} - 1 \right\|_{[0,a]}, \tag{1.5}$$

where the infimum is taken over all polynomials p, q with $\deg p \leq \alpha n$, $\deg q \leq \beta n$. We also define

$$a^* := a^*(\alpha, \beta) := \sup\{a : \delta_n(a; \alpha, \beta) \to 0 \text{ as } n \to \infty\}. \tag{1.6}$$

Theorem 3. *We have the inequality $a^* \leq \hat{a}$, where $\hat{a} := 2\pi\alpha$ for $\alpha = \beta$ and*

$$\hat{a} := \frac{2(\alpha - \beta)}{1 - 2\hat{y}}, \tag{1.7}$$

where $\hat{y} = \hat{y}(\alpha, \beta)$ is the root of the equation

$$g(y) := \frac{\sqrt{y(1-y)}}{1-2y} - \sin^{-1}\sqrt{y} = \frac{\pi}{2}\,\frac{\beta}{\alpha - \beta} \tag{1.8}$$

for $\alpha \neq \beta$.

Remark D. It can be shown that $\hat{a}(\alpha, \beta) \to 2\pi\alpha$ as $\beta \to \alpha$ (for fixed α), so the function $\hat{a}(\alpha, \beta)$ is continuous.

Remark E. Both functions $a^*(\alpha, \beta)$ and $\hat{a}(\alpha, \beta)$ are symmetric and therefore we need only consider the case $\alpha \geq \beta$ (the symmetry of a^* can be seen on making the change of variables $x \to a - x$; the symmetry of $\hat{a}$ follows from the identity $g(1-y) + g(y) = -\pi/2$). The case $\alpha \geq \beta$ is equivalent to $\hat{y} \in [0, 1/2]$. In this interval we have

$$g(y) = 2\int_0^y \frac{\sqrt{t(1-t)}}{(1-2t)^2}\,dt, \tag{1.9}$$

and therefore $g(y)$ is increasing from 0 to ∞ on $[0, 1/2]$. Hence the equation $g(y) = (\pi/2)\,\beta/(\alpha - \beta)$ has a unique root for any $\alpha \geq \beta \geq 0$ $(\alpha + \beta > 0)$.

Remark F. For $\alpha + \beta = 1$ (i.e., for a fixed number of free parameters in p_n/q_n) the function $\hat{a}(\alpha, \beta) = \hat{a}(1 - \beta, \beta)$ takes its maximum value over $\beta \in [0, 1]$ at $\beta = 1/2$. This means that diagonal approximations ($al = \beta$) are the most effective among all ray sequences with the same number of free parameters.

1.4. For fixed $\theta > 0, \alpha, \beta \geq 0, \alpha + \beta > 0$, we define

$$\Delta_n(b, \theta; \alpha, \beta) := \inf_{p,q} \left\| x^{n\theta} \frac{p(x)}{q(x)} - 1 \right\|_{[b,1]}, \tag{1.10}$$

where the infimum is taken over all polynomials p, q satisfying $\deg p \leq \alpha n$, $\deg q \leq \beta n$, and we set

$$b^* = b^*(\theta; \alpha, \beta) := \inf\{b : \Delta_n(b, \theta; \alpha, \beta) \to 0 \text{ as } n \to \infty\}. \tag{1.11}$$

Theorem 4. *We have the estimate $b^* \geq \hat{b}$, where $\hat{b} = \hat{b}(\theta; \alpha, \beta)$ is the unique root of the equation*

$$f(b) := \frac{1}{\pi}\int_0^b \frac{\sqrt{(\xi - \sqrt{t})(1 - \xi\sqrt{t})}}{t^{3/4}(1-t)}\,dt = 1 - \frac{\beta}{\theta}; \qquad \xi := 1 + \frac{\alpha}{\theta} - \frac{\beta}{\theta}, \tag{1.12}$$

when $\beta/\theta \leq 1$ and $\hat{b} := 0$ when $\beta/\theta > 1$.

We note that for $\beta/\theta > 1$ the fact that $b^* = 0$ is easily seen.

We shall also obtain the following representation for f:

$$f(b) = 1 - \frac{2}{\pi}\sin^{-1}\sqrt{\frac{1-\xi\sqrt{b}}{1-b}} + \xi\frac{2}{\pi}\sin^{-1}\sqrt{\frac{\sqrt{b}}{\xi}\,\frac{1-\xi\sqrt{b}}{1-b}}, \tag{1.13}$$

when $\sqrt{b} \leq \xi \leq 1/\sqrt{b}$. (We shall see that $\hat{b}$ satisfies these inequalities.) The two important particular cases $\alpha = 0$ and $\beta = 0$ have already been considered in [SV], [G], and [BS] and the following results obtained:

- if $\beta = 0$, then $b^* = (1 + \alpha/\theta)^{-2}$; and
- if $\alpha = 0$, then $b^* = (1 - \beta/\theta)^2$ for $\beta/\theta < 1$ and $b^* = 0$ if $\beta/\theta \geq 1$.

Note that the corresponding lower estimates are included in Theorem 4.

2. Proofs of Theorems 1.1 and 1.3

We denote by $V(x, \mu)$ the logarithmic potential for the measure $d\mu$:

$$V(x, \mu) := \int \log\frac{1}{|x-t|}\,d\mu(t).$$

We fix $a > 0$ and define the two distributions:

$$\sigma_1(t) := \frac{1}{\pi}\sqrt{\frac{a-t}{t}}, \qquad t \in [0, a], \tag{2.1}$$

$$\sigma_0(t) := \frac{1}{\pi}\frac{1}{\sqrt{t(a-t)}}, \qquad t \in (0, a). \tag{2.1a}$$

The following properties of the corresponding logarithmic potentials are easily verified (see Appendix):

$$V(x, \sigma_1\,dt) = -x + \text{const}, \qquad x \in [0, a], \tag{2.2}$$

$$V(x, \sigma_0\,dt) = \log(4/a), \qquad x \in [0, a]. \tag{2.3}$$

For $x \in \mathbf{R}$ we define the function

$$\sigma(t, x) := \sigma_1(t) - x\sigma_0(t), \qquad 0 \leq t \leq a. \tag{2.4}$$

For each fixed x, let $\sigma(t, x) = \sigma^+(t, x) - \sigma^-(t, x)$ be the Jordan decomposition of the measure $\sigma(t, x)\,dt$ in $[0, a]$ and set

$$p(x) := p(x, a) := \int \sigma^+(t, x)\,dt, \tag{2.5}$$

$$n(x) := n(x, a) := \int \sigma^-(t, x)\,dt. \tag{2.6}$$

Lemma 5. *With the notation of Theorem 3 we have the following implication: if $a = a(\alpha, \beta) < a^* = a^*(\alpha, \beta)$, then there exists an $x \in \mathbf{R}$ such that*

$$p(x, a) \le \beta \qquad \text{and} \qquad n(x, a) \le \alpha.$$

Proof. The condition $a < a^*$ means that the (α, β)-approximation to e^{nx} on $[0, a]$ is possible. In other words, there exist two sequences of polynomials

$$p_n \in \mathcal{P}_{[\alpha n]}, \qquad q_n \in \mathcal{P}_{[\beta n]}, \tag{2.7}$$

with

$$\delta_n := \left\| e^{nx}\frac{p_n(x)}{q_n(x)} - 1 \right\|_{[0,a]} \to 0 \qquad \text{as} \quad n \to \infty. \tag{2.8}$$

Set

$$\chi_{n,\alpha} := \frac{1}{n}\sum_{p_n(z)=0} \delta(z), \qquad \chi_{n,\beta} := \frac{1}{n}\sum_{q_n(z)=0} \delta(z),$$

where $\delta(z)$ denotes the unit point measure at z, and define $\mu_{n,\alpha}$ and $\mu_{n,\beta}$ to be the balayage (see [La]) of $\chi_{n,\alpha}$ and $\chi_{n,\beta}$ into $[0, a]$, respectively. Then we have (see [La])

$$\frac{1}{n}\log\frac{1}{|p_n(x)|} = V(x, \mu_{n,\alpha}) + \omega_{n,\alpha}, \qquad x \in [0, a], \tag{2.9}$$

$$\frac{1}{n}\log\frac{1}{|q_n(x)|} = V(x, \mu_{n,\beta}) + \omega_{n,\beta}, \qquad x \in [0, a], \tag{2.10}$$

where $\omega_{n,\alpha}$ and $\omega_{n,\beta}$ are constants depending on n. It follows from these two representations and (2.2) that

$$\frac{1}{n}\log\left|e^{nx}\frac{p_n(x)}{q_n(x)}\right| = V(x, \mu_{n,\beta} - \mu_{n,\alpha} - \sigma_1 dt) + \omega_n, \qquad x \in [0, a], \tag{2.11}$$

where ω_n is a constant.

From (2.8), we deduce that

$$\left|\frac{1}{n}\log\left(e^{nx}\frac{p_n(x)}{q_n(x)}\right)\right| \le \frac{1}{n}\left|\log(1 - \delta_n)\right| \to 0$$

uniformly on $[0, a]$ as $n \to \infty$. Therefore, with

$$\mu_n := \mu_{n,\beta} - \mu_{n,\alpha} - \sigma_1\, dt$$

we have, uniformly on $[0, a]$,

$$V(x, \mu_n) + \omega_n \to 0. \tag{2.12}$$

Furthermore, we see from (2.7) that

$$\|\mu_{n,\alpha}\| = \|\chi_{n,\alpha}\| = \frac{1}{n}\deg p_n \le \alpha,$$

$$\|\mu_{n,\beta}\| = \|\chi_{n,\beta}\| = \frac{1}{n}\deg q_n \le \beta.$$

Hence, we can find a subsequence $\Lambda \subset \mathbf{N}$ and positive measures μ_α, μ_β such that as $n \to \infty$, $n \in \Lambda$,

$$\mu_{n,\alpha} \xrightarrow{*} \mu_\alpha, \quad \|\mu_\alpha\| \leq \alpha; \qquad \mu_{n,\beta} \xrightarrow{*} \mu_\beta, \quad \|\mu_\beta\| \leq \beta; \tag{2.13}$$

where $\xrightarrow{*}$ denotes weak-star convergence.

It follows from (2.13) that $\mu_n \xrightarrow{*} \mu := \mu_\beta - \mu_\alpha - \sigma_1\, dt$ and therefore as $n \to \infty$, $n \in \Lambda$,

$$\mu_n(\mathbf{C}) \to \mu(\mathbf{C}), \qquad V(z, \mu_n) \to V(z, \mu), \qquad z \in \mathbf{C}\backslash[0, a]. \tag{2.14}$$

Furthermore, on integrating (2.12) with respect to the (unit) equilibrium measure $\sigma_0(x)\, dx$ and utilizing (2.3) we obtain

$$\begin{aligned} \int_0^a V(x, \mu_n)\sigma_0(x)\, dx + \omega_n &= \int_0^a V(t, \sigma_0\, dx)\, d\mu_n(t) + \omega_n \\ &= \mu_n(\mathbf{C}) \log\left(\frac{4}{a}\right) + \omega_n \to 0, \end{aligned}$$

as $n \to \infty$, $n \in \Lambda$, and so from (2.14) we have

$$\lim_{\substack{n\to\infty \\ n\in\Lambda}} \omega_n = -\mu(\mathbf{C}) \log\left(\frac{4}{a}\right). \tag{2.15}$$

Next observe that for n sufficiently large, $p_n(x)$ and $q_n(x)$ do not vanish on $[0, a]$; hence from (2.9) and (2.10) it follows that $V(x, \mu_n)$ is finite and continuous on $\text{supp}(\mu_n)$ and therefore $V(z, \mu_n)$ is continuous on $\mathbf{C}$. Consequently,

$$h_n(z) := V(z, \mu_n) - \mu_n(\mathbf{C})V(z, \sigma_0\, dt)$$

is continuous on $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ and harmonic in $\mathbf{C}\backslash[0, a]$. Thus, by (2.12), (2.14), and (2.15), and the maximum principle, we have

$$\lim_{\substack{n\to\infty \\ n\in\Lambda}} h_n(z) = 0, \qquad z \in \overline{\mathbf{C}}.$$

On the other hand, (2.14) yields

$$\lim_{\substack{n\to\infty \\ n\in\Lambda}} h_n(z) = V(z, \mu) - \mu(\mathbf{C})V(z, \sigma_0\, dt), \qquad z \in \mathbf{C}\backslash[0, a],$$

and so

$$V(z, \mu) = V(z, \lambda\sigma_0\, dt), \qquad z \in \mathbf{C}\backslash[0, a],$$

where $\lambda := \mu(\mathbf{C})$. Since the potential of the signed measure $\mu - \lambda\sigma_0\, dt$ vanishes outside a set of two-dimensional Lebesgue measure zero, we have $\mu = \lambda\sigma_0\, dt$ and we obtain

$$\mu_\beta - \mu_\alpha = \sigma_1\, dt - \lambda\sigma_0\, dt = \sigma(t, \lambda)\, dt$$

(see (2.4)). Due to the minimizing property of the Jordan decomposition we then have

$$\begin{aligned} \alpha &\geq \|\mu_\alpha\| \geq \|\sigma^-(t, \lambda)\, dt\| = n(\lambda, a), \\ \beta &\geq \|\mu_\beta\| \geq \|\sigma^+(t, \lambda)\, dt\| = p(\lambda, a), \end{aligned}$$

which completes the proof. ∎

Lemma 6. *The following properties of the functions* $p(x) := p(x,a)$ *and* $n(x) := n(x,a)$ *(see* (2.5) *and* (2.6)*) hold for any fixed* $a > 0$:

(i) $$p(x) - n(x) = \frac{a}{2} - x, \qquad x \in \mathbf{R};$$

(ii) $$\begin{aligned} p(x) &= \frac{a}{2} - x \quad \textit{and} \quad n(x) = 0 \quad \textit{for} \quad x \le 0, \\ p(x) &= 0 \quad \textit{and} \quad n(x) = x - \frac{a}{2} \quad \textit{for} \quad x \ge a; \end{aligned}$$

(iii) $$p(x) = \frac{a}{\pi}\left\{\left(1 - 2\frac{x}{a}\right)\sin^{-1}\sqrt{1 - \frac{x}{a}} + \sqrt{\frac{x}{a}\left(1 - \frac{x}{a}\right)}\right\}, \qquad x \in [0,a];$$

(iv) $$\begin{aligned} p'(x) &= -\frac{2}{\pi}\sin^{-1}\sqrt{1 - \frac{x}{a}}, \qquad x \in [0,a], \\ n'(x) &= \frac{2}{\pi}\cos^{-1}\sqrt{1 - \frac{x}{a}}, \qquad x \in [0,a]. \end{aligned}$$

Proof. We have from definitions (2.1), (2.2), and (2.4)–(2.6)

$$p(x) - n(x) = \int_0^a \sigma(t,x)\,dt = \frac{1}{\pi}\int_0^a \sqrt{\frac{a-t}{t}}\,dt - x\frac{1}{\pi}\int_0^a \frac{dt}{\sqrt{t(a-t)}} = \frac{a}{2} - x.$$

Furthermore, we have

$$\sigma(t,x) = \frac{1}{\pi}\frac{a-t-x}{\sqrt{t(a-t)}} \ge 0 \qquad \text{for} \quad t \in [0,a], \qquad x \le 0,$$

and therefore $n(x) = 0$ for $x \le 0$. Likewise $\sigma(t,x) \le 0$ for $t \in [0,a]$ and $x \ge a$; therefore, $p(x) = 0$ for $x \ge a$. Assertions (i) and (ii) immediately follow from these remarks.

Next, we see from the representation $\sigma(t,x) = (1/\pi)(a-t-x)/\sqrt{t(a-t)}$ that for $x \in (0,a)$ the function $\sigma(t,x)$ is positive for $t \in [0, a-x)$ and negative in $(a-x, a]$. Hence,

$$\begin{aligned} p(x,a) &= \frac{1}{\pi}\int_0^{a-x} \sigma(t,x)\,dt = \frac{1}{\pi}\int_0^{a-x}\sqrt{\frac{a-t}{t}}\,dt - \frac{x}{\pi}\int_0^{a-x}\frac{dt}{\sqrt{t(a-t)}} \\ &= \frac{a}{\pi}\int_0^b \sqrt{\frac{1-t}{t}}\,dt - \frac{x}{\pi}\int_0^b \frac{dt}{\sqrt{t(1-t)}}, \end{aligned} \tag{2.16}$$

where

$$b := 1 - \frac{x}{a}.$$

Using the identities

$$\int_0^b \sqrt{\frac{1-t}{t}}\,dt = \sin^{-1}\sqrt{b} + \sqrt{b(1-b)}$$

and

$$\int_0^b \frac{dt}{\sqrt{t(1-t)}} = 2\sin^{-1}\sqrt{b},$$

we obtain assertion (iii) from (2.16).

The first equality in (iv) may be obtained by differentiation of (iii); the second inequality then follows from the first and (i). ∎

We now investigate the set of values of $a > 0$ satisfying the condition in the assertion of Lemma 5. For given $\alpha, \beta \geq 0$, $\alpha + \beta > 0$, we set

(2.17) $$A := A(\alpha, \beta) := \{a : \exists x \in \mathbf{R} \text{ with } p(x, a) \leq \beta,\ n(x, a) \leq \alpha\}.$$

Lemma 7. *For any $\alpha, \beta \geq 0$, $\alpha + \beta > 0$, and $a > 0$ there exists a unique root $\bar{x} = \bar{x}(a; \alpha, \beta)$ of the equation*

(2.18) $$\alpha p(x, a) = \beta n(x, a)$$

in $[0, a]$. *Furthermore,*

(2.19a) $$A = \{a > 0 : p(\bar{x}, a) \leq \beta\} \quad \text{for} \quad \beta > 0,$$

(2.19b) $$A = \{a > 0 : n(\bar{x}, a) \leq \alpha\} \quad \text{for} \quad \alpha > 0,$$

where A is defined in (2.17).

Proof. Consider first the case $\alpha, \beta > 0$. It follows by Lemma 6 that for fixed $a > 0$ the function $p(x, a)$ decreases on $(-\infty, a)$ from $+\infty$ to 0 and $p(x, a) = 0$ for $x \geq a$. Also, the function $n(x, a)$ increases from 0 to $+\infty$ on $(0, \infty)$ and $n(x, a) = 0$ for $x \leq 0$. The same is true for $(\beta/\alpha)n(x, a)$ and therefore there exists a unique root $\bar{x}$ of the equation

(2.20) $$p(x, a) = \frac{\beta}{\alpha} n(x, a), \qquad x \in (0, a),$$

which is equivalent to (2.18) for $\alpha, \beta > 0$. It also follows from the behavior of $n(x, a)$, $p(x, a)$ that

$$p(\bar{x}, a) = \frac{\beta}{\alpha} n(\bar{x}, a) = \min_{x \in \mathbf{R}} \max \left\{ p(x, a), \frac{\beta}{\alpha} n(x, a) \right\}.$$

On the other hand, the definition (2.17) may be written as

$$A = \left\{ a : \min_{x \in \mathbf{R}} \max \left\{ p(x, a), \frac{\beta}{\alpha} n(x, a) \right\} \leq \beta \right\}$$

provided $\alpha, \beta > 0$. Assertions (2.19) and (2.19) follow by these remarks.

It remains to notice that for $\alpha = 0$, $\beta > 0$, the representation (2.19a) holds with $\bar{x} = 0$, which is the unique root of (2.18) in $[0, a]$ for $\alpha = 0$. Similarly, for $\alpha > 0$, $\beta = 0$, the representation (2.19b) is true with $\bar{x} = a$, which is the unique root of (2.18) in $[0, a]$ for $\alpha > 0$, $\beta = 0$. ∎

We define the function $G(y)$ for $y \in [0, 1]$ by

$$\pi G(y) := (1 - 2y)\sin^{-1}\sqrt{1-y} + \sqrt{y(1-y)}, \qquad y \in [0, 1]. \tag{2.21}$$

Lemma 8. *The equation*

$$(\alpha - \beta)G(y) = \beta\big(y - \tfrac{1}{2}\big) \tag{2.22}$$

has a unique root $\bar{y} = \bar{y}(\alpha, \beta)$ *in* $[0, 1]$. *Furthermore, with* $\bar{x}$ *as in Lemma* 7

$$\bar{x}(a; \alpha, \beta) = a\bar{y}(\alpha, \beta), \tag{2.23}$$

$$p(x, a) = aG(x/a), \qquad a > 0, \quad x \in [0, a]. \tag{2.24}$$

Proof. For $\alpha = \beta$ the unique root of (2.22) is $\bar{y} = 1/2$. For $\alpha \neq \beta$ we can rewrite (2.22) as

$$G(y) = \big(y - \tfrac{1}{2}\big)\frac{\beta}{\alpha - \beta}. \tag{2.25}$$

The range of $\kappa := \beta/(\alpha - \beta)$ for $\alpha, \beta \geq 0$ is $(-\infty, -1] \cup [0, \infty)$. We also note that

$$G'(y) = -\frac{2}{\pi}\sin^{-1}\sqrt{1-y} \in [-1, 0] \qquad \text{for} \quad y \in [0, 1].$$

Now, if $\kappa \in (-\infty, -1]$, then the function

$$G_1(y) := G(y) - \kappa\big(y - \tfrac{1}{2}\big)$$

is increasing in $[0, 1]$ since $G_1'(y) = G'(y) - \kappa \geq 0$. We have also $G_1(0) = 1/2 + \kappa/2 \leq 0$ and $G_1(1/2) = 1/2\pi > 0$. Thus (2.25) has a unique root $\bar{y}$ for $\beta/(\alpha - \beta) \in (-\infty, -1]$. Next, if $\kappa \in [0, \infty)$, then $G_1'(y) = G'(y) - \kappa \leq 0$ for $y \in [0, 1]$ and so $G_1(y)$ decreases on $[0, 1]$. Since $G_1(1/2) = 1/2\pi$ and $G_1(1) = -\kappa/2 \leq 0$, (2.25) has in $[0, 1]$ a unique root $\bar{y}$ which actually belongs to $[1/2, 1]$.

The equality (2.24) follows by (2.21) and (iii) of Lemma 6. It remains to compare equations (2.22) and (2.18). Using (i) of Lemma 6 we may rewrite (2.18) as

$$(-\beta + \alpha)p(x, a) = \beta\left(x - \frac{a}{2}\right)$$

or, using (2.24), as

$$(-\beta + \alpha)G\left(\frac{x}{a}\right) = \beta\left(\frac{x}{a} - \frac{1}{2}\right).$$

The last equation coincides with (2.22) for $y = x/a$. Since both have a unique solution we obtain the assertion (2.23). ■

Lemma 9. *For any* $\alpha, \beta \geq 0$, $\alpha + \beta > 0$, *we have* (*see* (2.17))

$$A = [0, \bar{a}], \tag{2.26}$$

where $\bar{a} := \bar{a}(\alpha, \beta)$ is defined by

$$\bar{a} = \frac{\beta}{G(\bar{y})} = \frac{\alpha - \beta}{\bar{y} - 1/2} \qquad \textit{for} \quad \alpha \neq \beta, \tag{2.27}$$

$$\bar{a} = 2\pi\beta \qquad \textit{for} \quad \alpha = \beta. \tag{2.28}$$

Proof. Let $\alpha = \beta$. Then (2.22) has the unique root $\bar{y} = 1/2$ and $G(\bar{y}) = G(1/2) = 1/2\pi$ by (2.21). Now, it follows by (2.23) and (2.24) that $p(\bar{x}, a) = a/2\pi$ and so (2.19) may be written as $A = \{a > 0\colon\ a/2\pi \leq \beta\}$. Assertion (2.26) follows.

Suppose $\alpha \neq \beta$ and $\beta > 0$. It follows by (2.23) and (2.24) that (2.19) may be written as $A = \{a\colon\ aG(\bar{y}) \leq \beta\}$. Then (2.26) follows from the second equality in (2.27) and the fact that $\alpha > \beta$ implies that $\bar{y} > 1/2$ (see the proof of Lemma 8).

In case $\beta = 0$ we rewrite (2.19b) using (i) of Lemma 6 as

$$A = \left\{a > 0\colon\ p(\bar{x}, a) + \bar{x} - \frac{a}{2} \leq \alpha\right\}. \tag{2.29}$$

On the other hand, we have by (2.23) and (2.22) that

$$\begin{aligned} p(\bar{x}, a) + \bar{x} - \frac{a}{2} &= a\{G(\bar{y}) + (\bar{y} - \tfrac{1}{2})\} = a\left(\frac{\beta}{\alpha - \beta} + 1\right)(\bar{y} - \tfrac{1}{2}) \\ &= a \cdot \frac{\alpha}{\alpha - \beta}(\bar{y} - \tfrac{1}{2}). \end{aligned}$$

Therefore (2.29) is equivalent to (2.26). ■

Proof of Theorem 1.3. Using the notation of (2.17), the assertion of Lemma 5 may be written as follows. If $a < a^*$, then $a \in A$. In view of Lemma 2.5 this means that

$$(a < a^*) \Rightarrow (a \leq \bar{a})$$

and therefore we have $a^* \leq \bar{a}$.

It remains to notice from (2.21) and (1.8) that $\hat{y} = 1 - \bar{y}$, and so from (2.27) and (1.7) we have $\hat{a} = \bar{a}$. ■

3. Proofs of Theorems 1.2 and 1.4

For a fixed $b \in (0, 1)$ we let

$$\tilde{\sigma}_0(t) := \frac{1}{\pi} \frac{1}{\sqrt{(t - b)(1 - t)}}, \qquad t \in [b, 1], \tag{3.1}$$

$$\tilde{\sigma}_1(t) := \sqrt{b}\, \frac{\tilde{\sigma}_0(t)}{t}, \qquad t \in [b, 1], \tag{3.2}$$

$$\tilde{\sigma}(t, x) := \tilde{\sigma}_1(t) - x\tilde{\sigma}_0(t), \qquad t \in [b, 1], \quad x \in \mathbf{R}. \tag{3.3}$$

We note that $\tilde{\sigma}_0\, dt$ is the equilibrium distribution for $[b, 1]$ and $\tilde{\sigma}_1\, dt$ is the balayage of the unit mass at $x = 0$ to $[b, 1]$. Thus we have

$$V(x, \tilde{\sigma}_0\, dt) = \log \frac{4}{1 - b}, \qquad x \in [b, 1], \tag{3.4}$$

$$V(x, \tilde{\sigma}_1\, dt) = \log \frac{1}{x} + \text{const}, \qquad x \in [b, 1]. \tag{3.5}$$

For the measure $\tilde{\sigma}(t, x)\, dt$ ($x \in \mathbf{R}$ is fixed) we consider its Jordan decomposition $\tilde{\sigma}\, dt = \tilde{\sigma}^+\, dt - \tilde{\sigma}^-\, dt$ and put

$$p(x) \;:=\; p(x, b) := \int_\Delta \tilde{\sigma}^+(t, x)\, dt, \tag{3.6}$$

$$n(x) \;:=\; n(x, b) := \int_\Delta \tilde{\sigma}^-(t, x)\, dt, \tag{3.7}$$

where $\Delta := [b, 1]$.

Following the scheme of proof of Lemma 5 we obtain the following:

Lemma 10. *Let b^* be defined as in (1.11). If $1 > b > b^*$, then there exists an $x \in \mathbf{R}$ such that*

$$p(x, b) \le \frac{\beta}{\theta} \qquad \textit{and} \qquad n(x, b) \le \frac{\alpha}{\theta}.$$

Lemma 11. *The functions $p(x, b)$ and $n(x, b)$ defined in (3.6) and (3.7) satisfy the following properties*:

(i) $$p(x, b) - n(x, b) = 1 - x, \qquad x \in \mathbf{R}, \quad b \in (0, 1);$$

(ii) $$\begin{aligned} p(x, b) \;&=\; 1 - x \quad \textit{and} \quad n(x, b) = 0 \qquad \textit{for} \quad x \le \sqrt{b}, \\ p(x, b) \;&=\; 0 \quad \textit{and} \quad n(x, b) = x - 1 \qquad \textit{for} \quad x \ge 1/\sqrt{b}; \end{aligned}$$

(iii) $$p(x, b) = \int_b^{\sqrt{b}/x} \tilde{\sigma}(t, x)\, dt, \qquad \sqrt{b} \le x \le \frac{1}{\sqrt{b}};$$

(iv) $$\frac{\partial}{\partial x} p(x, b) = -\frac{2}{\pi} \sin^{-1} \sqrt{\frac{\sqrt{b}}{x} \frac{1 - x\sqrt{b}}{1 - b}}, \qquad \sqrt{b} < x \le \frac{1}{\sqrt{b}};$$

(v) $$p(x, b) = \frac{2}{\pi} \int_x^{1/\sqrt{b}} \sin^{-1} \sqrt{\frac{\sqrt{b}}{t} \frac{1 - t\sqrt{b}}{1 - b}}\, dt, \qquad \sqrt{b} \le x \le \frac{1}{\sqrt{b}};$$

(vi) $$p(x, b) = \frac{2}{\pi} \sin^{-1} \sqrt{\frac{1 - x\sqrt{b}}{1 - b}} - x \frac{2}{\pi} \sin^{-1} \sqrt{\frac{\sqrt{b}}{x} \frac{1 - x\sqrt{b}}{1 - b}}, \qquad \sqrt{b} \le x \le \frac{1}{\sqrt{b}};$$

(vii) $$p(x, b) = \frac{2}{\pi} \tan^{-1} \sqrt{\frac{1}{\sqrt{b}} \frac{1 - x\sqrt{b}}{x - \sqrt{b}}} - x \frac{2}{\pi} \tan^{-1} \sqrt{\sqrt{b} \frac{1 - x\sqrt{b}}{x - \sqrt{b}}}, \qquad \sqrt{b} \le x \le \frac{1}{\sqrt{b}};$$

(viii) $$\frac{\partial}{\partial b} p(x, b) = -\frac{1}{\pi} \frac{\sqrt{(x - \sqrt{b})(1 - x\sqrt{b})}}{b^{3/4}(1 - b)}, \qquad \sqrt{b} \le x \le \frac{1}{\sqrt{b}};$$

(ix) $$p(x, b) = 1 - \frac{1}{\pi} \int_0^b \frac{\sqrt{(x - \sqrt{t})(1 - x\sqrt{t})}}{t^{3/4}(1 - t)}\, dt, \qquad \sqrt{b} \le x \le \frac{1}{\sqrt{b}}.$$

Proof. (i) With $\Delta = [b, 1]$, we have

$$p(x,b) - n(x,b) = \int_\Delta \tilde{\sigma}(t,x)\,dt = \int_\Delta \tilde{\sigma}_1(t)\,dt - x\int_\Delta \tilde{\sigma}_0(t)\,dt = 1 - x.$$

(ii) The function $\tilde{\sigma}(t,x) = (\sqrt{b}/t - x)\tilde{\sigma}_0(t)$ satisfies the inequalities $\tilde{\sigma}(t,x) \geq 0$ for $t \in \Delta$ if $x \leq \sqrt{b}$ and $\tilde{\sigma}(t,x) \leq 0$ for $t \in \Delta$ if $x \geq 1/\sqrt{b}$. Hence (ii) follows from (i).

(iii) This property follows immediately from the definition of p.

(iv) We use (iii) to take derivative with respect to x. Since $\tilde{\sigma}(\sqrt{b}/x, x) = 0$, we obtain

$$\begin{aligned}\frac{\partial p}{\partial x}(x,b) &= -\int_b^{\sqrt{b}/x} \tilde{\sigma}_0(t)\,dt = -\frac{1}{\pi}\int_b^{\sqrt{b}/x} \frac{dt}{\sqrt{(t-b)(1-t)}}\\ &= -\frac{1}{\pi}\int_0^\lambda \frac{d\tau}{\sqrt{\tau(1-\tau)}} = -\frac{2}{\pi}\sin^{-1}\sqrt{\lambda},\end{aligned}$$

where

$$\lambda := \lambda(x,b) = \frac{\sqrt{b}/x - b}{1-b} = \frac{\sqrt{b}}{x}\,\frac{1 - x\sqrt{b}}{1-b} \tag{3.8}$$

(we use this notation hereafter).

(v) Since $p(1/\sqrt{b}, b) = 0$, (v) follows from (iv).

(vi) We integrate by parts in (v) to obtain

$$\begin{aligned}\frac{\pi}{2}p(x,b) &= \int_x^{1/\sqrt{b}} \sin^{-1}\sqrt{\lambda}\,dt = t\sin^{-1}\sqrt{\lambda}\Big|_x^{1/\sqrt{b}} - \int_x^{1/\sqrt{b}} t\frac{d}{dt}(\sin^{-1}\sqrt{\lambda})\,dt\\ &= -x\sin^{-1}\sqrt{\lambda(x,b)} - \frac{1}{2}\int_x^{1/\sqrt{b}} \frac{t(\partial\lambda/\partial t)(t,b)}{\sqrt{\lambda(1-\lambda)}}\,dt.\end{aligned}$$

For the integrand in the last term we have

$$\frac{t\,\partial\lambda/\partial t}{\sqrt{\lambda(1-\lambda)}} = \frac{-1}{\sqrt{(1/\sqrt{b} - t)(t - \sqrt{b})}},$$

and after the substitution $\tau = (1/\sqrt{b} - t)/(1/\sqrt{b} - \sqrt{b})$ in the integral we obtain

$$\begin{aligned}-\frac{1}{2}\int_x^{1/\sqrt{b}} \frac{t\,\partial\lambda/\partial t\,dt}{\sqrt{\lambda(1-\lambda)}} &= \frac{1}{2}\int_x^{1/\sqrt{b}} \frac{dt}{\sqrt{(1/\sqrt{b} - t)(t-\sqrt{b})}}\\ &= \frac{1}{2}\int_0^{(1-x\sqrt{b})/(1-b)} \frac{d\tau}{\sqrt{\tau(1-\tau)}} = \sin^{-1}\sqrt{\frac{1-x\sqrt{b}}{1-b}}\end{aligned}$$

and (vi) follows.

(vii) Since $\sin^{-1}\alpha = \tan^{-1}(\alpha/\sqrt{1-\alpha^2})$, (vii) follows from (vi).

(viii) We use (v): $p(x,b) = (2/\pi)\int_x^{1/\sqrt{b}} \sin^{-1}\sqrt{\lambda(t,b)}\,dt$ to take the derivative with respect to b. We have

$$\begin{aligned}\frac{\partial}{\partial b}p(x,b) &= \frac{2}{\pi}\int_x^{1/\sqrt{b}} \frac{\partial}{\partial b}(\sin^{-1}\sqrt{\lambda(t,b)})\,dt\\ &= \frac{1}{\pi}\int_x^{1/\sqrt{b}} \frac{(\partial\lambda/\partial b)(t,b)\,dt}{\sqrt{\lambda(t,b)(1-\lambda(t,b))}},\end{aligned}$$

since $\lambda(t,b)$ equals zero for $t = 1/\sqrt{b}$ (see (3.8)). The integrand in the last term is

$$\begin{aligned}\frac{\partial\lambda/\partial b}{\sqrt{\lambda(1-\lambda)}} &= \frac{(1/\sqrt{b}+\sqrt{b})-2t}{2t(1-b)^2} \Bigg/ \sqrt{\frac{\sqrt{b}(1-t\sqrt{b})(t-\sqrt{b})}{t^2(1-b)^2}}\\ &= \frac{1}{2\sqrt{b}(1-b)}\frac{\gamma-2t}{\sqrt{\gamma t-t^2-1}},\end{aligned}$$

where

$$\gamma := \frac{1}{\sqrt{b}}+\sqrt{b}.$$

Hence, we have

$$\begin{aligned}\frac{\partial}{\partial b}p(x,b) &= \frac{1}{2\pi\sqrt{b}(1-b)}\int_x^{1/\sqrt{b}} \frac{\gamma-2t}{\sqrt{\gamma t-t^2-1}}\,dt\\ &= -\frac{\sqrt{\gamma x-x^2-1}}{\pi\sqrt{b}(1-b)}\end{aligned}$$

(since $\gamma t - t^2 - 1 = 0$ for $t = 1/\sqrt{b}$). The representation (viii) now follows.

(ix) We have $p(x,0) = 1$ from (vi). Now if $\sqrt{b} \le x \le 1/\sqrt{b}$, then $\sqrt{t} \le x \le 1/\sqrt{t}$ for $t \in [0,b]$ and we can integrate (viii) with respect to t instead of b over $[0,b]$.

Next we define

$$\begin{aligned}B &= B(\theta;\alpha,\beta)\\ &:= \left\{b\in(0,1) : \exists x\in\mathbf{R} \text{ with } p(x,b)\le\frac{\beta}{\theta},\ n(x,b)\le\frac{\alpha}{\theta}\right\}\end{aligned} \tag{3.9}$$

and we set

$$\bar{b} = \bar{b}(\theta,\alpha,\beta) := \inf B. \tag{3.10}$$

■

Lemma 12.

(i) *If* $\beta/\theta \ge 1$, *then* $\bar{b} = 0$.
(ii) *If* $\beta/\theta < 1$, *then* $\bar{b}$ *is the unique root of the equation*

$$p\left(1-\frac{\beta}{\theta}+\frac{\alpha}{\theta},\ b\right) = \frac{\beta}{\theta}$$

satisfying $\sqrt{b} \le 1-\beta/\theta+\alpha/\theta \le 1/\sqrt{b}$.

Proof. We consider first the case $\alpha,\ \beta > 0$. Using the notation

(3.11) $$h(b) := \min_{x\in\mathbf{R}} \max\left\{p(x,\ b),\ \frac{\beta}{\alpha}n(x,\ b)\right\}$$

we can rewrite the definition (3.9) of B as

(3.12) $$B = \left\{b \in (0,1) : h(b) \le \frac{\beta}{\theta}\right\}.$$

It follows by (i) and (viii) of Lemma 11 that

(3.13) $$\frac{\partial}{\partial b}p(x,b) = \frac{\partial}{\partial b}n(x,b) < 0 \qquad \text{for} \quad \sqrt{b} < x < \frac{1}{\sqrt{b}};$$

hence both functions (of b) $p(x,b)$ and $n(x,b)$ decrease for fixed x in the indicated domain.

On the other hand, it follows by (i), (ii), and (iv) of Lemma 11 that for fixed $b \in (0,1)$ the function $p(x,b)$ decreases from $+\infty$ at $x = -\infty$ to 0 at $x = 1/\sqrt{b}$ and $n(x,b)$ increases from 0 at $x = \sqrt{b}$ to $+\infty$ at $x = +\infty$. This means that the equation

(3.14) $$\alpha p(x,b) = \beta n(x,b)$$

has a unique root $x_1(b) \in (\sqrt{b}, 1/\sqrt{b})$ and

(3.15) $$h(b) = p(x_1(b), b).$$

We note that $x_1(b)$ is a continuous function of b and we can conclude that for any fixed $b \in (0,1)$, definition (3.11) may be written as

(3.16) $$h(b) = \min_{x\in[x_1-\varepsilon,\, x_1+\varepsilon]} \max\left\{p(x,b), \frac{\beta}{\alpha}n(x,b)\right\},$$

where $x_1 = x_1(b)$. Moreover, this equality holds in some neighborhood of b with the same value of x_1 as defined by the original value of b. If $\varepsilon > 0$ is small enough, then $(x_1-\varepsilon,\ x_1+\varepsilon) \times (b-\varepsilon,\ b+\varepsilon) \subset D := \{(x,b)\colon\ b \in (0,1), \sqrt{b} < x < 1/\sqrt{b}\}$ since $(x_1(b), b) \in D$. Hence, it follows by (3.13) and (3.16) that $h(b)$ is decreasing in some neighborhood of $b \in (0,1)$. Since $b \in (0,1)$ is arbitrary, we conclude that $h(b)$ is decreasing on $(0,1)$.

Next, we observe that $p(x,b) \to 1$ as $b \to 0$ uniformly over any interval $x \in [0,R]$ (see (ii) and (vi) of Lemma 11). Also, as $b \to 0$, $(\beta/\alpha)n(x,b) \to (\beta/\alpha)x$ (see (i) of Lemma 11), so that $x_1(b) \to \alpha/\beta$ and $h(b) = p(x_1(b), b) \to 1$. On the other hand, we have $0 \le p(x_1(b), b) \le p(\sqrt{b}, b) = 1-\sqrt{b}$. Therefore, $h(b) = p(x_1(b), b) \to 0$ as $b \to 1$. Hence $h(b)$ decreases from 1 to 0 on $(0,1)$.

Now, if $\beta/\theta \ge 1$, then $h(b) \le \beta/\theta$ for any $b \in (0,1)$ and assertion (i) of the lemma follows by (3.12). If $\beta/\theta < 1$, then from the properties of h and p described above, the value of $\bar{b}$ is determined by the system of equations

(3.17) $$\alpha p(x,b) = \beta n(x,b), \qquad p(x,b) = \frac{\beta}{\theta},$$

which has the unique solution $(x_1(\bar{b}), \bar{b})$. Using (i) of Lemma 11 the first equation may be written as $p(x) = (\beta/\alpha)\{p(x)+x-1\}$ or $p(x) = \beta(x-1)/(\alpha-\beta)$. If we substitute this expression for $p(x)$ in the second equation of (3.17) we see that the system (3.17) is equivalent to

$$\begin{cases} x = 1 - \beta/\theta + \alpha/\theta, \\ p(x, b) = \beta/\theta. \end{cases}$$

Hence assertion (ii) of the lemma follows.

Now suppose that $\alpha = 0$, $\beta > 0$. The requirement $n(x, b) \leq 0$ is included in the definition (3.9) of the set B, which implies that $x \leq \sqrt{b}$. The minimal value for $p(x, b)$ over $(-\infty, \sqrt{b}]$ is achieved at $x = \sqrt{b}$; hence (3.9) may be written in this case as

$$B = \left\{ b \in (0, 1) : p(\sqrt{b}, b) \leq \frac{\beta}{\theta} \right\}.$$

Since $p(\sqrt{b}, b) = 1 - \sqrt{b}$, this yields for $\bar{b} = \inf\{b \in B\}$,

$$\bar{b} = 0 \quad \text{if } \beta/\theta \geq 1, \qquad \bar{b} = \left(1 - \frac{\beta}{\theta}\right)^2 \quad \text{if } \beta/\theta < 1. \tag{3.18}$$

For $\beta/\theta < 1$, it is clear that $\bar{b} = (1 - \beta/\theta)^2$ is a root of the equation

$$p(1 - \beta/\theta, b) = \beta/\theta.$$

It remains to show that this root is unique; i.e.,

$$p(x, b) \neq 1 - x \qquad \text{for} \quad \sqrt{b} < x \leq 1/\sqrt{b}. \tag{3.19}$$

It follows from (iv) of Lemma 11 that

$$\frac{\partial}{\partial x} p(x, b) > -1 \qquad \text{for} \quad \sqrt{b} < x \leq 1/\sqrt{b}.$$

Therefore,

$$\begin{aligned} p(x, b) &= p(\sqrt{b}, b) + \int_{\sqrt{b}}^{x} \frac{\partial}{\partial t} p(t, b)\, dt \\ &> 1 - \sqrt{b} - (x - \sqrt{b}) = 1 - x \qquad \text{for} \quad \sqrt{b} < x \leq 1/\sqrt{b}, \end{aligned}$$

which yields (3.19). Thus assertions (i) and (ii) of the lemma hold when $\alpha = 0$.

Finally, for $\alpha > 0$, $\beta = 0$, we have from definition (3.9) that for $b \in B$, there exists an x such that $p(x, b) \leq 0$, and so $x \geq 1/\sqrt{b}$. The minimal value for $n(x, b)$ over $x \geq 1/\sqrt{b}$ is achieved only at $x = 1/\sqrt{b}$ and thus $B = \{b \in (0, 1)\colon n(1/\sqrt{b}, b) \leq \alpha/\theta\}$ or

$$\bar{b} = \inf B = \frac{1}{(1 + \alpha/\theta)^2}. \tag{3.20}$$

Clearly $\bar{b} = (1 + \alpha/\theta)^{-2}$ is a root of $p(1 + \alpha/\theta, b) = 0$ satisfying $\sqrt{b} \leq (1 + \alpha/\theta) \leq 1/\sqrt{b}$. Moreover, according to (ii) and (v) of Lemma 11, $\bar{b}$ is the only root of $p(1 + \alpha/\theta, b) = 0$ satisfying $\sqrt{b} \leq 1 + \alpha/\theta \leq 1/\sqrt{b}$. This completes the proof of Lemma 12. ∎

Proof of Theorem 4. The assertion of Lemma 10 combined with the definition of (3.9) may now be written as $(1 > b > b^*) \Rightarrow (b \in B) \Rightarrow (b \geq \bar{b})$. This implies that

$$b^* \geq \bar{b}.$$

Using (ix) of Lemma 11 the equation for $\bar{b}$ in (ii) of Lemma 3.3 may be written as

$$\frac{1}{\pi}\int_0^{\bar{b}} \frac{\sqrt{(\xi - \sqrt{t})(1 - \xi\sqrt{t})}}{t^{3/4}(1-t)}\, dt = 1 - \frac{\beta}{\theta} \tag{3.21}$$

with $\xi = 1 - \beta/\theta + \alpha/\theta$. Thus $\bar{b} = \hat{b}$ and the proof of Theorem 4 is complete. ■

Proof of Theorem 2. In the case $\alpha = \beta$ we have $\xi = 1 - \beta/\theta + \alpha/\theta = 1$ and the integral on the left-hand side of (3.21) becomes

$$\frac{1}{\pi}\int_0^{\bar{b}} \frac{1-\sqrt{t}}{t^{3/4}(1-t)} dt = \frac{1}{\pi}\int_0^{\bar{b}} \frac{dt}{t^{3/4}(1+\sqrt{t})} = \frac{4}{\pi}\int_0^{\bar{b}} \frac{d(t^{1/4})}{1+t^{1/2}}$$
$$= \frac{4}{\pi}\tan^{-1}(\bar{b}^{1/4}).$$

Hence, (3.21) has the solution

$$\bar{b} = \tan^4\left(\frac{\pi}{4}\left(1 - \frac{\beta}{\theta}\right)\right).$$

Theorem 2 now follows from Theorem 4 (with $\alpha = \beta = 1$). ■

4. Appendix

Here we prove the identity (2.2):

$$u(x) := \frac{1}{\pi}\int_0^a \log\frac{1}{|x-t|}\sqrt{\frac{a-t}{t}}\, dt = -x + c, \qquad x \in [0, a], \tag{A.1}$$

where c is a constant.

The integral in (A.1) defines a function $u(x)$ continuous in the whole plane $\mathbf{C}$. In the upper half-plane, we have

$$u(x) = \operatorname{Re} U(x), \qquad \operatorname{Im} x > 0, \tag{A.2}$$

where

$$U(z) := -\frac{1}{\pi}\int_0^a \log(z-t)\sqrt{\frac{a-t}{t}}\, dt, \qquad \operatorname{Im} z > 0, \tag{A.3}$$

and the branch of log is determined by the normalization $0 < \arg(z-t) < \pi$, $t \in [0, a]$, $\operatorname{Im} z > 0$. The derivative

$$U'(z) = \frac{1}{\pi}\int_0^a \sqrt{\frac{a-t}{t}}\,\frac{dt}{t-z}, \qquad z \in D := \overline{\mathbf{C}}\setminus[0, a], \tag{A.4}$$

is a single-valued analytic function in D. Let

$$f(z) := \sqrt{\frac{z-a}{z}}, \qquad z \in D, \quad f(\infty) := 1,$$

and denote by $f^+(x)$, $f^-(x)$, $x \in [0, a]$, the boundary values of $f(z)$ from the upper and lower half-planes, respectively. Then we have

$$f^{\pm}(x) = \pm i \sqrt{\frac{a-x}{x}}, \tag{A.5}$$

Using (A.5) we can rewrite (A.4) as

$$U'(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(t)}{t-z}\, dt, \qquad z \in D, \tag{A.6}$$

where ∂D is the boundary of D with positive orientation with respect to D. The integral in (A.6) is the Cauchy integral for $f(z)$ in D and therefore

$$U'(z) = f(z) - f(\infty) = \sqrt{\frac{z-a}{z}} - 1,$$

so that

$$U(z) = \int_0^z \sqrt{\frac{\zeta - a}{\zeta}} d\zeta - z + \text{const}, \qquad \operatorname{Im} z > 0.$$

Now (A.1) follows from the last representation and (A.2).

Finally, we remark that (2.3) is well known (see [Ts]) since $\sigma_0\, dt$ is the equilibrium distribution for the interval $[0, a]$, which has logarithmic capacity $a/4$.

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