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Asymptotic distribution of the zeros of Faber polynomials

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Abstract

Using potential theoretic methods and exploiting the connection with eigenvalues of Toeplitz matrices, we investigate the limiting behaviour of zeros of Faber polynomials generated by a Laurent series. Our results build upon fundamental work of J. L. Ullman. For example, we show that if E is a compact set with simply connected complement and connected interior whose boundary is either (i) not a piecewise analytic curve or (ii) a piecewise analytic curve but with a singularity other than an outward cusp, then the equilibrium distribution for E is a limit measure of the sequence of normalized zero counting measures for the Faber polynomials associated with E.

1. *Introduction*

Let $g(w) = w + \sum_{k=0}^{\infty} b_k w^{-k}$ be a Laurent series expansion of a function g. We assume that

$$\rho_0 := \limsup_{b\to\infty} |b_k|^{1/k} < \infty, \tag{1·1}$$

so that the series converges in $|w| > \rho_0$. The Faber polynomials $F_k(z)$ associated with g are defined by the generating function

$$\frac{g'(w)}{g(w)-z} = \sum_{k=0}^{\infty} \frac{F_k(z)}{w^{k+1}}.$$

For every k, $F_k(z)$ is a monic polynomial of exact degree k.

Let ν_k be the normalized counting measure of the zeros of $F_k(z)$. That is,

$$\nu_k := \frac{1}{k} \sum_{j=1}^{k} \delta_{\zeta_{j,k}},$$

where $\zeta_{1,k}, \ldots, \zeta_{k,k}$ are the zeros of $F_k(z)$ and δ_ζ is the unit point measure at ζ. Our

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results concern the limit behaviour of the measures ν_k. To state the results we need to define two sets. For every $z \in \mathbf{C}$ we write $g^{-1}(z) := \{w \mid |w| > \rho_0, g(w) = z\}$.

Definition 1·1. (a) $C_0 := \{z \mid g^{-1}(z) = \varnothing\}$.

(b) C_1 is the set of all points z such that $g^{-1}(z)$ contains exactly one point of largest absolute value, say w, and $g'(w) \neq 0$.

We remark that C_0 is compact and C_1 is an open set containing a neighbourhood of infinity. These sets were introduced by Ullman [**8**], see also [**10**]. He proved

THEOREM 1·2 [**8**]. *All limit points§ of the zeros of* $\{F_k(z)\}_1^\infty$ *are in* $\mathbf{C}\backslash C_1$. *Every boundary point of* C_1 *is a limit point of zeros.*

Here we establish the following results.

THEOREM 1·3. *If the interior of* C_0 *is empty, then the sequence* $\{\nu_k\}_1^\infty$ *converges in the weak-star topology to a measure* μ *whose support is equal to* ∂C_1, *the boundary of* C_1.

If, in addition, $\mathbf{C} = C_0 \cup C_1$, *then* μ *is the equilibrium distribution* (*with respect to the logarithmic potential, see* [**7**, Section III·2]) *of the compact set* C_0.

THEOREM 1·4. *If the interior of* C_0 *is connected, then there is a subsequence of* $\{\nu_k\}_1^\infty$ *that converges in the weak-star topology to a measure* μ *whose support is* ∂C_1.

If, in addition, $\mathbf{C} = C_0 \cup C_1$, *then* μ *is the equilibrium distribution of the compact set* C_0.

Let $E \subset \mathbf{C}$ be a compact set, not a single point, such that $\bar{\mathbf{C}}\backslash E$ is simply-connected. By the Riemann mapping theorem there is an $r > 0$ and a conformal mapping f from $\mathbf{C}\backslash E$ onto $\{|w| > r\}$ such that in a neighbourhood of infinity we have

$$f(z) = z + \sum_{k=0}^{\infty} a_k z^{-k}.$$

Let $g(w)$ denote the inverse mapping. The Faber polynomials associated with g as in (1·2) are in this context also called the Faber polynomials for E. We denote by $\mathrm{int}(E)$ the set of interior points of E.

THEOREM 1·5. (a) *If* $\mathrm{int}(E) = \varnothing$, *then the sequence* $\{\nu_k\}_1^\infty$ *converges in the weak-star topology to the equilibrium distribution of* E.

(b) *Suppose* $\mathrm{int}(E)$ *is connected and that either* (i) ∂E *is not a piecewise analytic curve*; *or* (ii) ∂E *is a piecewise analytic curve that has a singularity other than an outward cusp. Then there is a subsequence of* $\{\nu_k\}_1^\infty$ *that converges in the weak-star topology to the equilibrium distribution of* E.

The outline of this paper is as follows. Section 2 describes properties of the boundary of C_1. In Section 3 we discuss the asymptotic limit

$$\rho(z) := \limsup_{k\to\infty} |F_k(z)|^{1/k}$$

and in Section 4 we prove that $-\log \rho(z)$ is, in fact, a logarithmic potential corresponding to a unique probability measure with support equal to ∂C_1. The proofs of Theorems 1·3 and 1·4 are given in Section 5 and the proof of Theorem 1·5 appears in Section 6. Finally, in Section 7, we consider Faber polynomials associated with Laurent series having only finitely many terms.

§ ζ is a limit point if there exist a strictly increasing sequence $\{k_j\}$ and for every j a zero ζ_j of F_{k_j} such that $\zeta = \lim_{j\to\infty} \zeta_j$.

2. *The boundary of C_1*

Following Ullman [**8**] we discuss a classification of the boundary points of C_1. The multiplicity of a point w with $|w| > \rho_0$ is m if $g'(w) = \cdots = g^{(m-1)}(w) = 0$ and $g^{(m)}(w) \neq 0$.

Definition 2·1. For every $p \geqslant 2$, C_p is the set of all points $z \in \mathbf{C}\backslash C_0$ such that the points of largest absolute value in $g^{-1}(z)$ have total multiplicity p. Points in C_p are called *p-points*. If all points in $g^{-1}(z)$ of largest absolute value have multiplicity 1, then z is called a *simple p-point*.

We shall need some results on the structure of the boundary of C_1. These results are analogous to those of Ullman in connection with the asymptotic behaviour of eigenvalues of Toeplitz matrices. In fact, there is a closed connection between Faber polynomials and certain Toeplitz matrices. Let $\mathscr{F}_n$ denote the leading $n \times n$ principal submatrix of the infinite matrix

$$\mathscr{F} := \begin{matrix} b_0 & 2b_1 & 3b_2 & 4b_3 \\ & b_0 & b_1 & b_2 \\ & & b_0 & b_1 \\ & & & b_0 \end{matrix}$$

Then it is known that the zeros of F_n are exactly the eigenvalues of $\mathscr{F}_n$, see [**1**]. This matrix is nearly Toeplitz. The zeros of F'_{n+1} are exactly the eigenvalues of $\mathscr{G}_n$, the leading $n \times n$ principal submatrix of

$$\mathscr{G} := \begin{matrix} b_0 & b_1 & b_2 & b_3 \\ & b_0 & b_1 & b_2 \\ & & b_0 & b_1 \\ & & & b_0 \end{matrix}$$

The asymptotic behaviour (for $n \to \infty$) of the eigenvalues of Toeplitz matrices like $\mathscr{G}_n$ has been studied in [**6**], [**3**], [**9**] for the banded case (i.e. $b_k = 0$ for $k \geqslant m$). Ullman [**10**] studied a more general case which includes Toeplitz matrices of the form (2·2). An important quantity in determining asymptotic properties of the eigenvalues of $\mathscr{G}_n$ is the number $\tilde{\rho}$ defined as the smallest number such that g has a *meromorphic* extension to $|w| > \tilde{\rho}$. It is possible that $\tilde{\rho}$ is smaller than ρ_0. To emphasize the difference we denote the meromorphic extension by $\tilde{g}$. Ullman introduced (using different notation) the sets $\tilde{C}_0$ and $\tilde{C}_1$ analogous to Definition 1·1, but with $\tilde{g}$ instead of g. He proved the following two lemmas for the boundary of $\tilde{C}_1$. The same arguments can be used to establish them for the boundary of C_1.

LEMMA 2·2 [**10**, lemma 7]. *Every $z_0 \in C_p$, $p \geqslant 2$ has a neighbourhood $B(z_0, \epsilon) := \{z \mid |z - z_0| < \epsilon\}$ such that $\partial C_1 \cap \overline{B(z_0, \epsilon)}$ consists of a finite number of analytic Jordan arcs each joining z_0 to a point on the circle $|z - z_0| = \epsilon$. Any two arcs intersect only at z_0. The remaining points of $B(z_0, \epsilon)$ are in C_1.*

COROLLARY 2·3. *The boundary of C_1 has the following representation:*

$$\partial C_1 = \partial C_0 \cup \bigcup_{p \geqslant 2} C_p.$$

LEMMA 2·4 [**10**, lemma 8]. *For every* $z_0 \in C_p$, $p \geqslant 2$ *and* $\epsilon_0 > 0$ *there are* $z_1 \in C_q$, $q \geqslant 2$ *and* $\epsilon_1 > 0$ *such that*

$$B(z_1, \epsilon_1) \subset B(z_0, \epsilon_0), \tag{2·3}$$

$$B(z_1, \epsilon_1) \cap C_0 \neq \varnothing, \tag{2·4}$$

$$B(z_1, \epsilon_1) \cap C_1 = D_1 \cup D_2, \tag{2·5}$$

where D_1 *and* D_2 *are disjoint non-empty domains. Moreover, there exist analytic functions* $f_1(z)$, $f_2(z)$ *on* $B(z_1, \epsilon_1)$ *such that*

$$|f_1(z)| > |f_2(z)| \quad z \in D_1, \tag{2·6}$$

$$|f_1(z)| = \rho(z), \quad z \in D_1, \tag{2·7}$$

$$|f_2(z)| = \rho(z), \quad z \in D_2. \tag{2·8}$$

3. *Asymptotics of* $|F_k(z)|^{1/k}$

For every $z \in \mathbf{C}$ we define

$$\rho(z) := \limsup_{k \to \infty} |F_k(z)|^{1/k}. \tag{3·1}$$

The following result is also due to Ullman [**8**].

LEMMA 3·1. (a) *For every* $z \in C_0$,

$$\rho(z) = \rho_0, \tag{3·2}$$

where ρ_0 *is defined by* (1·1).

(b) *For every* $z \notin C_0$,

$$\rho(z) = \max\{|w| \mid w \in g^{-1}(z)\} \tag{3·3}$$

(c) *For every* $z \in C_1$

$$\rho(z) = \lim_{k \to \infty} |F_k(z)|^{1/k}, \tag{3·4}$$

i.e. the lim sup *in* (3·1) *can be replaced by* lim.

From Lemma 3·1 we obtain two important properties of $\rho(z)$.

LEMMA 3·2. $\rho(z)$ *is a continuous function on* $\mathbf{C}$.

Proof. This follows easily from (3·2) and (3·3).

LEMMA 3·3. (a) *The function* $-\log \rho(z)$ *is superharmonic on* $\mathbf{C}$.

(b) $-\log \rho(z)$ *is harmonic on* $C_1 \cup \operatorname{int}(C_0)$, *but not at points of* ∂C_1.

Proof. Write $U(z) := -\log \rho(z)$. (a) Recall that $U(z)$ is superharmonic on $\mathbf{C}$ if it satisfies the following three conditions.

(i) $U(z) \in (-\infty, +\infty]$ for every $z \in \mathbf{C}$; U is not identically $+\infty$;

(ii) U is lower semi-continuous;

(iii) For $z \in \mathbf{C}$ and $r > 0$ we have

$$U(z) \geqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} U(z + r\, e^{i\phi})\, d\phi.$$

Property (i) is obvious and property (ii) follows from Lemma 3·2. So we need only prove property (iii).

For $n\in\mathbf{N}$ write

$$U_n(z) := \inf_{k\geqslant n}\left(-\frac{1}{k}\log|F_k(z)|\right)$$

By (3·1) we have
$$U(z) = \lim_{n\to\infty} U_n(z), \quad z\in\mathbf{C}.$$

Since $-(1/k)\log|F_k(z)|$ is superharmonic, we find for $z\in\mathbf{C}$, $r>0$ and $k\geqslant n$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} U_n(z+r\,e^{i\phi})\,d\phi \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(-\frac{1}{k}\log|F_k(z+r\,e^{i\phi})|\right)d\phi \leqslant -\frac{1}{k}\log|F_k(z)|$$

Since this holds for every $k\geqslant n$ we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} U_n(z+r\,e^{i\phi})\,d\phi \leqslant U_n(z).$$

Next note that $\{U_n\}_1^\infty$ is an increasing sequence of functions with pointwise limit U. Thus for each n,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} U_n(z+r\,e^{i\phi})\,d\phi \leqslant U(z), \quad z\in\mathbf{C}.$$

By the monotone convergence theorem it follows that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} U(z+r\,e^{i\phi})\,d\phi \leqslant U(z).$$

Hence U satisfies property (iii) and we have proved that it is superharmonic.

(b) On C_1 we define $f(z)$ by $f(z) = w$ if and only if $|w| = \rho(z)$ and $g(w) = z$. It is easily seen that $f(z)$ is analytic on C_1 and non-zero. Since $U(z) = -\log|f(z)|$ for $z\in C_1$, it follows that $U(z)$ is harmonic on C_1. Also, since $U(z)$ is constant on C_0, it is harmonic on $\operatorname{int}(C_0)$.

Let $z_0\in\partial C_0$. Since $U(z)\leqslant U(z_0)$ for every z with equality only for $z\in C_0$, it follows from the maximum principle for harmonic functions that U is not harmonic at z_0.

Now suppose that U is harmonic in a neighbourhood $B(z_0,\epsilon_0)$ of some p-point z_0 with $p\geqslant 2$. Let z_1, ϵ_1, D_1, D_2, f_1 and f_2 be as in Lemma 2·4. Taking a smaller ϵ_1 if necessary, we may assume that $f_1(z)\neq 0$ for $z\in B(z_1,\epsilon_1)$. Then $h(z) := U(z)+\log|f_1(z)|$ is harmonic on $B(z_1,\epsilon_1)$. From (2·7) it follows that $h(z) = 0$ for $z\in D_1$, and from (2·6) and (2·8) it follows that $h(z)>0$ for $z\in D_2$. This is clearly impossible for a harmonic function.

This completes the proof in view of Corollary 2·3.

4. *The measure* μ

For a measure ν on $\mathbf{C}$ we denote its logarithmic potential by

$$U^\nu(z) := \int \log\frac{1}{|z-t|}\,d\nu(t), \quad z\in\mathbf{C}.$$

THEOREM 4· *There is a unique measure μ on $\mathbf{C}$ such that*

$$-\log\rho(z) = U^\mu(z), \quad z\in\mathbf{C}. \tag{4·1}$$

μ is a probability measure with support equal to ∂C_1.

'his measure is uniquely determined by g; we call it the measure associated with g.

Proof. By the Riesz Decomposition Theorem for superharmonic functions [**7**, theorem II·21], [**4**, §I·5] and by Lemma 3·3 there is a unique measure μ on **C** with the property that for every open bounded set D, there is a harmonic function u_D such that

$$-\log \rho(z) = u_D(z) + \int_D \log \frac{1}{|z-t|} d\mu(t) \quad z \in D. \tag{4·2}$$

Let $D = B_r = \{|z| < r\}$ and take r such that $\partial C_1 \subset B_r$. Since $-\log \rho(z)$ is harmonic off of ∂C_1, it follows from (4·2) that $\operatorname{supp}\mu \cap B_r \subset \partial C_1$, see [**7**, theorem II·25]. Since r can be taken arbitrarily large, it follows that $\operatorname{supp}\mu \subset \partial C_1$. Then we obtain from (4·2) that there is a function u which is harmonic on **C** such that

$$-\log \rho(z) = u(z) + U^\mu(z) \quad z \in \mathbf{C} \tag{4·3}$$

Let M be the total mass of μ. Then we have

$$\lim_{|z|\to\infty} (U^\mu(z) + M \log|z|) = 0.$$

Also from the definition of $\rho(z)$

$$\lim_{|z|\to\infty} (-\log \rho(z) + \log|z|) = 0.$$

Thus from (4·3)

$$\lim_{|z|\to\infty} (u(z) - (M- \quad \log|z|) = 0.$$

If $M > 1$, then it would follow that $\lim_{|z|\to\infty} u(z) = +\infty$, which is impossible by the minimum principle for harmonic functions. Similarly $M < 1$ is impossible. Thus $M = 1$, which means that μ is a probability measure. Then it also follows from (4·4) that u vanishes identically. So (4·1) holds.

By Lemma 3·3, $U^\mu(z)$ is not harmonic on ∂C_1. This implies $\partial C_1 \subset \operatorname{supp}\mu$. We have already proved that $\operatorname{supp}\mu \subset \partial C_1$; hence $\operatorname{supp}\mu = \partial C_1$.

5. *Proofs of Theorems* 1·3 *and* 1·4

Let ν_k denote the normalized counting measure of the zeros of $F_k(z)$ Then

$$U^{\nu_k}(z) = -\frac{1}{k}\log|F_k(z)|$$

From (3·1) and Theorem 4·1 it follows that

$$\liminf_{k\to\infty} U^{\nu_k}(z) = -\log \rho(z) = U^\mu(z \quad z \in \mathbf{C} \tag{5·1}$$

From Lemma 3·1 (*c*), we find

$$\lim_{k\to\infty} U^{\nu_k}(z) = -\log \rho(z) = U^\mu(z), \quad z \in C_1 \tag{5·2}$$

Lemma 5· *Let ν be any weak-star limit of the sequence $\{\nu_k\}_1^\infty$. Then*

$$U^\nu(z) = U^\mu(z), \quad z \in \overline{C}_1.$$

Furthermore,
$$U^\nu(z) \geqslant U^\mu(z), \quad z \in \mathbf{C}.$$

Proof. From (5·2) and the Lower Envelope Theorem [**4**, theorem 3·8] it follows that

$$U^\nu(z) = U^\mu(z)$$

quasi-everywhere on C_1, where quasi-everywhere means 'except for a set of logarithmic capacity zero'. By Theorem 1·2 the support of ν is contained in $\mathbf{C}\backslash C_1$. Hence U^ν is continuous on C_1. Since U^μ is also continuous (Lemma 3·2), it follows that

$$U^\nu(z) = U^\mu(z; \quad z \in C_1$$

Let $z_0 \in C_p$ with $p \geqslant 2$ and let $B(z_0, \epsilon)$ be as in Lemma 2·2. We see from Lemma 2·2 that z_0 is a regular point for each component of $C_1 \cap B(z_0, \epsilon)$. Then it follows from (5·3) and the continuity of potentials in the fine topology [**4**, §V.3] that $U^\nu(z_0) = U^\mu(z_0)$. Thus

$$U^\nu(z) = U^\mu(z; \quad z \notin C_0.$$

Since U^μ is continuous and U^ν is lower semi-continuous, it follows that

$$U^\nu(z) \leqslant U^\mu(z), \quad z \in \partial C_0.$$

It remains to show that $U^\nu(z) \geqslant U^\mu(z)$ for $z \in C_0$. For every $\epsilon > 0$, the set

$$F_\epsilon := \{z \mid U^\mu(z) = -\log \rho_0 - \epsilon\}$$

is a closed subset of $\mathbf{C}\backslash C_0$ such that every $z \in C_0$ belongs to a bounded component of $\mathbf{C}\backslash F_\epsilon$. Since $U^\nu(z) = -\log \rho_0 - \epsilon$ on F_ϵ, it follows by the minimum principle that $U^\nu(z) \geqslant -\log \rho_0 - \epsilon$ on C_0. Since $\epsilon > 0$ is arbitrary we find

$$U^\nu(z) \geqslant -\log \rho_0 = U^\mu(z \quad z \in C$$

This proves the lemma

Proof of Theorem 1·3. Let ν be any weak-star limit of $\{\nu_k\}_1^\infty$. If C_0 has empty interior we have by the previous lemma that $U^\nu(z) = U^\mu(z)$ for every $z \in \mathbf{C}$. The unicity theorem for logarithmic potentials (see [**4**, theorem 1·12]) then implies $\nu = \mu$.

The support of μ is equal to ∂C_1 by Theorem 4·1.

If $\mathbf{C} = C_0 \cup C_1$, then the support of μ is equal to ∂C_0. Since U^μ is constant on C_0, μ is the equilibrium distribution of C_0, cf. [7, theorem III·15]. ∎

Lemma 5·2. *Suppose the interior of C_0 is non-empty. Let G be a connected component of the interior of C_0. Then there exists a subsequence of $\{\nu_k\}_1^\infty$ that converges to a measure ν such that*

$$U^\nu(z) = U^\mu(z), \quad z \in G.$$

Proof. Pick $z_0 \in G$. By (1·1) there is a strictly increasing sequence $\{n_k\}_1^\infty$ such that

$$\lim_{k\to\infty} |F_{n_k}(z_0)|^{1/n_k} = \rho(z_0) = \rho_0.$$

Then it follows that

$$\lim_{k\to\infty} U^{\nu_{n_k}}(z_0) = \quad \log \rho_0.$$

Let ν be a weak-star limit of $\{\nu_{n_k}\}_k$. By the Principle of Descent [4, theorem 1·3] $U^{\nu}(z_0) \leqslant -\log\rho_0$. We know from Lemma 5·1 that $U^{\nu}(z) = -\log\rho_0$ for $z \in \partial G$. Then the minimum principle shows that $U^{\nu}(z) = -\log\rho_0$ for every $z \in G$. ∎

Proof of Theorem 1·4. If the interior of C_0 is connected, then Lemma 5·2 shows that there is a weak-star limit ν of a subsequence of $\{\nu_k\}_1^\infty$ such that

$$U^{\nu}(z) = U^{\mu}(z), \quad z \in \operatorname{int}(C_0).$$

Combined with Lemma 5·1 this gives that $U^{\nu}(z) = U^{\mu}(z)$ for every $z \in \mathbf{C}$. Then the unicity of theorem for logarithmic potentials gives $\nu = \mu$.

The support of μ is equal to ∂C_1 by Theorem 4·1.

If $\mathbf{C} = C_0 \cup C_1$, then the support of μ is equal to ∂C_0. Since U^{μ} is constant on C_0, μ is the equilibrium distribution of C_0.

6. *Proof of Theorem* 1·5

The function $g(w)$ maps $|w| > r$ one-to-one onto $\mathbf{C}\backslash E$. Therefore $\mathbf{C}\backslash E \subset C_1$. Let ρ_0 be defined by (1·1). We have $\rho_0 \leqslant r$.

Proof of Theorem 1·5. (a) Suppose $\operatorname{int}(E) = \varnothing$. Then $\operatorname{int}(C_0) = \varnothing$ and it follows from Theorem 1·3 that the sequence $\{\nu_k\}_1^\infty$ converges in the weak-star topology to μ. By Lemma 5·1 this measure μ satisfies (4·1).

If $\rho_0 = r$, then $C_0 = E$. If $\rho_0 < r$ then every point of E corresponds to at least 2 points on $|w| = r$ (because $\operatorname{int}(E) = \varnothing$) and then $E = \bigcup_{p \geqslant 2} C_p$. In either case we have $E = \partial C_1$.

From Theorem 4·1 it follows that $\operatorname{supp}\mu = E$ and that U^{μ} is constant on E. Therefore μ is the equilibrium distribution for E.

(b) Suppose $\operatorname{int}(E)$ is connected. We will show that either of the conditions (i) and (ii) implies that $\rho_0 = r$. Then it will follow that $C_0 = E$ and $C_1 = \mathbf{C}\backslash E$. An application of Theorem 1·4 then completes the proof.

Assume that $\rho_0 < r$. We show that both (i) and (ii) cannot hold. $g(w)$ is analytic on the circle $|w| = r$ and it maps $|w| = r$ onto ∂E. There are at most a finite number of points on $|w| = r$ for which $g'(w) = 0$.

If there are no such points then ∂E is an analytic curve and (i) and (ii) do not hold. If there are k distinct points on $|w| = r$ with $g'(w) = 0$, we order them as $w_1, w_2, \ldots, w_k$ in such a way that $0 \leqslant \arg(w_1) < \arg(w_2) < \cdots < \arg(w_k) < 2\pi$. Let $z_j := g(w_j)$, $j = 1, \ldots, k$. If $m \geqslant 2$ is the multiplicity of the zero of $g(w) - z_j$ at w_j then the image under g of the circle $|w| = r$ makes an angle $m\pi$ at z_j. Since g is univalent in $|w| > r$ it follows that $m = 2$ and ∂E has an outward cusp at z_j.

The image of the arc $|w| = r, \arg(w_j) < \arg(w) < \arg(w_{j+1})$ is an analytic arc that joins z_j and z_{j+1}. Thus ∂E consists of a finite number of analytic arcs. Two arcs meet each other at common endpoints to form an outward cusp. It follows that (i) and (ii) do not hold.

Remarks 6·1. (a) Let g map $|w| > r$ one-to-one onto the complement of E. It is easily seen that the balayage of every limit measure of $\{\nu_k\}_1^\infty$ to the boundary of E is equal to the equilibrium distribution of E. Also the balayage of the measure μ (associated with g, see Theorem 4·1) to the boundary of E is equal to the equilibrium distribution of E.

(b) He and Saff[2] considered a region E (an m-cusped hypocycloid) whose boundary has m outward cusps and for which no boundary points except for the cusps is a limit point of the zeros of the Faber polynomials.

(c) If E is a regular m-gon then Theorem 1·5(b) says that a subsequence of $\{\nu_k\}_1^\infty$ converges to the equilibrium distribution of E. Using different methods one can show that in this case the full sequence converges. We do not know if this is true for every E that satisfies the conditions of Theorem 1·5(b).

(d) Let

$$g(w) = w \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} w^{-2k} = (w^2+1)^{\frac{1}{2}},$$

where the branch of the square root is chosen such that w is positive for w positive. The series converges for $|w| > 1$ and g maps $|w| > 1$ one-to-one onto the complement of $E = \{z \in \mathbf{C} \mid |z^2-1| \leqslant 1\}$ (a lemniscate). Ullman[**8**] proved that the Faber polynomial of degree $2k$ is $(z^2-1)^k$, so that $\nu_{2k} = (\delta_1 + \delta_{-1})/2$, while the sequence $\{\nu_{2k-1}\}_1^\infty$ converges to the equilibrium distribution of E. Thus the full sequence $\{\nu_k\}_1^\infty$ does not converge. Note that for this case $\operatorname{int}(E)$ is not connected.

7. *Laurent series with finitely many terms*

Let $m \geqslant 1$ and let

$$g(w) = w + \sum_{k=0}^{m} b_k w^{-k}. \tag{7·1}$$

We assume $b_m \neq 0$. It is clear that $\rho_0 = 0$. For every $z \in \mathbf{C}$ the equation $g(w) = z$ has $m+1$ solutions $w_0(z), \ldots, w_m(z)$. We order them in such a way that $|w_0(z)| \geqslant |w_1(z)| \geqslant \cdots \geqslant |w_m(z)|$.

We note that the Faber polynomials F_k have a simple expression in terms of the solutions $w_j(z), j = 0, \ldots, m$. This representation was given in [2] for the special case $b_0 = b_1 = \cdots = b_{m-1} = 0$.

PROPOSITION 7·1. *For every* $k \geqslant 1$,

$$F_k(z) = \sum_{j=0}^{m} w_j(z)^k.$$

Proof. It is easy to see that

$$g(w) - z = w^{-m} \prod_{j=0}^{m} (w - w_j(z)).$$

Taking logarithmic derivatives with respect to w, we obtain

$$\frac{g'(w)}{g(w)-z} = -\frac{m}{w} + \sum_{j=0}^{m} \frac{1}{w - w_j(z)} = -\frac{m}{w} + \sum_{k=0}^{\infty} \sum_{j=0}^{m} w_j(z)^k w^{-k-1}$$

The proposition now follows because of (1·2). ▌

We will now consider the normalized counting distributions ν_k of the zeros of the Faber polynomials. It is clear from the above that $C_0 = \varnothing$. Therefore Theorem 1·3 immediately gives:

PROPOSITION 7·2. *The sequence* $\{\nu_k\}_1^\infty$ *converges in the weak-star topology to a measure* μ *with support* ∂C_1.

The structure of ∂C_1 was determined by Schmidt and Spitzer[6], see also [3]. These results were proved for eigenvalues of banded Toeplitz matrices, but apply also to the zeros of Faber polynomials associated with a function g having a Laurent expansion with finitely many terms. cf. the discussion in Section 2.

THEOREM 7·3. *∂C_1 is a finite union of closed analytic Jordan arcs, where either distinct arcs are disjoint or, if not. their intersection consists of common endpoints.*

Ullman[9] proved that ∂C_1 is connected. We do not know if ∂C_1 is necessarily simply connected.

Hirschman[3] gave a formula for the limit distribution of the eigenvalues of banded Toeplitz matrices. The same formula holds for the limit measure μ of Proposition 7·2. Hirschman showed that μ is absolutely continuous with respect to arc length on every analytic arc and he obtained an expression for the density. Here we will give a different proof of this result, using a general theorem on the reconstruction of a measure from its potential.

Let γ be one of the analytic Jordan arcs of ∂C_1. Let $z(s)$ be a parametrization of γ by means of arc length measured from an arbitrary point on γ. From Lemma 2·4 it follows that there exists an open set B such that $B \cap \partial C_1$ is the interior of γ and $B \cap C_1 = D_1 \cup D_2$, where D_1 and D_2 are non-empty disjoint domains. For every $z \in C_1$ the mapping $z \mapsto w_0(z)$ is well-defined and analytic. We denote it by f_1 on D_1 and by f_2 on D_2. These functions have an analytic continuation across the interior of γ such that $|f_1(z)| = |f_2(z)| > 0$ on the interior of γ. The result of Hirschman is:

THEOREM 7·4 [3]. *μ is absolutely continuous with respect to arc length on every analytic arc γ of ∂C_1 and we have*

$$d\mu = \frac{1}{2\pi}\left|\frac{d}{dz}\frac{f_2(z)}{f_1(z)}\right|_{z=z(s)} ds \tag{7·2}$$

Moreover, μ has no atoms

Proof. The potential of μ is (see Theorem 4·1)

$$U^\mu(z) = \begin{cases} -\log|f_1(z)| & z \in D_1 \\ -\log|f_2(z)| & z \in D_2 \end{cases}$$

Since f_1 and f_2 have analytic continuations across the interior of γ, it follows that U^μ is Lip 1 in B. Let $n_1(s)$ and $n_2(s)$ denote the normals to γ at $z(s) \in \gamma$ pointing into D_1 and D_2, respectively. Then from [5, chapter II, theorem 1·5] it follows that μ is absolutely continuous with respect to ds and

$$d\mu = -\frac{1}{2\pi}\left(\frac{\partial U^\mu}{\partial n_1}(z(s)) + \frac{\partial U^\mu}{\partial n_2}(z(s))\right) ds.$$

From (7·3)

$$\frac{\partial U^\mu}{\cdots}(z(s)) = \mathrm{Re}\left[-n_j(s)\frac{f_j'(z(s))}{f_j(z(s))}\right], \quad j = \quad 2$$

Since $n_2(s) = -n_1(s)$ it follows from this and (7·4) that

$$d\mu = \frac{1}{2\pi}\mathrm{Re}\left[n_1(s)\left(\frac{f_1'(z(s))}{f_1(z(s))} - \frac{f_2'(z(s))}{f_2(z(s))}\right)\right] ds$$

Next for $z = z(s)$ we have $|f_1(z)| = |f_2(z)|$, or $\mathrm{Re}\,[\log f_1(z) - \log f_2(z)] = 0$. Differentiating this with respect to s we find

$$\mathrm{Re}\left[z'(s)\left(\frac{f_1'(z(s))}{f_1(z(s))} - \frac{f_2'(z(s))}{f_2(z(s))}\right)\right] = 0.$$

Then it follows since $n_1(s) = \pm iz'(s)$ that

$$n_1(s)\left(\frac{f_1'(z(s))}{f_1(z(s))} - \frac{f_2'(z(s))}{f_2(z(s))}\right)$$

is real and therefore it is equal to

$$\pm\left|\frac{f_1'(z(s))}{f_1(z(s))} - \frac{f_2'(z(s))}{f_2(z(s))}\right|$$

In view of (7·5) we need the plus sign and so

$$d\mu = \frac{1}{2\pi}\left|\frac{f_1'(z(s))}{f_1(z(s))} - \frac{f_2'(z(s))}{f_2(z(s))}\right| ds. \tag{7·6}$$

Since $|f_1(z)| = |f_2(z)|$ for $z = z(s)$, (7·2) follows from (7·6).

Since $U^{\mu}(z)$ is finite for every $z \in \mathbf{C}$, it follows that μ has no atoms.

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