

## Behavior of Lagrange interpolants to the absolute value function in equally spaced points

X. LI – E.B. SAFF

*Dedicated to Aldo Ghizzetti*

**RIASSUNTO:** *Viene trovato il limite “star” debole della successione delle misure “counting” normalizzate degli zeri delle interpolanti di Lagrange, associate a nodi equidistanti in  $[-1, 1]$  e relativi alla funzione  $f_s(x) = |x - s|$ , con  $s \in (-1, 1)$ . Questo risultato viene poi utilizzato per stabilire la regione esatta, in cui le interpolanti di Lagrange convergono geometricamente.*

**ABSTRACT:** *We find the weak star limit of the sequence of normalized counting measures of the zeros of the Lagrange interpolants to  $f_s(x) = |x - s|$  ( $-1 < s < 1$ ) associated with equidistant nodes on  $[-1, 1]$ . We use this to establish the exact region in which the Lagrange interpolants converges geometrically.*

**KEY WORDS:** *Lagrange interpolation – Equidistant nodes – Zero distribution – Potential theory.*

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### 1 – Introduction and statements of main results

For a function  $f$  defined on  $[-1, 1]$ , let  $L_n(f)$  denote the Lagrange

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interpolating polynomial of degree at most  $n$  to  $f$  at the equidistant nodes

$$x_k^{(n)} = -1 + 2k/n, \quad k = 0, 1, \dots, n.$$

Bernstein proved that (cf. [8]) for  $f(x) = |x|$ , the sequence  $L_n(|t|; x)$  diverges if  $0 < |x| < 1$ . Recently, BYRNE, MILLS and SMITH [9] considered the rate of this divergent sequence. They proved, if  $0 < |x| < 1$ , then

$$\limsup_{n \rightarrow \infty} |L_n(|t|; x) - |x|^{1/n}| = (1+x)^{(1+x)/2} (1-x)^{(1-x)/2}.$$

Li and Mohapatra [6] further improved this result by showing that

$$(1) \quad \lim_{n \rightarrow \infty} \left| \frac{L_n(|t|; x) - |x|^{1/n}}{w_n(x)} \right| = e$$

for all  $x \in \mathbf{R}$  (the set of real numbers), where  $w_n(x) := \prod_{k=0}^n (x - x_k^{(n)})$ .

In contrast to the above results, under the assumption that  $f$  is bounded on  $[-1, 1]$  and analytic at  $x = 0$ , the authors proved in [7] that the sequence  $L_n(f; x)$  converges to  $f$  geometrically in a neighborhood (in the complex plane) of  $x = 0$ . This leads to the question of finding the exact region where  $L_n(f; \cdot)$  converges to  $f$ . Although the answer in the general situation is still unknown, we try in this note to gain some insight by considering the special but interesting case when

$$f(x) = f_s(x) := |x - s| \quad (-1 < s < 1).$$

We determine the exact region in which  $L_n(f_s; \cdot)$  converges to (an analytic continuation of)  $f_s$  geometrically. This is done by studying the zero distribution of  $L_n(f_s; \cdot)$ , which is equivalent to the  $n$ th root asymptotics of  $L_n(f_s; z)$  in  $\mathbf{C}$ . Furthermore, we will show that (1) has an extension to all  $z$  in the complex plane  $\mathbf{C}$ .

To state our results, we first introduce some notation. The *potential* corresponding to the uniform distribution  $\frac{1}{2} dt$  on  $[-1, 1]$  is given by

$$U(z) = \frac{1}{2} \int_{-1}^1 \log |z - t| dt.$$

The level curves of  $U(z)$  are denoted by  $\Gamma_s := \{z \in \mathbf{C} \mid U(z) = U(s)\}$   $s \in \mathbf{R}$ . Let

$$\Omega_s := [-1, -s] \cup \Gamma_s \cup [s, 1] \text{ for } |s| < 1.$$

Let  $\nu_n(t)$  be the normalized counting measure of the zeros of  $L_n(f_s; \cdot)$ , i.e.,

$$\int_B d\nu_n(t) = \frac{\text{the number of the zeros of } L_n(f_s; z) \text{ in } B}{n},$$

for every Borel set  $B \subseteq \mathbf{C}$ . For a compact set  $S \subseteq \mathbf{C}$ , we will use  $\text{Ext}(S)$  and  $\text{Int}(S)$  to denote the unbounded and the (union of) bounded components of  $\overline{\mathbf{C}} \setminus S$ , respectively, where  $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ . We need one more concept from potential theory. A measure  $b_s$  supported on  $\Omega_s$  is called a *balayage* of the uniform distribution  $\frac{1}{2}dt$  on  $[-1, 1]$  to  $\Omega_s$  if

$$\int_{\Omega_s} \log |z - t| db_s(t) = U(z) \text{ for all } z \in \text{Ext}(\Omega_s)$$

On using the fact that  $\Gamma_s$  is regular with respect to the Dirichlet problem for  $\text{Int}(\Gamma_s)$ , one can show that at least one such balayage  $b_s$  exists (cf. [5, §4.2]), and since  $U(z)$  is continuous in  $\mathbf{C}$ , such a measure  $b_s$  must be unique ([5, Theorem 4.6, Corollary 2]).

We now state our results. Their proofs are given in Section 3.

**THEOREM 1.** *The sequence of the normalized counting measures  $\{\nu_n\}$  of the zeros of  $L_n(f_s; \cdot)$  converges, in the weak star topology, to the balayage  $b_s$  of  $\frac{1}{2}dt$  on  $[-1, 1]$  to  $\Omega_s$ , as  $n \rightarrow \infty$  through a subsequence  $\Lambda$  of positive integers.*

**REMARK 1.** Our proof shows that, in Theorem 1,  $\Lambda$  can be any sequence for which

$$(2) \quad \lim_{\substack{n \rightarrow \infty \\ n \in \Lambda}} |a_n|^{1/n} = e^{-U(s)}$$

where  $a_n$  denotes the leading coefficient of  $L_n(f_s; \cdot)$ .

**REMARK 2.** It can be shown (by using Khinchine's theorem [10] in Lemma 3 below) that for almost all  $s \in (-1, 1)$ ,

$$\lim_{n \rightarrow \infty} |a_n|^{1/n} = e^{-U(s)}$$

So, by Remark 1, the whole sequence  $\{\nu_n\}_{n=1}^\infty$  converges to  $b_s$  for almost all  $s$ .

**THEOREM 2.** For  $s \in (-1, 1)$ , we have

$$(3) \quad \limsup_{n \rightarrow \infty} |L_n(f_s; z)|^{1/n} = e^{U(z) - U(s)}$$

quasi-everywhere in  $\text{Ext}(\Omega_s)$  and

$$(4) \quad \lim_{n \rightarrow \infty} L_n(f_s; z) = (s - z) \operatorname{sgn}(s)$$

geometrically for every  $z \in \text{Int}(\Gamma_s)$

Here we use "quasi-everywhere" to mean that the property holds except on a set having logarithmic capacity zero.

**REMARK 3.** Note that  $U(z) > U(s)$  for  $z \in \text{Ext}(\Omega_s)$ , so (3) implies that for quasi-every  $z \in \text{Ext}(\Omega_s)$  a subsequence of  $L_n(f_s; z)$  tends to  $\infty$  geometrically. It is also possible to show that (3) holds for almost all  $z \in \text{Ext}(\Gamma_s) \setminus \{-1, 1\}$  and  $s \in (-1, 1)$ .

**REMARK 4.** Using Remarks 1 and 2, it can be shown that for almost all  $s$ ,  $\limsup$  can be replaced by  $\lim$  in (3).

**REMARK 5.** The relation (4) also follows from the general theorem proved in [7].

**THEOREM 3** For all  $z \in \mathbb{C}$ , there holds

$$(5) \quad \lim_{n \rightarrow \infty} \left| \frac{L_n(|t|; z) - |z|}{w_n(z)} \right|^{1/n} = e.$$

Readers familiar with the subject of asymptotic zero distributions of best polynomial approximants (cf. [2], [3], [11]) will recognize that the above results and their proofs have a flavor similar to those for the best polynomial approximants. However, our proofs are a bit more involved because in the present situation, unlike the case for best polynomial approximations, the limit measure  $b_s$  is not the equilibrium measure on its support.

## 2 – Lemmas

Define

$$\phi_s(x) := \begin{cases} x - s, & s \leq x \leq 1, \\ 0, & -1 \leq x \leq s. \end{cases}$$

Then, if  $x < s$ ,

$$\frac{L_n(f_s; x) - |x - s|}{2} = L_n(\phi_s; x)$$

By Newton's formula (cf. [8, p.14]),

$$L_n(\phi_s; x) = \sum_{k=0}^n \frac{\Delta_n^k \phi_s(-1)}{k!} \left(\frac{n}{2}\right)^k (x+1) \left(x+1 - \frac{2(k-1)}{n}\right),$$

where

$$\Delta_n^k \phi_s(-1) = \sum_{r=0}^k (-1)^{k-r} \binom{k}{r} \phi_s\left(-1 + \frac{2r}{n}\right), \quad k = 0, 1, \dots, n.$$

Set  $k(s) := \max\{k : x_k^{(n)} \leq s\}$ . Then  $k(s) : [n(s+1)/2]$  and  $x_{k(s)}^{(n)}$  is the closest node to the left of  $s$  (or equal to  $s$ ).

LEMMA 1. (i) If  $0 \leq k \leq k(s)$ , then  $\Delta_n^k \phi_s(-1) = 0$ .  
(ii) If  $k(s) + 1 \leq k \leq n$ , then

$$\Delta_n^k \phi_s(-1) = \frac{(-1)^{k-k(s)} (k-2)!}{(k-k(s)-1)! k(s)! n} \left\{ \frac{n(s+1)}{2} (k-1) - \left[ \frac{n(s+1)}{2} \right] k \right\}$$

PROOF. Assertion (i) is obvious. To prove (ii), we need the following two formulae [4]:

$$(6) \quad \sum_{k=0}^m (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m} \quad (n \geq 1)$$

and

$$(7) \quad \sum_{k=0}^m (-1)^k \binom{n}{k} k = (-1)^m n \binom{n-2}{m-1} \quad (n \geq 2)$$

Now

$$\begin{aligned}
 \Delta_n^k \phi_s(-1) &= \sum_{r=k(s)+1}^k (-1)^{k-r} \binom{k}{r} \phi_s\left(-1 + \frac{2r}{n}\right) \\
 &= \sum_{r=k(s)+1}^k (-1)^{k-r} \binom{k}{r} \left(-s - \frac{2r}{n}\right) = \\
 &= \sum_{l=0}^{k-k(s)-1} (-1)^l \binom{k}{l} \left(-s - \frac{2(k-l)}{n}\right) = \\
 &= \left(-s - 1 + \frac{2k}{n}\right) \sum_{l=0}^{k-k(s)-1} (-1)^l \binom{k}{l} - \frac{2}{n} \sum_{l=0}^{k-k(s)-1} (-1)^l \binom{k}{l} = \\
 &= \left(-s - 1 + \frac{2k}{n}\right) (-1)^{k-k(s)-1} \binom{k-1}{k-k(s)-1} + \\
 &= -\frac{2}{n} (-1)^{k-k(s)-1} k \binom{k-2}{k-k(s)-2} = \\
 &= \frac{(-1)^{k-k(s)-1} (k-2)!}{(k-k(s)-1)! k(s)!} \left\{ \left(-s - 1 + \frac{2k}{n}\right) (k-1) - \frac{2}{n} k (k-k(s)-1) \right\} = \\
 &= \frac{(-1)^{k-k(s)} (k-2)!}{(k-k(s)-1)! k(s)!} \frac{2}{n} \left\{ \frac{n(s+1)}{2} (k-1) - \left[ \frac{n(s+1)}{2} \right] k \right\}
 \end{aligned}$$

This concludes the proof of Lemma 1 □

$$d_k^{(n)}(s) := \frac{n(s+1)}{2} (k-1) - \left[ \frac{n(s+1)}{2} \right] k$$

for  $k = k(s) + 1, \dots, n$ . Then we have the following simple lemma.

LEMMA 2 For  $s \in (-1, 1)$  and  $n \geq 2$ , the coefficient of  $x^n$  in  $L_n(f_s; x)$  is

$$a_n := \frac{2(-1)^{n-k(s)}}{(n-1)n!} \left(\frac{n}{2}\right)^{n-1} \binom{n-1}{k(s)} d_n^{(n)}(s)$$

PROOF. Note that

$$a_n = 2 \times (\text{the coefficient of } x^n \text{ in } L_n(\phi_s; x))$$

when  $n \geq 2$ , and

$$\text{the coefficient of } x^n \text{ in } L_n(\phi_s; x) = \frac{\Delta_n^n \phi_s(-1)}{n!} \left(\frac{n}{2}\right)^n$$

We can now apply Lemma 1 (ii) to establish this lemma.  $\square$

LEMMA 3. For  $s \in (-1, 1)$ , we have

$$\limsup_{n \rightarrow \infty} |d_n^{(n)}(s)|^{1/n} = 1$$

PROOF. Since

$$|d_n^{(n)}(s)| = \left| \frac{n(s+1)}{2}(n-1) - \left[ \frac{n(s+1)}{2} \right] n \right| \leq n^2(|s|+1) \leq 2n^2$$

we have

$$\limsup_{n \rightarrow \infty} |d_n^{(n)}(s)|^{1/n} \leq 1.$$

Write

$$d_n^{(n)}(s) = n(n-1) \left\{ \frac{s+1}{2} - \frac{1}{2} \left[ \frac{n(s+1)}{2} \right] \right\} =: n(n-1)I_n(s)$$

Then, to prove the lemma, it suffices to show

$$(8) \quad \limsup_{n \rightarrow \infty} |I_n(s)|^{1/n} \geq 1$$

for all  $s \in (-1, 1)$ . Assume, to the contrary, (8) is not true for some  $s \in (-1, 1)$ . Then, there exist  $r \in (0, 1)$  and  $N > 0$  such that

$$(9) \quad |I_n(s)| < r^n$$

for all  $n \geq N$ . Consequently,

$$(10) \quad |I_n(s) - I_{n+1}(s)| < 2r^n$$

for  $n \geq N$ . But, with  $t = (s + 1)/2 \in (0, 1)$ ,

$$\begin{aligned} n(n-1)|I_n(s) - I_{n+1}(s)| &= n(n-1) \left| \frac{[(n+1)t]}{n} - [nt]n - 1 \right| \\ &= |(n-1)[(n+1)t] - n[nt]| \end{aligned}$$

If there are infinitely many  $n$  such that

$$(n-1)[(n+1)t] - n[nt] \neq 0,$$

then for those  $n$ ,  $n(n-1)|I_n(s) - I_{n+1}(s)| \geq 1$ . So  $\limsup_{n \rightarrow \infty} |I_n(s) - I_{n+1}(s)|^{1/n} \geq 1$ , contradicting (10). Hence there are only finitely many  $n$  such that (11) holds. Therefore, there is a constant  $M > 0$  such that  $(n-1)[(n+1)t] = n[nt]$  for all  $n \geq M$ . Then  $I_n(s) = I_{n+1}(s)$  for  $n \geq M$ . This, together with (9), tells us that  $I_n(s) = 0$  for  $n \geq M$ . That is,

$$t = \frac{[nt]}{n}$$

for all  $n \geq M$ , which implies that  $(n-1)t$  is an integer for every  $n \geq M$ . This can happen only if  $t = 0$  or  $t = 1$ , which is impossible. This completes the proof.  $\square$

LEMMA 4. *Let  $s \in (-1, 1)$ . Then*

$$\limsup_{n \rightarrow \infty} |a_n|^{1/n} = e^{-U(s)}$$

PROOF. From Lemmas 2 and 3 and Stirling's formula we obtain

$$\limsup_{n \rightarrow \infty} |a_n|^{1/n} = \frac{e}{(1+s)^{(1+s)/2}(1-s)^{(1-s)/2}} = e^{-U(s)}$$

□

LEMMA 5 Let  $x \leq s$  Then,

$$|L_n(\phi_s; x)| \leq \frac{2n^2}{n!} \left(\frac{n}{2}\right) \binom{n}{k(s)+1} |w_n(x)| =: c(n; s) |w_n(x)|,$$

$$\lim_{n \rightarrow \infty} c(n; s)^{1/n} = e^{-U(s)}$$

PROOF. Write  $d_k$  for  $d_k^{(n)}(s)$ ,  $k = k(s)+1, \dots, n$ . Then, using Lemma 1, we have

$$\begin{aligned} L_n(\phi_s; x) &= \sum_{k=k(s)+1}^n \frac{(-1)^{k-k(s)}(k-2)!}{k!(k-k(s)-1)!k(s)!} \frac{2}{n} d_k \left(\frac{n}{2}\right)^k (x+1) \cdot \left(x+1 - \frac{2(k-1)}{n}\right) = \\ &= \sum_{k=k(s)+1}^n \frac{(-1)^{k-k(s)} w_n(x) \left(\frac{n}{2}\right)^{k-1} d_k}{k(k-1)(k-k(s)-1)!k(s)!(x+1 - \frac{2k}{n}) \cdots (x+1 - \frac{2(n-1)}{n})(x-1)} \end{aligned}$$

Now, note that

$$\left|x+1 - \frac{2(k(s)+1)}{n}\right| \geq \left|s+1 - \frac{2(k(s)+1)}{n}\right| = \frac{2|d_{k(s)+1}|}{nk(s)}$$

$$\left|x+1 - \frac{2k}{n}\right| \geq -1 + \frac{2k}{n} \left(-1 + \frac{2(k(s)+1)}{n}\right) = \frac{2}{n}(k - k(s) - 1)$$

for  $k = k(s) + 2, \dots, n$ .

$$\begin{aligned}
& |L_n(\phi_s; x)| \\
& \leq \frac{|w_n(x)| \left(\frac{n}{2}\right)^n}{n!} \binom{n}{k(s)+2} \\
& \quad + \sum_{k=k(s)+2}^n \frac{|w_n(x)| \left(\frac{n}{2}\right)^{k-1} |d_k|}{k(k-1)(k-k(s)-1)!k(s)! \frac{2}{n}(k-k(s)-1) \cdots \frac{2}{n}(n-k(s)-1)} \\
& \quad \frac{|w_n(x)| \left(\frac{n}{2}\right)^n}{n!} \binom{n}{k(s)+2} \\
& \quad + \sum_{k=k(s)+2}^n \frac{|w_n(x)| \left(\frac{n}{2}\right)^{k-1} |d_k|}{k(k-1)(k-k(s)-1)(n-k(s)-1)k(s)! \left(\frac{2}{n}\right)^{n-k+1}} \leq \\
& \leq \frac{|w_n(x)| \left(\frac{n}{2}\right)^n}{n!} \binom{n}{k(s)+1} + \\
& \quad + \sum_{k=k(s)+2}^n \frac{|w_n(x)| \left(\frac{n}{2}\right)^n 2n}{(k(s)+1)(n-k(s)-1)k(s)!} \leq \\
& \leq \frac{2n^2}{n!} \left(\frac{n}{2}\right) \binom{n}{k(s)+1} |w_n(x)|
\end{aligned}$$

Equation (13) follows directly from an application of Stirling's formula. This completes the proof of Lemma 5.  $\square$

LEMMA 6. For  $z \in [-1, 1]$ , there holds

$$|w_n(z)|e^{-nU(z)} \leq 3n^2$$

PROOF. To simplify the notation, we write  $x_k$  for  $x_k^{(n)}$ ,  $k = 0, 1, \dots, n$ . Using the monotonicity of  $\log x$  on  $(0, \infty)$ , we have for  $k = 1, 2, \dots, k(z)$ ,

$$\frac{1}{n} \log(z - x_k) \leq \frac{1}{2} \int_{x_{k-1}}^{x_k} \log |z - t| dt;$$

and for  $k = k(z) + \dots, n - 1$ ,

$$\frac{1}{n} \log(x_k - z) \leq \frac{1}{2} \int_{x_k}^{x_{k+1}} \log|z - t| dt$$

Summing the above inequalities, we get

$$\frac{1}{n} \sum_{k=1}^{n-1} \log|z - x_k| \leq \frac{1}{2} \left( \int_{-1}^{x_{k(z)}} + \int_{x_{k(z)+}^1 \right) \log|z - t| dt,$$

or, equivalently,

$$(15) \quad \frac{1}{n} \log|w_n(z)| - \frac{1}{n} \log(1 - x^2) \leq U(z) - \frac{1}{2} \int_{x_{k(z)}}^{x_{k(z)+1}} \log|z - t| dt$$

Now

$$\int_{x_{k(z)}}^{x_{k(z)+1}} \log|z - t| dt = g(z - x_{k(z)}) + g(x_{k(z)+1} - z),$$

where  $g(u) := u \log u - u$ . Note that  $g'(u) = \log u < 0$  for  $u \in (0, 1)$ , so  $g$  is decreasing and  $g(u) < g(0+) = 0$  on the interval  $(0, 1)$ . Since

$$x_{k(z)+1} - z \in \left(0, \frac{2}{n}\right),$$

it then follows that

$$\left| \int_{x_{k(z)}}^{x_{k(z)+1}} \log|z - t| dt \right| \leq 2 \left| g\left(\frac{2}{n}\right) \right| = \frac{4}{n} \left(1 + \log \frac{n}{2}\right)$$

Using this together with (15), we obtain

$$\frac{1}{n} \log|w_n(z)| \leq U(z) + \frac{2}{n} \left(1 + \log \frac{n}{2}\right),$$

which implies (14) □

### 3 - Proofs of Theorems

We are now ready to prove the theorems of Section 1.

PROOF OF THEOREM 1. Let  $s \in (-1, 1)$ . If  $x \leq s$ , then, from Lemma 5, we have  $|L_n(\phi_s; x)| \leq c(n; s)|w_n(x)|$ , and so

$$|L_n(f_s; x) - |x - s|| = 2|L_n(\phi_s; x)| \leq 2c(n; s)|w_n(x)|.$$

If  $x \geq s$ , then, since

$$L_n(|t - s|; x) = L_n(|-t + s|; x) = L_n(|t - (-s)|; -x)$$

it follows from (16) and the fact that  $|w_n(-x)| = |w_n(x)|$

$$|L_n(f_s; x) - |x - s|| = |L_n(|t - (-s)|; -x) - |-x - (-s)|| \leq 2c(n; -s)|w_n(x)|$$

Hence, with  $\hat{c}(n; s) := \max\{c(n; s), c(n; -s)\}$

$$|L_n(f_s; x) - |x - s|| \leq 2\hat{c}(n; s)|w_n(x)|$$

for all  $x \in [-1, 1]$ . Note that, from (13),

$$\lim_{n \rightarrow \infty} \hat{c}(n; s)^{1/n} = e^{-U(s)}.$$

Next we need to estimate  $L_n(f_s; z)$  for  $z \in \mathbf{C}$ . Let us first estimate  $L_n(f_s; z) - |z - s|$  by extending (17) to  $\mathbf{C}$ . For definiteness, we assume  $s > 0$ ; the case when  $s < 0$  can be handled similarly. Define for  $n \geq 2$

$$p(z) := \log |L_n(f_s; z) - (s - z)| - nU(z)$$

Then  $p(z)$  is a subharmonic function in  $\overline{\mathbf{C}} \setminus [-1, 1]$  with  $p(\infty) = \log |a_n|$ . Using the maximum principle, we have

$$p(z) \leq \max_{z \in [-1, 1]} p(z) = \max\left\{ \max_{z \in [-1, s]} p(z), \max_{z \in [s, 1]} p(z) \right\}, \quad z \in \mathbf{C}.$$

By (17),

$$\begin{aligned} \max_{z \in [-1, s]} p(z) &= \max_{z \in [-1, s]} \left\{ \log \left| \frac{L_n(f_s; z) - |z - s|}{w_n(z)} \right| + \log |w_n(z)| - nU(z) \right\} \leq \\ &\leq \max_{z \in [-1, s]} \log \left\{ 2\hat{c}(n; s)|w_n(z)|e^{-nU(z)} \right\} \leq \log \{6\hat{c}(n; s)n^2\}. \end{aligned}$$

where in the last inequality we used (14). On the other hand, for  $z \in [s, 1]$

$$\begin{aligned} e^{p(z)} &= |L_n(f_s; z) - |z - s| - 2(s - z)| e^{-nU(z)} \leq \\ &\leq |L_n(f_s; z) - |z - s|| e^{-nU(z)} + 2|z - s| e^{-nU(z)} \leq \\ &\leq 2\hat{c}(n; s)|w_n(z)| e^{-nU(z)} + 4e^{-nU(z)} \leq 6\hat{c}(n; s)n^2 + 4e^{-nU(z)} \end{aligned}$$

where in the last inequality we used (14) again. So,

$$\max_{z \in [s, 1]} p(z) \leq \log\{6\hat{c}(n; s)n^2 + 4e^{-nU(s)}\}.$$

for  $z \in \mathbf{C}$ ,

$$\begin{aligned} p(z) &\leq \max\left\{\log(6\hat{c}(n; s)n^2), \log(6\hat{c}(n; s)n^2 + 4e^{-nU(s)})\right\} \\ &= \log\{6\hat{c}(n; s)n^2 + 4e^{-nU(s)}\} =: \log K(n; s); \end{aligned}$$

and, by using (18), it is easy to verify that

$$\lim_{n \rightarrow \infty} K(n; s)^{1/n} = e^{-U(s)}$$

An important consequence of (20) and (21) is the following: For  $s > 0$ ,

$$\begin{aligned} \limsup_{n \rightarrow \infty} |L_n(f_s; z) - (s - z)|^{1/n} &\leq \lim_{n \rightarrow \infty} K(n; s)^{1/n} e^{U(z)} \\ &= e^{U(z) - U(s)} < 1 \end{aligned}$$

for all  $z \in \text{Int}(\Gamma_s)$ .

Now, we are ready to estimate  $L_n(f_s; z)$ . Let  $G_s(t)$  be the Green's function for  $\text{Ext}(\Omega_s)$  with pole at  $\infty$ . Set  $\hat{\Gamma}_\rho := \{z \in \mathbf{C} : G_s(z) = \rho\}$ ,  $\rho > 0$ . Since

$$\lim_{n \rightarrow \infty} \{K(n; s)e^{nU(z)}\}^{1/n} = e^{U(z) - U(s)} > 1$$

uniformly for  $z \in \hat{\Gamma}_\rho$ , we have, for  $n$  sufficiently large and  $z \in \hat{\Gamma}_\rho$ ,  $|z - s| \leq K(n; s)e^{nU(z)}$ . Thus, using (20) we obtain, for  $n$  large and  $z \in \hat{\Gamma}_\rho$ ,

$$|L_n(f_s; z)| \leq |z - s| + K(n; s)e^{nU(z)} \leq 2K(n; s)e^{nU(z)}$$

Next, define

$$P(z) = \log |L_n(f_s; z)| - nU(z).$$

Then  $P(z)$  is subharmonic in  $\mathbf{C} \setminus [-1, 1]$  with  $P(\infty) = \log |a_n|$ . From (23), for each  $\rho > 0$ , there is a constant  $N(\rho) > 0$  such that when  $n \geq N(\rho)$ ,

$$P(z) \leq \log\{2K(n; s)\}, \text{ for } z \in \hat{\Gamma}_\rho.$$

Fix  $\rho^* > 0$ , and let  $I_{\rho^*}$  denote the set of all the zeros of  $L_n(f_s; z)$  that lie in  $\text{Ext}(\hat{\Gamma}_{\rho^*})$ . Choose  $\rho \in (0, \rho^*)$ . Let  $G(z; \zeta)$  be the Green's function for  $\text{Ext}(\hat{\Gamma}_\rho)$  with pole at  $\zeta$ . Then  $G(z; \infty) \equiv G_s(z) - \rho$ . Define

$$h(z) := P(z) + \sum_{\zeta \in I_{\rho^*}} G(z; \zeta)$$

The function  $h(z)$  is subharmonic in  $\text{Ext}(\hat{\Gamma}_\rho)$ , and by (24)

$$\limsup_{z \rightarrow \xi \in \hat{\Gamma}_\rho} h(z) = P(\xi) \leq \log\{2K(n; s)\}$$

Hence, the maximum principle for subharmonic functions gives

$$h(z) \leq \log\{2K(n; s)\}$$

for all  $z \in \text{Ext}(\hat{\Gamma}_\rho)$ . Note that  $G(\infty; \zeta) = G(\zeta; \infty) \geq \rho^* - \rho$  for  $\zeta \in \text{Ext}(\hat{\Gamma}_{\rho^*})$ . Thus

$$h(\infty) = \log |a_n| + \sum_{\zeta \in I_{\rho^*}} G(\infty; \zeta) \geq \log |a_n| + n\nu_n\{\text{Ext}(\hat{\Gamma}_{\rho^*})\}(\rho^* - \rho),$$

where  $\nu_n$  is the normalized counting measure of the zeros of  $L_n(f_s; z)$ . It then follows from (25) that

$$(\rho^* - \rho)n\nu_n\{\text{Ext}(\hat{\Gamma}_{\rho^*})\} \leq \log \frac{2K(n; s)}{|a_n|}$$

Now, from Lemma 4, we can find an infinite subsequence of positive integers, say  $\Lambda$ , such that

$$\lim_{\substack{n \rightarrow \infty \\ n \in \Lambda}} |a_n|^{1/n} = e^{-U(s)}$$

Thus, (26) and (21) imply that

$$\limsup_{\substack{n \rightarrow \infty \\ n \in \Lambda}} \nu_n \{ \text{Ext}(\hat{\Gamma}_{\rho^*}) \} \leq (\rho^* - \rho)^{-1} \lim_{\substack{n \rightarrow \infty \\ n \in \Lambda}} \log \left\{ \frac{2K(n; s)}{|a_n|} \right\}^{1/n} = 0,$$

and so

$$\lim_{\substack{n \rightarrow \infty \\ n \in \Lambda}} \nu_n \{ \text{Ext}(\hat{\Gamma}_{\rho^*}) \} = 0 \text{ for every } \rho^* > 0.$$

Let  $\nu$  be a weak star limit of  $\{\nu_n\}_{n=1}^\infty$ . Then, from (27),  $\text{supp}(\nu) \subseteq \mathbf{C} \setminus \text{Ext}(\Omega_s)$ . But (22) implies that  $L_n(f_s; z) \rightarrow z - s$  for  $z \in \text{Int}(\Gamma_s)$ , and therefore  $L_n(f_s; z)$  has only finitely many zeros in each compact subset of  $\text{Int}(\Gamma_s)$ . Hence, we must have  $\text{supp}(\nu) \subseteq \Omega_s$ .

We now show that the sequence  $\{\nu_n\}_{n \in \Lambda}$  converges in the weak star topology to the measure  $b_s$ . Suppose that for some infinite sequence  $\Lambda_0 \subseteq \Lambda$ ,  $\nu_n \rightarrow \nu$  in the weak star topology as  $n \rightarrow \infty$  and  $n \in \Lambda_0$ . We claim that

$$\int_{\Omega_s} \log |z - t| d\nu(t) \leq U(z)$$

for all  $z \in \text{Ext}(\Omega_s)$ . Indeed, fix  $z \in \text{Ext}(\Omega_s)$ . Then, by (24), we have for  $n \geq n_z$

$$\int_{\mathbf{C}} \log |z - t| d\nu_n(t) - U(z) \leq \log \left\{ \frac{2K(n; s)}{|a_n|} \right\}^{1/n}$$

Let  $R > |z| + 1$ , so that

$$\int_{|t| \geq R} \log |z - t| d\nu_n(t) \geq 0.$$

Then (29) yields

$$\int_{|t| \leq R} \log |z - t| d\nu_n(t) \leq U(z) + \log \left\{ \frac{2K(n; s)}{|a_n|} \right\}^{1/n}$$

and so, on letting  $n \rightarrow \infty$ ,  $n \in \Lambda_0$ , we obtain

$$\limsup_{\substack{n \rightarrow \infty \\ n \in \Lambda_0}} \int_{|t| \leq R} \log |z - t| d\nu_n(t) \leq U(z)$$

By the lower envelope theorem (cf. [5]), we then get

$$\int_{|t| \leq R} \log |z - t| d\nu(t) \leq U(z),$$

for quasi-every  $z \in \text{Ext}(\Omega_s)$ , with  $R > |z| + 1$ . But since both potentials in (30) are continuous in  $\text{Ext}(\Omega_s)$ , (recall that  $\text{supp}(\nu) \subseteq \Omega_s$ ), this inequality holds for every  $z \in \text{Ext}(\Omega_s)$ ,  $R > |z| + 1$ . Letting  $R \rightarrow \infty$  gives

$$\int_{\mathbb{C}} \log |z - t| d\nu(t) \leq U(z),$$

which is equivalent to claim (28).

Since the difference of the two sides in (28) is a harmonic function in  $\text{Ext}(\Omega_s)$  even at  $\infty$  with value 0, the equality in (28) must hold for all  $z \in \text{Ext}(\Omega_s)$ . Thus  $\nu$  is a balayage of  $dt/2$  on  $[-1, 1]$  to  $\Omega_s$ . By the uniqueness of  $b_s$ ,  $\nu = b_s$ . Therefore, the sequence  $\{\nu_n\}_{n \in \Lambda}$  has only one weak star limit and so it converges in the weak star topology, and the limit measure is  $b_s$ .  $\square$

PROOF OF THEOREM 2. Equation (4) follows from (22). We now verify (3). Inequality (23) implies that

$$\limsup_{n \rightarrow \infty} |L_n(f_s; z)|^{1/n} \leq \lim_{n \rightarrow \infty} \{2K(n; s)\}^{1/n} e^{U(z)} e^{U(z) - U(s)}$$

for  $z \in \hat{\Gamma}_\rho$ ,  $\rho > 0$ . By the arbitrariness of  $\rho > 0$ , (31) holds for all  $z \in \text{Ext}(\Omega_s)$ . On the other hand, if  $\Lambda$  is chosen such that (2) holds, then  $\nu_n \rightarrow b_s$  as  $n \rightarrow \infty$  and  $n \in \Lambda$  by Theorem 1. So, for quasi-every  $z \in \text{Ext}(\Omega_s)$ , we have by the lower envelope theorem

$$\begin{aligned} \limsup_{\substack{n \rightarrow \infty \\ n \in \Lambda}} |L_n(f_s; z)|^{1/n} &= \limsup_{\substack{n \rightarrow \infty \\ n \in \Lambda}} \left\{ \exp \left( \int \log |z - t| d\nu_n(t) \right) |a_n|^{1/n} \right\} \\ &= e^{U(z) - U(s)} \end{aligned}$$

Therefore, equality holds in (31) quasi-everywhere in  $\text{Ext}(\Omega_s)$ , and so (3) is true.  $\square$

PROOF OF THEOREM 3. Equation (5) is valid for  $z \in \mathbf{R}$  by [6]. So we assume  $z \in \mathbf{C} \setminus \mathbf{R}$ . First, note that  $U(z) > U(0) = -1$  and  $\lim_{n \rightarrow \infty} |w_n(z)|^{1/n} = e^{U(z)}$  for  $z \in \mathbf{C} \setminus \mathbf{R}$ . Next, note that (5) is a consequence of the following:

$$\text{For } z \in \mathbf{C} \setminus \mathbf{R}, \lim_{n \rightarrow \infty} |L_n(|t|; z)|^{1/n} = e^{U(z)+1}$$

Hence, we need only show (32).

Let  $n' := [n/2]$ . Since  $L_n(|t|; x)$  is an even function, we can write  $L_n(|t|; x) = P_{n'}(x^2)$  for some  $P_{n'} \in \mathcal{P}_{n'}$ . It is easy to verify that  $P_{n'}$  is the polynomial of degree at most  $n'$  which interpolates  $\sqrt{x}$  at the points  $(0 \leq) t_{n'} < t_{n'-1} < \dots < t_1 < t_0 = 1$  with  $t_k = (x_k^{(n)})^2$ ,  $k = 0, 1, \dots, n'$ . Define  $w_n^*(x) := \prod_{k=0}^{n'} (x - t_k)$ . We now claim that

$$P_{n'}(z) - \sqrt{z} = \frac{w_n^*(z)}{\pi} \int_0^\infty \frac{\sqrt{t} dt}{w_n^*(-t)(t+z)}, \quad z \in \mathbf{C} \setminus (-\infty, 0]$$

We check (33) only for the case when  $n$  is even. The proof for the case when  $n$  is odd follows the same line and is simpler. When  $n$  is even,  $t_{n'} = 0$ . So, the point 0 is a point of interpolation and  $P_{n'}(z)/z$  is the polynomial of degree  $n' - 1$  that interpolates  $1/\sqrt{z}$  at points  $(0 <) t_{n'-1} < \dots < t_1 < t_0 = 1$ . Then, using the Hermite formula:

$$(34) \quad \frac{P_{n'}(z)}{z} - \frac{1}{\sqrt{z}} = \frac{w_n^*(z)}{2z\pi i} \int_\gamma \frac{d\zeta}{\sqrt{\zeta}(w_n^*(\zeta)/\zeta)(z-\zeta)}, \quad z \in \text{Int}(\gamma),$$

where  $\gamma$  is an arbitrary positively oriented contour in  $\mathbf{C} \setminus (-\infty, 0]$  that contains  $[t_{n'-1}, 1]$  in its interior. Let  $\gamma$  deform to the boundary of  $A(\varphi, r, R) := \{z : |\arg(z)| \leq \varphi \text{ and } r \leq |z| \leq R\}$  with  $0 < \varphi < \pi$  and  $0 < r < 1/n^2 < 1 < R$ . Now, let the inner radius  $r$  tend to 0 and the outer radius  $R$  tend to  $\infty$  and then let the angle  $\varphi$  tend to  $\pi$ . Then the integral in (34) converges to the integral in (33) multiplied by  $-2/i$ , from which our claim (33) follows.

In terms of  $L_n(|t|; z)$ , (33) yields

$$L_n(|t|; z) - z = \frac{w_n^*(z^2)}{\pi} \int_0^\infty \frac{\sqrt{t} dt}{w_n^*(-t)(t+z^2)}, \quad \text{Re}(z) > 0.$$

Define

$$S_n(z) := \frac{(-1)^{n'+1}}{\pi} \int_0^\infty \frac{\sqrt{t} dt}{w_n^*(-t)(t+z)} \quad z \in \mathbf{C} \setminus (-\infty, 0]$$

Then

$$S_n(z) = \int_0^\infty \frac{\psi_n(t) dt}{t+z}, \quad z \in \mathbf{C} \setminus (-\infty, 0],$$

where

$$\frac{(-1)^{n'+1}\sqrt{t}}{\pi w_n^*(-t)} \geq 0 \text{ for } t \geq 0.$$

$$\operatorname{Im}(z) \cdot \operatorname{Im}(S_n(z)) < 0 \text{ if } \operatorname{Im}(z) \neq 0$$

and

$$S_n(z) > 0 \text{ if } z > 0.$$

Now, we can define an analytic function  $H_n(z) := \operatorname{Log}(S_n(z)/S_n(1))$  for  $z \in \mathbf{C} \setminus (-\infty, 0]$ . Since  $|\operatorname{Im}(H_n(z))| = |\operatorname{Arg}(S_n(z)/S_n(1))| < \pi$ , then we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\operatorname{Im}(H_n(z))| = 0.$$

locally uniformly for  $z \in \mathbf{C} \setminus (-\infty, 0]$ . By Schwarz's integral formula,  $H_n(z)$  can be expressed in terms of  $\operatorname{Im}(H_n(z))$  and  $\operatorname{Re}(H_n(z_0))$  in any disk with center  $z_0$  contained in  $\mathbf{C} \setminus (-\infty, 0]$ . In particular, from (36) we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \operatorname{Re}(H_n(z)) = 0,$$

uniformly for  $|z-1| \leq \rho$  ( $\rho < 1$ ). Then, by using a chain of circles we can extend (37) to all points contained in  $\mathbf{C} \setminus (-\infty, 0]$ . It then follows that

$$\lim_{n \rightarrow \infty} \left( \frac{1}{n} \log |S_n(z)| - \frac{1}{n} \log S_n(1) \right) = 0$$

locally uniformly for  $z \in \mathbb{C} \setminus (-\infty, 0]$ . Using (1) with  $x = 1$  in (35), we get

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log S_n(1) = 1,$$

and so (38) implies that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log |S_n(z)| = 1$$

locally uniformly for  $z \in \mathbb{C} \setminus (-\infty, 0]$ . This, together with (35), gives us

$$\lim_{n \rightarrow \infty} \left| \frac{L_n(|t|; z) - z}{w_n(z)} \right|^{1/n} = e$$

locally uniformly for  $\operatorname{Re}(z) > 0$ . Similarly,

$$\lim_{n \rightarrow \infty} \left| \frac{L_n(|t|; z) + z}{w_n(z)} \right|^{1/n} = e$$

locally uniformly for  $\operatorname{Re}(z) < 0$ . Now, from (39) and (40), we see that (32) holds if, in addition, we assume  $\operatorname{Re}(z) \neq 0$ .

Finally, we verify that (32) holds when  $\operatorname{Re}(z) = 0$ . The proof for this case turns out to be very lengthy. We will give here only a sketch of the proof and leave the details to the reader. Assume  $z = bi$  for some real number  $b \neq 0$ . It is easy to see that

$$w'_n(x_k^{(n)}) = \left(\frac{2}{n}\right)^n (-1)^{n-k} k!(n-k)!$$

So, using Lagrange's formula, we have

$$L_n(|t|; bi) = \left(\frac{n}{2}\right)^n \frac{(-1)^n}{n!} w_n(bi) \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{|x_k^{(n)}|}{bi - x_k^{(n)}}$$

The summation in (41) (let's call it  $S_n$ ) can be written as

$$\sum_{k=0}^{n'} (-1)^k \binom{n}{k} \frac{-2bi|x_k^{(n)}|}{b^2 + (x_k^{(n)})^2} \text{ for even } n,$$

and

$$\sum_{k=0}^{n'} (-1)^k \binom{n}{k} \frac{2(x_k^{(n)})^2}{b^2 + (x_k^{(n)})^2} \text{ for odd } n.$$

For odd  $n$ , we apply the residue theorem to write  $S_n$  as

$$S_n = \frac{1}{2\pi i} \int_{C_{\delta, M}} \frac{(-1)^{n'} 2\Gamma(n+1)}{\Gamma(\frac{n+1}{2} + 1 + z)\Gamma(\frac{n+1}{2} - z)} \frac{(\frac{1+2z}{n})^2}{b^2 + (\frac{1+2z}{n})^2} \frac{\pi dz}{\sin \pi z}$$

where  $C_{\delta, M}$  denotes the rectangle formed by lines  $\operatorname{Re}(z) = -\delta/2$  ( $0 < \delta < 1$ ),  $\operatorname{Re}(z) = n/2$ , and  $\operatorname{Im}(z) = \pm M$  ( $M > 0$ ). (This integral representation of  $S_n$  can be verified by noting that the integrand is analytic in  $C_{\delta, M}$  except at  $z = 0, 1, \dots, (n-1)/2$  where it has simple poles and the residue at  $z = (n-1)/2 - k$  is the  $k$ th term in the summation form of  $S_n$ .) Let  $\Omega(z)$  denote the integrand. It can be verified that

- (i) for fixed  $n$ , the integral along lines  $\operatorname{Im}(z) = \pm M$  tends to 0 as  $M \rightarrow \infty$ ,
- (ii) the integral along  $\operatorname{Re}(z) = n/2$  tends to 0 as  $n \rightarrow \infty$ ,
- (iii) the absolute value of the integral along  $\operatorname{Re}(z) = -\delta/2$  is greater than  $c|\Omega(-\delta/2)|$  for some positive constant  $c$  independent of  $n$ .

Indeed, (i) follows from the (crude) estimate  $\Omega(z) = O(|z|^{-2})$  ( $n$  fixed), while (ii) is proved by showing  $\Omega(z) = O(n^{-1/2}(1/4 + t^2)^{-1})$  ( $n \rightarrow \infty$ ) with  $z = n/2 + it$  ( $t$  real). Assertion (iii) is verified by the saddle point method (cf. [1]). Note that the integrand  $\Omega(z)$  can be written as

$$\Omega(z) = \frac{2n!(1+2z)^2}{(z + \frac{n+1}{2})(z + \frac{n+1}{2} - 1) \cdots (z + \frac{n+1}{2} - n)(n^2 b^2 + (1+2z)^2)}$$

$n$  is even can be handled similarly, and (32) holds. This completes our proof.  $\square$

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### INDIRIZZO DEGLI AUTORI:

Xin Li – Department of Mathematics – University of Central Florida – Orlando, FL 32816

E.B. Saff – Department of Mathematics – University of South Florida – Tampa, FL 33620