Behavior of Lagrange interpolants to the absolute value function in equally spaced points

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Dedicated to Aldo Ghizzetti

1 – Introduction and statements of main results

For a function $f$ defined on $[-1,1]$, let $L_n(f)$ denote the Lagrange

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interpolating polynomial of degree at most \(n\) to \(f\) at the equidistant nodes
\[
x_k^{(n)} = -1 + 2k/n, \quad k = 0, 1, \ldots, n.
\]

Bernstein proved that (cf. [8]) for \(f(x) = |x|\), the sequence \(L_n(|t|; x)\) diverges if \(0 < |x| < 1\). Recently, Byrne, Mills and Smith [9] considered the rate of this divergent sequence. They proved, if \(0 < |x| < 1\), then
\[
\limsup_{n \to \infty} \left| L_n(|t|; x) - |x| \right|^{1/n} = (1 + x)^{(1+x)/2}(1 - x)^{(1-x)/2}.
\]

Li and Mohapatra [6] further improved this result by showing that
\[
\lim_{n \to \infty} \left| L_n(|t|; x) - |x| \right|^{1/n} = e
\]
for all \(x \in \mathbb{R}\) (the set of real numbers), where \(w_n(x) := \prod_{k=0}^{n} (x - x_k^{(n)})\).

In contrast to the above results, under the assumption that \(f\) is bounded on \([-1, 1]\) and analytic at \(x = 0\), the authors proved in [7] that the sequence \(L_n(f; x)\) converges to \(f\) geometrically in a neighborhood (in the complex plane) of \(x = 0\). This leads to the question of finding the exact region where \(L_n(f; \cdot)\) converges to \(f\). Although the answer in the general situation is still unknown, we try in this note to gain some insight by considering the special but interesting case when
\[
f(x) = f_s(x) := |x - s| \quad (-1 < s < 1).
\]

We determine the exact region in which \(L_n(f_s; \cdot)\) converges to (an analytic continuation of) \(f_s\) geometrically. This is done by studying the zero distribution of \(L_n(f_s; \cdot)\), which is equivalent to the \(n\)th root asymptotics of \(L_n(f_s; z)\) in \(\mathbb{C}\). Furthermore, we will show that (1) has an extension to all \(z\) in the complex plane \(\mathbb{C}\).

To state our results, we first introduce some notation. The potential corresponding to the uniform distribution \(\frac{1}{2} dt\) on \([-1, 1]\) is given by
\[
U(z) = \frac{1}{2} \int_{-1}^{1} \log |z - t| \, dt.
\]
The level curves of $U(z)$ are denoted by $\Gamma_s := \{ z \in \mathbb{C} \mid U(z) = U(s) \}$ for $s \in \mathbb{R}$. Let

$$\Omega_s := [-1, -s] \cup \Gamma_s \cup [s, 1] \text{ for } |s| < 1.$$ 

Let $\nu_n(t)$ be the normalized counting measure of the zeros of $L_n(f_s; \cdot)$, i.e.,

$$\frac{1}{n} \int_B d\nu_n(t) = \frac{\text{the number of the zeros of } L_n(f_s; z) \text{ in } B}{n},$$

for every Borel set $B \subseteq \mathbb{C}$. For a compact set $S \subseteq \mathbb{C}$, we will use $\text{Ext}(S)$ and $\text{Int}(S)$ to denote the unbounded and the (union of) bounded components of $\mathbb{C} \setminus S$, respectively, where $\mathbb{C} = \mathbb{C} \cup \{\infty\}$. We need one more concept from potential theory. A measure $b_s$ supported on $\Omega_s$ is called a balayage of the uniform distribution $\frac{1}{2}dt$ on $[-1, 1]$ to $\Omega_s$ if

$$\int_{\Omega_s} \log |z-t|db_s(t) = U(z) \text{ for all } z \in \text{Ext}(\Omega_s).$$

On using the fact that $\Gamma_s$ is regular with respect to the Dirichlet problem for $\text{Int}(\Gamma_s)$, one can show that at least one such balayage $b_s$ exists (cf. [5, §4.2]), and since $U(z)$ is continuous in $\mathbb{C}$, such a measure $b_s$ must be unique ([5, Theorem 4.6, Corollary 2]).

We now state our results. Their proofs are given in Section 3.

**Theorem 1.** The sequence of the normalized counting measures $\{\nu_n\}$ of the zeros of $L_n(f_s; \cdot)$ converges, in the weak star topology, to the balayage $b_s$ of $\frac{1}{2}dt$ on $[-1, 1]$ to $\Omega_s$, as $n \to \infty$ through a subsequence $\Lambda$ of positive integers.

**Remark 1.** Our proof shows that, in Theorem 1, $\Lambda$ can be any sequence for which

$$\lim_{n \to \infty} |a_n|^{1/n} = e^{-U(s)},$$

where $a_n$ denotes the leading coefficient of $L_n(f_s; \cdot)$.

**Remark 2.** It can be shown (by using Khinchine's theorem [10] in Lemma 3 below) that for almost all $s \in (-1, 1)$,

$$\lim_{n \to \infty} |a_n|^{1/n} = e^{-U(s)}.$$
So, by Remark 1, the whole sequence \( \{v_n\}_{n=1}^\infty \) converges to \( b_s \) for almost all \( s \).

**Theorem 2**  
For \( s \in (-1, 1) \), we have

\[
\limsup_{n \to \infty} |L_n(f_s; z)|^{1/n} = e^{U(z) - U(s)}
\]

quasi-everywhere in \( \text{Ext}(\Omega_s) \) and

\[
\lim_{n \to \infty} L_n(f_s; z) = (s - z) \text{sgn}(s)
\]

geometrically for every \( z \in \text{Int}(\Gamma_s) \).

Here we use “quasi-everywhere” to mean that the property holds except on a set having logarithmic capacity zero.

**Remark 3.** Note that \( U(z) > U(s) \) for \( z \in \text{Ext}(\Omega_s) \), so (3) implies that for quasi-every \( z \in \text{Ext}(\Omega_s) \) a subsequence of \( L_n(f_s; z) \) tends to \( \infty \) geometrically. It is also possible to show that (3) holds for almost all \( z \in \text{Ext}(\Gamma_s) \setminus \{-1, 1\} \) and \( s \in (-1, 1) \).

**Remark 4.** Using Remarks 1 and 2, it can be shown that for almost all \( s \), \( \limsup \) can be replaced by \( \lim \) in (3).

**Remark 5.** The relation (4) also follows from the general theorem proved in [7].

**Theorem 3**  
For all \( z \in \mathbb{C} \), there holds

\[
\lim_{n \to \infty} \left| L_n(f_s; z) - |z| \right|^{1/n} = e.
\]

Readers familiar with the subject of asymptotic zero distributions of best polynomial approximants (cf. [2], [3], [11]) will recognize that the above results and their proofs have a flavor similar to those for the best polynomial approximants. However, our proofs are a bit more involved because in the present situation, unlike the case for best polynomial approximations, the limit measure \( b_s \) is not the equilibrium measure on its support.
2 - Lemmas

Define

\[ \phi_s(x) := \begin{cases} 
  x - s, & s \leq x \leq 1, \\
  0, & -1 \leq x \leq s.
\end{cases} \]

Then, if \( x < s \)

\[ \frac{L_n(f_s;x) - |x-s|}{2} = L_n(\phi_s;x) \]

By Newton's formula (cf. [8, p.14]),

\[ L_n(\phi_s;x) = \sum_{k=0}^{n} \Delta_n^k \phi_s(-1) \frac{(n)}{k!} (x + 1)^k (x + 1 - 2(k-1)) \]

where

\[ \Delta_n^k \phi_s(-1) = \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} \phi_s(-1 + \frac{2r}{n}), \quad k = 0, 1, \ldots, n. \]

Set \( k(s) := \max\{k : x_k^n \leq s\} \). Then \( k(s) : [n(s+1)/2] \) and \( x_k^n \) is the closest node to the left of \( s \) (or equal to \( s \)).

**Lemma 1.** (i) If \( 0 \leq k \leq k(s) \), then \( \Delta_n^k \phi_s(-1) = 0 \).

(ii) If \( k(s) + 1 \leq k \leq n \), then

\[ \Delta_n^k \phi_s(-1) = \frac{(-1)^{k-k(s)}(k-2)!}{(k-k(s)-1)!k(s)!} \frac{2}{n} \left( \frac{n(s+1)}{2} - \left[ \frac{n(s+1)}{2} \right] k \right) \]

**Proof.** Assertion (i) is obvious. To prove (ii), we need the following two formulae [4]:

\[ \sum_{k=0}^{m} (-1)^{k} \binom{n}{k} = (-1)^m \binom{n-1}{m} \quad (n \geq 1) \tag{6} \]

and

\[ \sum_{k=0}^{m} (-1)^{k} \binom{n}{k} k = (-1)^m n \binom{n-2}{m-1} \quad (n \geq 2) \tag{7} \]
Now

\[ \Delta_n^k \phi_s(-1) = \sum_{r=k(s)+1}^{k} -1^{k-r} \binom{k}{r} \phi_s(-1 + 2r/n) \]

\[ \sum_{r=k(s)+1}^{k} (-1)^{k-r} \binom{k}{r} (-s - 2r/n) = \]

\[ \sum_{i=0}^{k-k(s)-1} (-1)^i \binom{k}{i} (-s - 2(k-l)/n) = \]

\[ = (-s - 1 + 2k/n) \sum_{i=0}^{k-k(s)-1} (-1)^i \binom{k}{i} - 2n \sum_{i=0}^{k-k(s)-1} (-1)^i \binom{k}{i} = \]

\[ = (-s - 1 + 2k/n)(-1)^{k-k(s)-1} \binom{k-1}{k-k(s)-1} + \]

\[ - \frac{2}{n} (-1)^{k-k(s)-1} k \binom{k-2}{k-k(s)-2} = \]

\[ = \frac{(-1)^{k-k(s)-1} (k-2)!}{(k-k(s)-1)! k(s)!} \left\{ (-s - 1 + 2k/n)(k-1) - \frac{2}{n} k(k-k(s)-1) \right\} = \]

\[ = \frac{(-1)^{k-k(s)} (k-2)!}{(k-k(s)-1)! k(s)!} \binom{n(s+1)}{2} (k-1) - \left[ \frac{n(s+1)}{2} \right] k \}

This concludes the proof of Lemma 1

\[ d_k^{(n)}(s) := \frac{n(s+1)}{2} (k-1) - \left[ \frac{n(s+1)}{2} \right] k \]

for \( k = k(s) + 1, \), then we have the following simple lemma.
LEMMA 2. For $s \in (-1, 1)$ and $n \geq 2$, the coefficient of $x^n$ in $L_n(f_s; x)$ is

$$a_n := \frac{2(-1)^{n-k(s)}}{(n-1)n!} \binom{n}{2} \binom{n-1}{k(s)} d_n^{(n)}(s)$$

PROOF. Note that

$$a_n = 2 \times \text{(the coefficient of } x^n \text{ in } L_n(\phi_s; x))$$

when $n \geq 2$, and

the coefficient of $x^n$ in $L_n(\phi_s; x)$

$$\Delta_n \phi_s(-1) \binom{n}{2}$$

We can now apply Lemma 1 (ii) to establish this lemma.

LEMMA 3. For $s \in (-1, 1)$, we have

$$\limsup_{n \to \infty} \left| d_n^{(n)}(s) \right|^{1/n} = 1$$

PROOF. Since

$$|d_n^{(n)}(s)| = \left| \frac{n(s+1)}{2} (n-1) - \left[ \frac{n(s+1)}{2} \right] n \right| \leq n^2 |s+1| \leq 2n^2$$

we have

$$\limsup_{n \to \infty} \left| d_n^{(n)}(s) \right|^{1/n} \leq 1$$

Write

$$d_n^{(n)}(s) = n(n-1) \left( \frac{s+1}{2} - \frac{1}{n} \left[ \frac{n(s+1)}{2} \right] \right) = n(n-1)I_n(s)$$

Then, to prove the lemma, it suffices to show

(8) $$\limsup_{n \to \infty} \left| I_n(s) \right|^{1/n} \geq 1$$
for all \( s \in (-1, 1) \). Assume, to the contrary, (8) is not true for some \( s \in (-1, 1) \). Then, there exist \( r \in (0, 1) \) and \( N > 0 \) such that

(9) \[ |I_n(s)| < r^n \]

for all \( n \geq N \). Consequently,

(10) \[ |I_n(s) - I_{n+1}(s)| < 2r^n \]

for \( n \geq N \). But, with \( t = (s+1)/2 \in (0, 1) \),

\[
n(n-1)|I_n(s) - I_{n+1}(s)| = n(n-1) \frac{[(n+1)t] - [nt]n - 1}{n} = |(n-1)((n+1)t) - n[nt]|\]

If there are infinitely many \( n \) such that

\[
(n-1)((n+1)t) - n[nt] \neq 0,
\]

then for those \( n \), \( n(n-1)|I_n(s) - I_{n+1}(s)| \geq 1 \). So \( \limsup_{n \to \infty} |I_n(s) - I_{n+1}(s)|^{1/n} \geq 1 \), contradicting (10). Hence there are only finitely many \( n \) such that (11) holds. Therefore, there is a constant \( M > 0 \) such that \( (n-1)((n+1)t) = n[nt] \) for all \( n \geq M \). Then \( I_n(s) = I_{n+1}(s) \) for \( n \geq M \). This, together with (9), tells us that \( I_n(s) = 0 \) for \( n \geq M \). That is,

\[
t = \frac{[nt]}{1}
\]

for all \( n \geq M \), which implies that \( (n-1)t \) is an integer for every \( n \geq M \). This can happen only if \( t = 0 \) or \( t = 1 \), which is impossible. This completes the proof. \( \Box \)

**Lemma 4.** Let \( s \in (-1, 1) \). Then

\[
\limsup_{n \to \infty} |a_n|^{1/n} e^{-U(s)}
\]
PROOF. From Lemmas 2 and 3 and Stirling's formula we obtain
\[
\lim_{n \to \infty} \sup |a_n|^{1/n} = \frac{e}{(1 + s)^{(1+s)/2}(1 - s)^{(1-s)/2}} = e^{-U(s)}
\]

**LEMMA 5** Let \( x \leq s \) Then,

\[
|L_n(\phi_s; x)| \leq \frac{2n^2}{n!} \left( \frac{n}{2} \right) \left( \frac{n}{k(s) + 1} \right) |w_n(x)| =: c(n; s) w_n(x),
\]

\[
\lim_{n \to \infty} c(n; s)^{1/n} = e^{-U(s)}
\]

**PROOF.** Write \( d_k \) for \( d^{(n)}_k(s) \), \( k = k(s) + 1 \), \( n \). Then, using Lemma 1, we have

\[
L_n(\phi_s; x)
\]

\[
= \sum_{k=k(s)+1}^{n} \frac{(-1)^{k-k(s)(k-2)!}}{k!(k-k(s)-1)!k(s)!} \frac{2}{n} d_k \left( \frac{n}{2} \right) (x + 1) \cdot (x + 1 - \frac{2(k - 1)}{n}) =
\]

\[
= \sum_{k=k(s)+1}^{n} \frac{(-1)^{k-k(s)} w_n(x) \left( \frac{n}{2} \right)^{k-1} d_k}{k(k-1)(k-k(s)-1)!k(s)!} (x+1-\frac{2k}{n}) \cdots (x+1-\frac{2(k-1)}{n})(x-1)
\]

Now, note that

\[
|x + 1 - \frac{2(k(s) + 1)}{n}| \geq |s + 1 - \frac{2(k(s) + 1)}{n}| = \frac{2|d_k(s)+1|}{nk(s)}
\]

\[
|x + 1 - \frac{2k}{n}| \geq -1 + \frac{2k}{n} \left( -1 + \frac{2(k(s) + 1)}{n} \right) = \frac{2}{n}(k - k(s) - 1)
\]
for \( k = k(s) + 2 \), \( n \\

|L_n(\phi; x)| \\
\leq \frac{|w_n(x)|}{n!} \left( \frac{n}{k(s)} \right)^n + \\
+ \sum_{k=k(s)+2}^{n} \frac{|w_n(x)|}{k(k-1)(k-k(s)-1)!k(s)! \frac{n}{k} (k-k(s)-1) \cdots \frac{n}{k} (n-k(s)-1)} \\
\leq \frac{|w_n(x)|}{n!} \left( \frac{n}{k(s)+1} \right)^n + \\
+ \sum_{k=k(s)+2}^{n} \frac{|w_n(x)|}{(k(s)+1)(n-k(s)-1)!k(s)!} \frac{2n}{(k(s)+1)(n-k(s)-1)!k(s)!} \\
\leq 2n^2 \left( \frac{n}{2} \right) \left( \frac{n}{k(s)+1} \right) |w_n(x)| \\

Equation (13) follows directly from an application of Stirling's formula. This completes the proof of Lemma 5. \( \square \)

**LEMMA 6** For \( z \in [-1, 1] \), there holds

\[ |w_n(z)| e^{-nU(z)} \leq 3n^2 \]

**PROOF.** To simplify the notation, we write \( x_k \) for \( x_k^{(n)} \), \( k = 0, 1, \ldots, n \)

Using the monotonicity of \( \log x \) on \((0, \infty)\), we have for \( k = 1, 2, \ldots, k(z) \),

\[ \frac{1}{n} \log(z - x_k) \leq \frac{1}{2} \int_{x_k-1}^{x_k} \log |z - t| dt; \]
and for \( k = k(z) + n - 1 \),

\[
\frac{1}{n} \log(x_k - z) \leq \frac{1}{2} \int_{x_k}^{x_{k+1}} \log |z - t| dt
\]

Summing the above inequalities, we get

\[
\frac{1}{n} \sum_{k=1}^{n-1} \log |z - x_k| \leq \frac{1}{2} \left( \int_{-1}^{x_k(z)} + \int_{-1}^{x_{k+1}(z)} \right) \log |z - t| dt,
\]

or, equivalently,

\[
15) \quad \frac{1}{n} \log |w_n(z)| - \frac{1}{n} \log(1 - x^2) \leq U(z) - \frac{1}{2} \int_{x_k(z)}^{x_{k+1}(z)} \log |z - t| dt
\]

Now

\[
\int_{x_k(z)}^{x_{k+1}(z)} \log |z - t| dt = g(z - x_k(z)) + g(x_{k+1}(z) - z),
\]

where \( g(u) := u \log u - u \). Note that \( g'(u) = \log u < 0 \) for \( u \in (0, 1) \), so \( g \) is decreasing and \( g(u) < g(0+) = 0 \) on the interval \((0, 1)\). Since

\[
x_{k+1}(z) - z \in \left(0, \frac{2}{n}\right),
\]

it then follows that

\[
\left| \int_{x_k(z)}^{x_{k+1}(z)} \log |z - t| dt \right| \leq 2 |g\left(\frac{2}{n}\right)| = \frac{4}{n} (1 + \log \frac{n}{2})
\]

Using this together with (15), we obtain

\[
\frac{1}{n} \log |w_n(z)| \leq U(z) + \frac{2}{n} (1 + \log \frac{n}{2}),
\]

which implies (14) \( \square \)
3 – Proofs of Theorems

We are now ready to prove the theorems of Section 1.

PROOF OF THEOREM 1. Let \( s \in (-1,1) \). If \( x \leq s \), then, from Lemma 5, we have |\( L_n(\phi_s; x) | \leq c(n; s)|w_n(x)| \), and so

\[
|L_n(f_s; x) - |x - s|| = 2|L_n(\phi_s; x)| \leq 2c(n; s)|w_n(x)|
\]

If \( x \geq s \), then, since

\[
L_n(|t - s|; x) = L_n(|-t + s|; x) = L_n(|t - (-s)|; -x)
\]

it follows from (16) and the fact that \(|w_n(-x)| = |w_n(x)|\)

\[
|L_n(f_s; x) - |x - s|| = |L_n(|t - (-s)|| - x) - | - x - (-s)|| \leq 2c(n; -s)|w_n(x)|
\]

Hence, with \( \hat{c}(n; s) := \max\{c(n; s), c(n; -s)\}\)

\[
|L_n(f_s; x) - |x - s|| \leq 2\hat{c}(n; s)|w_n(x)|
\]

for all \( x \in [-1,1] \). Note that, from (13),

\[
\lim_{n \to \infty} \hat{c}(n; s)^{1/n} = e^{-U(s)}.
\]

Next we need to estimate \( L_n(f_s; z) \) for \( z \in C \). Let us first estimate \( L_n(f_s; z) - |z - s| \) by extending (17) to \( C \). For definiteness, we assume \( s > 0 \); the case when \( s < 0 \) can be handled similarly. Define for \( n \geq 2 \)

\[
p(z) := \log |L_n(f_s; z) - (s - z)| - nU(z)
\]

Then \( p(z) \) is a subharmonic function in \( C \setminus [-1,1] \) with \( p(\infty) = \log |a_n| \)

Using the maximum principle, we have

\[
p(z) \leq \max_{z \in [-1,1]} p(z) = \max_{z \in [-1,s]} \{ \max_{z \in [-1,s]} p(z), \max_{z \in [s,1]} p(z) \}, \ z \in C
\]

By (17),

\[
\max_{z \in [-1,s]} p(z) = \max_{z \in [-1,s]} \left\{ \log \frac{|L_n(f_s; z) - |z - s||}{w_n(z)} + \log |w_n(z)| - nU(z) \right\} \leq \max_{z \in [-1,s]} \log \left\{ 2\hat{c}(n; s)|w_n(z)|e^{-nU(z)} \right\} \leq \log \{ 6\hat{c}(n; s)n^2 \}.
\]
where in the last inequality we used (14). On the other hand, for \( z \in [s, 1 \] \\
\begin{align*}
e^{|p(z)|} &= |L_n(f_s; z) - (z - s)| e^{-nU(z)} \\
&\leq |L_n(f_s; z) - (z - s)| e^{-nU(z)} + 2|z - s| e^{-nU(z)} \\
&\leq 2\hat{c}(n; s)|w_n(z)| e^{-nU(z)} + 4e^{-nU(z)} \leq 6\hat{c}(n; s)n^2 + 4e^{-nU(z)}
\end{align*}

where in the last inequality we used (14) again. So,

\[
\max_{z \in [s, 1]} p(z) \leq \log\{6\hat{c}(n; s)n^2 + 4e^{-nU(s)}\}
\]

for \( z \in \mathbb{C} \),

\[
p(z) \leq \max \left\{ \log (6\hat{c}(n; s)n^2), \log \left( 6\hat{c}(n; s)n^2 + 4e^{-nU(s)} \right) \right\}
\]

\[
= \log \{6\hat{c}(n; s)n^2 + 4e^{-nU(s)}\} =: \log K(n; s);
\]

and, by using (18), it is easy to verify that

\[
\lim_{n \to \infty} K(n; s)^{1/n} = e^{-U(s)}
\]

An important consequence of (20) and (21) is the following. For \( s > 0 \),

\[
\limsup_{n \to \infty} |L_n(f_s; z) - (s - z)|^{1/n} \leq \lim_{n \to \infty} K(n; s)^{1/n} e^{U(s)}
\]

\[
= e^{U(z)-U(s)} < 1
\]

for all \( z \in \text{Int}(\Gamma_s) \).

Now, we are ready to estimate \( L_n(f_s; z) \). Let \( G_s(t) \) be the Green's function for \( \text{Ext}(\Omega_s) \) with pole at \( \infty \). Set \( \hat{\Gamma}_\rho := \{ z \in \mathbb{C} : G_s(z) = \rho \} \), \( \rho > 0 \). Since

\[
\lim_{n \to \infty} \{ K(n; s)e^{nU(z)} \}^{1/n} = e^{U(z)-U(s)} > 1
\]

uniformly for \( z \in \hat{\Gamma}_\rho \), we have, for \( n \) sufficiently large and \( z \in \hat{\Gamma}_\rho \), \( |z - s| \leq K(n; s)e^{nU(z)} \). Thus, using (20) we obtain, for \( n \) large and \( z \in \hat{\Gamma}_\rho \),

\[
|L_n(f_s; z)| \leq |z - s| + K(n; s)e^{nU(z)} \leq 2K(n; s)e^{nU(z)}
\]
Next, define

$$P(z) = \log |L_n(f_s; z)| - nU(z).$$

Then $P(z)$ is subharmonic in $\mathbb{C} \setminus [-1, 1]$ with $P(\infty) = \log |a_n|.$ From (23), for each $\rho > 0,$ there is a constant $N(\rho) > 0$ such that when $n \geq N(\rho),$

$$P(z) \leq \log \{2K(n; s)\}, \quad \text{for } z \in \Gamma^\rho.$$

Fix $\rho^* > 0,$ and let $I_{\rho^*}$ denote the set of all the zeros of $L_n(f_s; z)$ that lie in $\text{Ext}(\Gamma^\rho).$ Choose $\rho \in (0, \rho^*).$ Let $G(z; \zeta)$ be the Green’s function for $\text{Ext}(\Gamma^\rho)$ with pole at $\zeta.$ Then $G(z; \infty) \equiv G_s(z) - \rho.$ Define

$$h(z) := P(z) + \sum_{\zeta \in I_{\rho^*}} G(z; \zeta)$$

The function $h(z)$ is subharmonic in $\text{Ext}(\Gamma^\rho),$ and by (24)

$$\limsup_{z \to \infty} h(z) = P(\infty) \leq \log \{2K(n; s)\}.$$

Hence, the maximum principle for subharmonic functions gives

$$h(z) \leq \log \{2K(n; s)\}$$

for all $z \in \text{Ext}(\Gamma^\rho).$ Note that $G(\infty; \zeta) = G(\zeta; \infty) \geq \rho^* - \rho$ for $\zeta \in \text{Ext}(\Gamma^\rho).$ Thus

$$h(\infty) = \log |a_n| + \sum_{\zeta \in I_{\rho^*}} G(\infty; \zeta) \geq \log |a_n| + n\nu_n\{\text{Ext}(\Gamma^\rho^*)\}(\rho^* - \rho).$$

where $\nu_n$ is the normalized counting measure of the zeros of $L_n(f_s; z).$ It then follows from (25) that

$$(\rho^* - \rho)n\nu_n\{\text{Ext}(\Gamma^\rho^*)\} \leq \log \frac{2K(n; s)}{|a_n|}.$$

Now, from Lemma 4, we can find an infinite subsequence of positive integers, say $\Lambda,$ such that

$$\lim_{n \to \infty, n \in \Lambda} |a_n|^{1/n} = e^{-U(s)}.$$
Thus, (26) and (21) imply that

\[
\limsup_{n \to \infty} \nu_n\{ \Ext(\Gamma_{\rho'}) \} \leq (\rho^* - \rho)^{-1} \lim_{n \to \infty} \left( \frac{2K(n; s)}{|a_n|} \right)^{1/n} = 0,
\]

and so

\[
\lim_{n \to \infty} \nu_n\{ \Ext(\Gamma_{\rho'}) \} = 0 \text{ for every } \rho' > 0.
\]

Let \( \nu \) be a weak star limit of \( \{\nu_n\}_{n=1}^{\infty} \). Then, from (27), \( \text{supp}(\nu) \subseteq C \setminus \Ext(\Omega_s) \). But (22) implies that \( \Ln(f_s; z) \to z - s \) for \( z \in \Int(\Gamma_s) \), and therefore \( \Ln(f_s; z) \) has only finitely many zeros in each compact subset of \( \Int(\Gamma_s) \). Hence, we must have \( \text{supp}(\nu) \subseteq \Omega_s \).

We now show that the sequence \( \{\nu_n\}_{n \in \Lambda} \) converges in the weak star topology to the measure \( b_s \). Suppose that for some infinite sequence \( \Lambda_0 \subseteq \Lambda \), \( \nu_n \to \nu \) in the weak star topology as \( n \to \infty \) and \( n \in \Lambda_0 \). We claim that

\[
\int_{\Omega_s} \log |z - t| d\nu(t) \leq U(z)
\]

for all \( z \in \Ext(\Omega_s) \). Indeed, fix \( z \in \Ext(\Omega_s) \). Then, by (24), we have for \( n \geq n_z \)

\[
\int_{\Omega_s} \log |z - t| d\nu_n(t) - U(z) \leq \log \left\{ \frac{2K(n; s)}{|a_n|} \right\}^{1/n}
\]

Let \( R > |z| + 1 \), so that

\[
\int_{|t| \geq R} \log |z - t| d\nu_n(t) \geq 0.
\]

Then (29) yields

\[
\int_{|t| \leq R} \log |z - t| d\nu_n(t) \leq U(z) + \log \left\{ \frac{2K(n; s)}{|a_n|} \right\}^{1/n}
\]
and so, on letting \( n \to \infty, \ n \in \Lambda_0 \), we obtain

\[
\limsup_{n \to \infty} \int_{|t| \leq R} \log |z - t| \, d\nu_n(t) \leq U(z)
\]

By the lower envelope theorem (cf. [5]), we then get

\[
\int_{|t| \leq R} \log |z - t| \, d\nu(t) \leq U(z),
\]

for quasi-every \( z \in \text{Ext}(\Omega_s) \), with \( R > |z| + 1 \). But since both potentials in (30) are continuous in \( \text{Ext}(\Omega_s) \), (recall that \( \text{supp}(\nu) \subseteq \Omega_s \)), this inequality holds for every \( z \in \text{Ext}(\Omega_s), \ R > |z| + 1 \). Letting \( R \to \infty \) gives

\[
\int_{C} \log |z - t| \, d\nu(t) \leq U(z),
\]

which is equivalent to claim (28).

Since the difference of the two sides in (28) is a harmonic function in \( \text{Ext}(\Omega_s) \) even at \( \infty \) with value 0, the equality in (28) must hold for all \( z \in \text{Ext}(\Omega_s) \). Thus \( \nu \) is a balayage of \( dt/2 \) on \([-1, 1]\) to \( \Omega_s \). By the uniqueness of \( b_s \), \( \nu = b_s \). Therefore, the sequence \( \{\nu_n\}_{n \in \Lambda} \) has only one weak star limit and so it converges in the weak star topology, and the limit measure is \( b_s \).

PROOF OF THEOREM 2. Equation (4) follows from (22). We now verify (3). Inequality (23) implies that

\[
\limsup_{n \to \infty} \left| L_n(f_s; z) \right|^{1/n} \leq \lim_{n \to \infty} \left\{ 2K(n; s) \right\}^{1/n} e^{U(z)} \ e^{U(s) - U(s)}
\]

for \( z \in \hat{\Omega}_\rho, \ \rho > 0 \). By the arbitrariness of \( \rho > 0 \), (31) holds for all \( z \in \text{Ext}(\Omega_s) \). On the other hand, if \( \Lambda \) is chosen such that (2) holds, then \( \nu_n \to b_s \) as \( n \to \infty \) and \( n \in \Lambda \) by Theorem 1. So, for quasi-every \( z \in \text{Ext}(\Omega_s) \), we have by the lower envelope theorem

\[
\limsup_{n \to \infty} \left| L_n(f_s; z) \right|^{1/n} = \limsup_{n \to \infty} \left\{ \exp \left( \int \log |z - t| \, d\nu_n(t) \right) \right\}^{1/n} \leq e^{U(z) - U(s)}
\]
Therefore, equality holds in (31) quasi-everywhere in \( \text{Ext}(\Omega_a) \), and so (3) is true.

**Proof of Theorem 3.** Equation (5) is valid for \( z \in \mathbb{R} \) by [6]. So we assume \( z \in \mathbb{C} \setminus \mathbb{R} \). First, note that \( U(z) > U(0) = -1 \) and \( \lim_{n \to \infty} |w_n(z)|^{1/n} = e^{U(z)} \) for \( z \in \mathbb{C} \setminus \mathbb{R} \). Next, note that (5) is a consequence of the following:

For \( z \in \mathbb{C} \setminus \mathbb{R} \),
\[
\lim_{n \to \infty} |L_n(|t|; z)|^{1/n} = e^{U(z) + 1}
\]

Hence, we need only show (32).

Let \( n' := \lfloor n/2 \rfloor \). Since \( L_n(|t|; x) \) is an even function, we can write \( L_n(|t|; x) = P_{n'}(x^2) \) for some \( P_{n'} \in \mathbb{P}_{n'} \). It is easy to verify that \( P_{n'} \) is the polynomial of degree at most \( n' \) which interpolates \( \sqrt{x} \) at the points \((0 \leq \xi < \xi_{n'-1} < \ldots < \xi_1 < \xi_0 = 1 \) with \( \xi_k = \left(x^{(n')}_k\right)^2 \), \( k = 0, 1, \ldots, n' \).

Define \( w_n(x) := 
\prod_{k=0}^{n'}(x - \xi_k) \). We now claim that

\[
P_{n'}(z) - \sqrt{z} = \frac{w_n^*(x)}{\pi} \int_0^\infty \frac{\sqrt{t} dt}{w_n^*(-t)(t + z)}, \quad z \in \mathbb{C} \setminus (-\infty, 0]
\]

We check (33) only for the case when \( n \) is even. The proof for the case when \( n \) is odd follows the same line and is simpler. When \( n \) is even, \( \xi_{n'-1} = 0 \). So, the point 0 is a point of interpolation and \( P_{n'}(z)/z \) is the polynomial of degree \( n' - 1 \) that interpolates \( 1/\sqrt{z} \) at points \((0 < \xi_{n'-1} < \ldots < \xi_1 < \xi_0 = 1 \). Then, using the Hermite formula:

\[
i \int_0^\infty \frac{\sqrt{t} dt}{w_n^*(-t)(t + z)} = \frac{1}{2 \pi i} \int_{\gamma} \frac{w_n^*(z)}{z - \zeta} \frac{d\zeta}{\sqrt{\zeta(w_n^*(\zeta)/\zeta)(z - \zeta)}}, \quad z \in \text{Int}(\gamma)
\]

where \( \gamma \) is an arbitrary positively oriented contour in \( \mathbb{C} \setminus (-\infty, 0] \) that contains \([\xi_{n'-1}, 1]\) in its interior. Let \( \gamma \) deform to the boundary of \( A(\varphi, r, R) := \{z : |\arg(z)| \leq \varphi \text{ and } r \leq |z| \leq R\} \) with \( 0 < \varphi < \pi \) and \( 0 < r < 1/n^2 < 1 < R \). Now, let the inner radius \( r \) tend to 0 and the outer radius \( R \) tend to \( \infty \) and then let the angle \( \varphi \) tend to \( \pi \). Then the integral in (34) converges to the integral in (33) multiplied by \(-2/i\), from which our claim (33) follows.

In terms of \( L_n(|t|; z) \), (33) yields

\[
L_n(|t|; z) - z = \frac{w_n^*(z^2)}{\pi} \int_0^\infty \frac{\sqrt{t} dt}{w_n^*(-t)(t + z^2)}, \quad \text{Re}(z) > 0
\]
Define

\[ S_n(z) := \frac{(-1)^{n'+1}}{\pi} \int_0^\infty \frac{\sqrt{t}}{w_n^*(t)(t + z)} \, dt \quad z \in \mathbb{C} \ (-\infty, 0] \]

Then

\[ S_n(z) = \int_0^\infty \frac{\psi_n(t)dt}{t + z}, \quad z \in \mathbb{C} \ (-\infty, 0], \]

where

\[ \frac{(-1)^{n'+1}\sqrt{t}}{\pi w_n^*(-t)} \geq 0 \quad \text{for} \quad t \geq 0. \]

\[ \Im(z) \cdot \Im(S_n(z)) < 0 \quad \text{if} \quad \Im(z) \neq 0 \]

and

\[ S_n(z) > 0 \quad \text{if} \quad z > 0. \]

Now, we can define an analytic function \( H_n(z) := \Log(S_n(z)/S_n(1)) \) for \( z \in \mathbb{C} \ (-\infty, 0] \). Since \( |\Im(H_n(z))| = |\Arg(S_n(z)/S_n(1))| < \pi \), then we have

\[ \lim_{n \to -\infty} \frac{1}{n} |\Im(H_n(z))| = 0. \]

locally uniformly for \( z \in \mathbb{C} \ (-\infty, 0] \). By Schwarz's integral formula, \( H_n(z) \) can be expressed in terms of \( \Im(H_n(z)) \) and \( \Re(H_n(z_0)) \) in any disk with center \( z_0 \) contained in \( \mathbb{C} \ (-\infty, 0] \). In particular, from (36) we have

\[ \lim_{n \to -\infty} \frac{1}{n} \Re(H_n(z)) = 0, \]

uniformly for \( |z - 1| \leq \rho \) (\( \rho < 1 \)). Then, by using a chain of circles we can extend (37) to all points contained in \( \mathbb{C} \ (-\infty, 0] \). It then follows that

\[ \lim_{n \to -\infty} \left( \frac{1}{n} \log |S_n(z)| - \frac{1}{n} \log S_n(1) \right) = 0. \]
locally uniformly for \( z \in \mathbb{C} \setminus (-\infty, 0] \). Using (1) with \( x = 1 \) in (35), we get

\[
\lim_{n \to \infty} \frac{1}{n} \log S_n(1) = 1,
\]

and so (38) implies that

\[
\lim_{n \to \infty} \frac{1}{n} \log |S_n(z)| = 1
\]

locally uniformly for \( z \in \mathbb{C} \setminus (-\infty, 0] \). This, together with (35), gives us

\[
\lim_{n \to \infty} \left| \frac{L_n([t]; z) - z}{w_n(z)} \right|^{1/n} = e
\]

locally uniformly for \( \Re(z) > 0 \). Similarly,

\[
\lim_{n \to \infty} \left| \frac{L_n([t]; z) + z}{w_n(z)} \right|^{1/n} = e
\]

locally uniformly for \( \Re(z) < 0 \). Now, from (39) and (40), we see that (32) holds if, in addition, we assume \( \Re(z) \neq 0 \).

Finally, we verify that (32) holds when \( \Re(z) = 0 \). The proof for this case turns out to be very lengthy. We will give here only a sketch of the proof and leave the details to the reader. Assume \( z = bi \) for some real number \( b \neq 0 \). It is easy to see that

\[
w'_n(x_k^{(n)}) = \left( \frac{2}{n} \right)^n (-1)^{n-k} k!(n-k)!
\]

So, using Lagrange's formula, we have

\[
L_n([t]; bi) = \left( \frac{n}{2} \right)^n (-1)^n \frac{(-1)^n}{n!} w_n(bi) \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{|x_k^{(n)}|}{b^2 + (x_k^{(n)})^2}
\]

The summation in (41) (let's call it \( S_n \)) can be written as

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{-2bi|x_k^{(n)}|}{b^2 + (x_k^{(n)})^2}
\]

for even \( n \),
and

\[
\sum_{k=0}^{n'} (-1)^k \binom{n}{k} \frac{2(x_k^{(n)})^2}{b^2 + (x_k^{(n)})^2} \quad \text{for odd } n.
\]

For odd \(n\), we apply the residue theorem to write \(S_n\) as

\[
S_n = \frac{1}{2\pi i} \int_{C_{\delta,M}} \frac{(-1)^n 2\Gamma(n+1)}{\Gamma(\frac{n+1}{2} + 1 + z)\Gamma(\frac{n+1}{2} - z)} \frac{(1+2z)^2}{b^2 + (1+2z)^2 \sin \pi z} \pi dz
\]

where \(C_{\delta,M}\) denotes the rectangle formed by lines \(\Re(z) = -\delta/2\) \((0 < \delta < 1)\), \(\Re(z) = n/2\), and \(\Im(z) = \pm M\) \((M > 0)\). (This integral representation of \(S_n\) can be verified by noting that the integrand is analytic in \(C_{\delta,M}\) except at \(z = 0, 1, ..., (n - 1)/2\) where it has simple poles and the residue at \(z = (n - 1)/2 - k\) is the \(k\)th term in the summation form of \(S_n\).) Let \(\Omega(z)\) denote the integrand. It can be verified that

(i) for fixed \(n\), the integral along lines \(\Im(z) = \pm M\) tends to 0 as \(M \to \infty\),

(ii) the integral along \(\Re(z) = n/2\) tends to 0 as \(n \to \infty\),

(iii) the absolute value of the integral along \(\Re(z) = -\delta/2\) is greater than \(c|\Omega(-\delta/2)|\) for some positive constant \(c\) independent of \(n\).

Indeed, (i) follows from the (crude) estimate \(\Omega(z) = O(|z|^{-2})\) \((n\) fixed), while (ii) is proved by showing \(\Omega(z) = O(n^{-1/2}(1/4 + t^2)^{-1})\) \((n \to \infty)\) with \(z = n/2 + it\) \((t\) real). Assertion (iii) is verified by the saddle point method (cf. [1]). Note that the integrand \(\Omega(z)\) can be written as

\[
\Omega(z) = \frac{2n!(1+2z)^2}{(z + \frac{n+1}{2})(z + \frac{n+1}{2} - 1) \cdots (z + \frac{n+1}{2} - n)(n^2b^2 + (1+2z)^2)}
\]
$n$ is even can be handled similarly, and (32) holds. This completes our proof.

REFERENCES


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