Estimating the Argument of Approximate Conformal Mappings*

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Let \( f \) be analytic in the unit disk \( E, f(0) = 0, f(z) \neq 0 \) otherwise. We consider the problem of estimating the argument of \( \frac{\arg(f(z)/z)}{1} \) for \( |f| < 1 \) given bounds for \( |f| \) on \( |z| = 1 \). Such problems arise in measuring the accuracy of approximate conformal mappings of simply connected domains onto the unit disk.

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1. INTRODUCTION

Let \( \Omega \) be a bounded simply connected domain in the complex \( z \)-plane, and \( \zeta \in \Omega \). According to the Riemann mapping theorem, there exists a unique analytic function \( w = F(z) \) that maps \( \Omega \) onto the unit disk \( |w| < 1 \) and satisfies \( F(\zeta) = 0, \ F'(\zeta) > 0 \). The determination of such a conformal mapping is of substantial practical importance and several techniques such as the Bergman kernel method, Trefethen's Schwarz–Christoffel algorithm and Symm's integral equation method exist for constructing approximations to \( F \) (cf. [2, 4]). In generating an approximate mapping \( \hat{F} \) by such methods, it is straightforward to check the accuracy of the modulus of \( \hat{F}(z) \) for \( z \) on the boundary \( \partial \Omega \) of \( \Omega \), i.e. to determine (or bound)

\[
\varepsilon(\hat{F}) := \max_{z \in \partial \Omega} |\hat{F}(z)| - |F(z)| = \max_{z \in \partial \Omega} |\hat{F}(z)| - 1.
\]

(1.1)

However, estimates for the error in the arg \( \hat{F}(z) \), that is, bounds for

\[
\delta(\hat{F}) := \sup_{z \in \Omega} |\arg \hat{F}(z) - \arg F(z)|
\]

(1.2)

are far more difficult to obtain. Thus a natural question that arises (and was posed to the authors by N. Papamichael) is the following.

Can one utilize a bound for \( \varepsilon(\hat{F}) \) to determine an estimate for \( \delta(\hat{F}) \)?

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Without further assumptions on \( \tilde{F} \), the answer is certainly negative. However, if some additional information is known about the geometric properties of the approximate \( \tilde{F} \) (such as the property that \( \tilde{F}(\Omega) \) is starlike with respect to the origin), then estimates for \( \delta(\tilde{F}) \) can indeed be determined in terms of \( \epsilon(\tilde{F}) \); see Corollary 2.4. This is one of the goals of the present paper.

In analyzing the above problem we shall study a subclass of bounded analytic functions \( f \) in the unit disk that is of independent interest. This class is defined as follows.

**Definition 1.1** Let \( E = \{ z : |z| < 1 \} \) denote the unit disk, \( f(z) \) be analytic on \( E \) and \( \partial(f(E)) \) denote the boundary of \( f(E) \). We say that \( f \) belongs to the class \( GS \) if it satisfies the following properties:

(a) \( f(0) = 0, f'(0) > 0; \)

(b) \( f(z)/z \neq 0 \) for \( z \in E; \)

(c) \( M_f := \sup\{|w| : w \in \partial(f(E))\} < \infty \) and \( m_f := \inf\{|w| : w \in \partial(f(E))\} > 0. \)

If \( f \in GS \), we introduce the quantity

\[
\Delta_f := \min \left\{ 1 - \inf_{z \in E} \text{Re} h_f(z), \sup_{z \in E} \text{Re} h_f(z) - 1 \right\}
\]

where

\[
h_f(z) := zf'(z)/f(z). \tag{1.4}
\]

Notice that for every \( f \in GS \) we have \( \Delta_f \geq 0 \) with \( \Delta_f = 0 \) if and only if \( f(z) = f'(0)z \).

Finally for each \( \Delta \geq 0 \) we consider the class

\[
GS(\Delta) := \{ f : f \in GS \text{ and } \Delta_f \leq \Delta \}. \tag{1.5}
\]

In Theorem 2.1 we shall show how \( GS(\Delta) \) and the value \( \epsilon \) from (1.1) are related to the well-known class \( S \) of all normalized univalent functions on \( E \). And, as one of our main results, we obtain in Theorem 2.2 an estimate for

\[
\sup_{z \in E} |\arg(f(z)/z)|, \quad f \in GS(\Delta),
\]

in terms of \( \Delta \) and the ratio \( m_f/M_f \). Section 3 is devoted to examples of the sharpness of our estimates and discusses related conjectures.

**2. Properties of Class GS**

We denote by \( S \) the collection of all normalized functions

\[
f(z) = z + c_2 z^2 + c_3 z^3 + \cdots \tag{2.1}
\]

that are analytic and univalent on \( E \). For each \( \alpha \in [0,1] \), we denote by \( S^*(\alpha) \) the set of all normalized analytic functions \( f \) on \( E \) that satisfy

\[
\text{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} \geq \alpha, \quad z \in E,
\]
that is, those functions $f$ that are starlike of order $\alpha$. We note that $S^*(\alpha) \subset S$ and that $S^* := S^*(0)$ consists of the normalized univalent functions whose range is starlike with respect to the origin. Finally, we shall denote by $S^*_\alpha(\alpha)$ the subset of bounded functions from $S^*(\alpha)$, and we let $S^*_0 := S^*_0(0)$.

From definition (1.5) and known lower bounds for the modulus of univalent functions the following two properties of the class $GS(\Delta)$ are immediate:

(i) $GS(\Delta_1) \subset GS(\Delta_2)$ if $\Delta_1 < \Delta_2$;
(ii) $S^*_\alpha(\alpha) \subset GS(1 - \alpha)$; in particular, $S^*_\alpha \subset GS(1)$.

A connection between the classes $S$ and $GS(\Delta)$ and the value $\varepsilon$ from (1.1) is revealed in the following theorem.

**Theorem 2.1** Let $f \in S$ and $\Delta \in (0, \infty)$. Then the function

$$f_\Delta(z) := \frac{f(rz)}{r}, \quad r := 1 - (1 + \Delta)^{-1/2}, \quad (2.3)$$

satisfies

$$|f_\Delta(\zeta) - 1| \leq \Delta \quad \text{for} \quad |\zeta| = 1, \quad (2.4)$$

and $\Delta_{f_\Delta} < \Delta$. Furthermore, $f_\Delta \in GS(\Delta)$.

**Proof** For functions $f \in S$, the following distortion theorems are well-known (cf. [3, Section V.8]):

$$\frac{r}{(1 + r)^2} \leq |f(z)| \leq \frac{r}{(1 - r)^2}, \quad \left| \frac{zf'(z)}{f(z)} \right| \leq \frac{1 + r}{1 - r}$$

for $r = |z| < 1$. Thus for $|\zeta| = 1$ and $r$ as defined in (2.3) we have

$$|f_\Delta(\zeta)| = \left| \frac{f(r\zeta)}{r} \right| \leq \frac{1}{(1 - r)^2} = 1 + \Delta,$$

$$|f_\Delta(\zeta)| \geq \frac{1}{(1 + r)^2} = \frac{1}{(2 - (1 + \Delta)^{-1/2})^2}$$

Since $x := (1 + \Delta)^{1/2} > 1$, it is straightforward to verify that

$$1 > (2 - x^2)(2 - x^{-1})^2,$$

from which it follows that

$$\frac{1}{(2 - (1 + \Delta)^{-1/2})^2} > 2 - x^2 = 1 - \Delta.$$

Hence, from (2.6) and (2.7) we obtain (2.4).

Next, we consider $\text{Re}\{zf'_\Delta(z)/f_\Delta(z)\}$. For $z \in E$, the second estimate in (2.5) yields

$$\left| \frac{zf'_\Delta(z)}{f_\Delta(z)} \right| = \left| \frac{rf'(rz)}{f(rz)} \right| \leq \frac{1 + rz}{1 - rz} \leq \frac{1 + r}{1 - r}$$

Since

$$\frac{1 + r}{1 - r} = 1 + \frac{2r}{1 - r} = 1 + \frac{2\Delta}{(1 + \Delta)^{1/2} + 1} < 1 + \Delta,$$
we obtain from (2.8) that
\[
\Delta f_\Delta \leq \sup_{z \in E} \text{Re} \left\{ \frac{zf'_\Delta(z)}{f_\Delta(z)} \right\} - 1 < \Delta,
\]
which completes the proof.

We now state one of our main results.

**Theorem 2.2** Let \( \Delta \in (0, \infty) \). If \( f \in GS(\Delta) \) and \( m_f, M_f \) are as in Definition 1.1, then for \( z \in E \)

\[
\left| \arg \left( \frac{f(z)}{z} \right) \right| \leq 2\Delta \arccos \left( \left( \frac{m_f}{M_f} \right)^{1/\Delta} \right)
\]

\[
\leq \Delta \min \{ \pi, 2[(M_f/m_f)^{1/\Delta} - 1]^{1/2} \}, \tag{2.9}
\]

with equality in both estimates if and only if \( f(z) = f'(0)z \).

In the proof of this result we shall appeal to the following lemma.

**Lemma 2.3** Let \( w(z) \neq 0 \) be analytic in \( E \) and satisfy \( w(0) = 0 \) and \( \Re{w(z)} < d \) in \( E \). Let

\[
\delta := \frac{1}{\inf_{z \in E} \Re{zw'(z)}} > 0.
\]

Then, for \( z \in E \),

\[
|\Im{w(z)}| < \frac{2}{\delta} \arccos(e^{-\delta d/2}). \tag{2.11}
\]

**Proof** Clearly \( d > 0 \) and \( \delta < \infty \). Let

\[
g(z) := ze^{\delta w(z)}, \quad z \in E,
\]

and note that \( g(0) = 0, g'(0) = 1 \) and

\[
|g(z)| < e^{\delta \Re{w(z)}} < e^{\delta d} =: M \quad \text{for} \quad z \in E.
\]

Furthermore, from the definition of \( \delta \), we have

\[
\Re\left\{ \frac{zg'(z)}{g(z)} \right\} = \Re\{1 + z\delta w'(z)\} > 0, \quad \text{for} \quad z \in E. \tag{2.13}
\]

Hence \( g \in S^* \).

Next we apply a result of R. Barnard [1] which asserts that for any nonconstant entire function \( \Phi \) and any fixed \( z \) in \( E \), the function that maximizes the quantity

\[
\Re\left\{ \Phi \left( \log \frac{f(z)}{z} \right) \right\}
\]

over the class of functions \( f \in S^*_\Delta \) bounded by \( M \) has its image domain equal to the disk \( \{ t : |t| < M \} \) minus one radial slit. In particular, for \( \Phi(s) = \pm is \) we have that

\[
|\Im{w(z)}| = \frac{1}{\delta} |\arg(g(z)/z)| < \frac{1}{\delta} \max_{|\zeta| = 1} |\arg(G(\zeta)/\zeta)|, \tag{2.14}
\]
where \( G \) is the extremal mapping defined by the relation

\[
\frac{k(\zeta)}{M} = k \left( \frac{G(\zeta)}{M} \right), \quad k(\zeta) := \frac{\zeta}{(1 - \zeta)^2}
\]

Now let \( \zeta := e^{i\alpha}, \theta(\alpha) := \text{arg}(G(\zeta)), \) where \( \alpha, \theta \in [-\pi, \pi] \). Then it is easy to verify that if

\[
|\alpha| \leq \alpha_0 := 2\arcsin(M^{-1/2}),
\]

then \( \theta(\alpha) = 2\arcsin(M^{1/2}\sin(\alpha/2)) \); while if \( \alpha_0 < |\alpha| \leq \pi \), then \( \theta(\alpha) = \pm \pi \). Since

\[
\frac{d}{d\alpha}[\theta(\alpha) - \alpha] = \frac{M^{1/2}\cos(\alpha/2)}{[1 - M\sin^2(\alpha/2)]^{1/2}} - 1 > 0 \quad \text{for} \quad |\alpha| \leq \alpha_0,
\]

we obtain that for \( \alpha \in [-\pi, \pi] \)

\[
|\theta(\alpha) - \alpha| \leq \pi - 2\arcsin(M^{-1/2}) = 2\arcsin(M^{-1/2}).
\]

Inequality (2.11) now follows from (2.14) and the definition of \( M \) in (2.12).

**Proof of Theorem 2.2** Let \( a := f'(0) > 0 \) and \( g(z) := f(z)/z = a + \cdots \). Then by Definition 1.1 we have \( g(z) \neq 0 \) in \( E \). Also \( 1/g(z) = 1/a + \cdots \) is analytic in \( E \). Since for \( \zeta \in \partial(g(E)) \) we have

\[
m_f \leq |\zeta| \leq M_f,
\]

then \( m_f \leq a \leq M_f \). If \( \Delta_f = 0 \), then \( f(z) \equiv az, M_f = m_f \) and \( \arg(f(z)/z) \equiv 0 \); thus equality holds throughout in (2.9).

So suppose \( \Delta_f > 0 \). Then \( w(z) := \log(g(z)/a) \) is analytic in \( E \), \( w(0) = 0 \), and \( w(z) \neq 0 \). Furthermore, we have

\[
|\text{Im} w(z)| = |\arg(f(z)/z)|,
\]

\[
|\text{Re} w(z)| = |\log|g(z)/a|| < \log \left( \frac{M_f}{m_f} \right) \quad z \in E,
\]

and

\[
zw'(z) = \frac{zf'(z)}{f(z)} - 1.
\]

From (2.19) and the definition of \( \Delta_f \) in (1.3) it follows that

\[
\Delta_f = -\inf_{z \in E} \text{Re}[zw'(z)] \quad \text{or} \quad \Delta_f = -\inf_{z \in E} \text{Re}[z(-w(z))']
\]

Hence with (2.18) we can apply Lemma 2.3 to \( w(z) \) or \(-w(z)\) with

\[
d := \log \left( \frac{M_f}{m_f} \right) \quad \text{and} \quad \delta := \frac{1}{\Delta_f} \geq \frac{1}{\Delta} > 0,
\]

and we obtain

\[
|\text{Im} w(z)| = |\arg(f(z)/z)| < 2\Delta f \arccos[(m_f/M_f)^{1/2\Delta_f}],
\]

where the strict inequality follows from the strict inequality in (2.11).
To obtain the first estimate in (2.9) we shall show that the function
\[ \varphi(x) := x \arccos(\alpha^{1/x}), \quad \alpha := (m_f/M_f)^{1/2} \in (0, 1), \]
(2.22)
is increasing for \( x > 0 \). For this purpose, let \( y := \arccos(\alpha^{1/x}) \) and
\[ \psi(y) := \varphi \circ x(y) = (\log \alpha) \frac{y}{\log (\cos y)}, \quad y \in (0, \pi/2). \]
(2.23)
Then
\[ \psi'(y) = \frac{(\log \alpha) \log (\cos y) + y \tan y}{\log^2 (\cos y)} = \frac{\log \alpha}{\log^2 (\cos y)} \int_0^y \frac{u}{\cos^2 u} \, du < 0, \]
and since
\[ \tan y = \frac{\tan \alpha}{y} \cos y, \]
we have \( \varphi'(x) > 0 \) for \( x > 0 \). Hence \( \varphi(\Delta_f) \leq \varphi(\Delta) \).

Finally, we use the inequality
\[ \arccos t < \min \left\{ \frac{\pi}{2}, \frac{1 - t^2}{t} \right\} \]
for \( t := \alpha^{1/\Delta} \in (0, 1) \),
to deduce from (2.21) that
\[ |\arg(f(z)/z)| < 2\varphi(\Delta_f) \leq 2\varphi(\Delta) \leq \Delta \min\{\pi, 2[\alpha^{-2/\Delta} - 1]^{1/2}\}, \]
which completes the proof. \( \square \)

As an immediate application of Theorem 2.2 we obtain the following estimate for the argument of approximate conformal mappings that are starlike.

**Corollary 2.4** Let \( \Omega \) be a simply connected domain bounded by the Jordan curve \( \Gamma \), \( z \in \Omega \), and \( w = F(z) \) be the conformal mapping of \( \Omega \) onto the unit disk \( |w| < 1 \) that satisfies \( F(\zeta) = 0 \), \( F'(\zeta) > 0 \). Let \( \tilde{F}(z) \) be analytic and univalent on \( \Omega \), \( \tilde{F}(\zeta) = 0 \), \( \tilde{F}'(\zeta) > 0 \), and assume that \( \tilde{F}(\Omega) \) is starlike with respect to \( w = 0 \). If
\[ ||\tilde{F}(z)|| < 1 \leq \varepsilon \leq \frac{1}{2}, \quad \text{for } z \in \Gamma, \]
(2.24)
then
\[ \delta(\tilde{F}) := \sup_{z \in \Omega} |\arg \tilde{F}(z) - \arg F(z)| \leq 4\varepsilon^{1/2}. \]
(2.25)

**Proof** Let \( f := \tilde{F} \circ F^{-1} \) for \( |w| < 1 \). Then \( f(0) = 0 \), \( f'(0) > 0 \), and \( f(w)/f'(0) \in S_b \subset GS(1) \) with
\[ M_f \leq 1 + \varepsilon, \quad 1 - \varepsilon \leq m_f. \]
Hence, by (2.9) with \( \Delta = 1 \), we have for \( |w| < 1 \)
\[ |\arg \left( \frac{f(w)}{f'(0)w} \right)| \leq \min \left\{ \pi, 2 \left[ \frac{1 + \varepsilon}{1 - \varepsilon} - 1 \right]^{1/2} \right\} \]
which yields (2.25). \( \square \)
3. SEQUENCES OF GS FUNCTIONS

Pertaining to the sharpness of Theorem 2.2 and its application to approximate conformal mappings we now consider sequences \( \{f_n\}_1^\infty \subset GS(\Delta) \setminus GS(0) \) that satisfy

\[
\lim_{n\to\infty} \frac{M_{f_n}}{m_{f_n}} = 1.
\]  

DEFINITION 3.1 The constant \( \delta > 0 \) is said to be admissible for the sequence \( \{f_n\}_1^\infty \) if for an integer \( N \)

\[
\sup_{n\geq N, z \in E} \log \left( \frac{M_{f_n}}{m_{f_n}} \right) < \infty
\]

The next result asserts that \( \delta = 1/2 \) is admissible for all such sequences from \( GS(\Delta) \), while \( \delta = 1 \) fails to be admissible.

THEOREM 3.2 For any \( \Delta \in (0, \infty) \):

(a) \( \delta = 1/2 \) is admissible for every sequence \( \{f_n\}_1^\infty \subset GS(\Delta) \setminus GS(0) \) satisfying (3.1).

(b) \( \delta = 1 \) is not admissible for “power sequences” \( \{f_n\}_1^\infty \subset GS, \Delta_{f_n} = \Delta \) (cf. (3.6) below) satisfying (3.1) that are generated by an arbitrary function \( f \in GS \) with

\[
\sup_{z \in E} |\arg(f(z)/z)| = \infty.
\]

Proof (a) Let \( \{f_n\}_1^\infty \subset GS(\Delta) \setminus GS(0) \) satisfy (3.1). From Theorem 2.2 we have

\[
|\arg(f_n(z)/z)| < 2\Delta \left( \frac{M_{f_n}}{m_{f_n}} \right)^{1/\Delta - 1/2}.
\]

Let \( \epsilon > 0 \). Then by (3.1) there exists an integer \( N \) such that

\[
\log \left( \frac{M_{f_n}}{m_{f_n}} \right) < \epsilon \quad \text{for} \quad n \geq N
\]

Thus from (3.3) we obtain

\[
\sup_{n\geq N, z \in E} \frac{|\arg(f_n(z)/z)|}{\log^{1/2}(M_{f_n}/m_{f_n})} \leq 2\Delta^{1/2} \sup_{n\geq N} \frac{\Delta \left( \frac{M_{f_n}}{m_{f_n}} \right)^{1/\Delta - 1}}{\log(M_{f_n}/m_{f_n})} \leq 2\Delta^{1/2} \left( \frac{\epsilon}{\epsilon/\Delta} - 1 \right)^{1/2} < \infty,
\]

so that \( \delta = 1/2 \) is admissible. In fact, we have shown that

\[
\limsup_{n \to \infty} \left\{ \sup_{z \in E} \frac{|\arg(f_n(z)/z)|}{\log^{1/2}(M_{f_n}/m_{f_n})} \right\} \leq 2\Delta^{1/2}.
\]

(b) Let \( f \in GS \) with

\[
\sup_{z \in E} |\arg(f(z)/z)| = \infty.
\]
for example, \( f(z) = z/(1-z)^i \) satisfies (3.5). Set \( M := M_f, m := m_f \). For fixed \( \Delta > 0 \), define the power sequence
\[
f_n(z) := z \left( \frac{f(r_n z)}{z} \right)^{c_n(\Delta)} \quad n = 1, 2,
\]
where \( r_n \in (0,1) \), \( r_n \to 1 \) as \( n \to \infty \), and
\[
c_n(\Delta) := \frac{\Delta}{\min(a,b)} > 0,
\]
where \( a := 1 - \inf_{|z|<r_n} \Re h_f(z), b := \sup_{|z|<r_n} \Re h_f(z) - 1 \), and \( h_f \) is given in (1.4). Then we have
\[
\log \left( \frac{M_f}{m_f} \right) = c_n(\Delta) \log \left[ \frac{\max_{|z|=r_n} |f(z)|}{\min_{|z|=r_n} |f(z)|} \right] > 0, \\
\arg(f_n(z)/z) = c_n(\Delta) \arg(f(r_n z)/z), \\
\frac{zf_n'(z)}{f_n(z)} = 1 + c_n(\Delta) \left[ \frac{r_n z f'(r_n z)}{f(r_n z)} - 1 \right].
\]
From (3.10) and (3.7) it follows that
\[
\Delta_n = \frac{\Delta}{\min(a,b)} \min(a,b) = \Delta,
\]
and so \( f_n \in GS, \Delta_n = \Delta \) for all \( n \).

Next we claim that
\[
\inf_{z \in E} \Re h_f(z) = -\infty, \quad \sup_{z \in E} \Re h_f(z) = \infty,
\]
and
\[
\lim_{n \to \infty} c_n(\Delta) = 0. \tag{3.12}
\]
Indeed if (3.11) does not hold, then \( \Delta_f < \infty \) and the result of Theorem 2.2 contradicts assumption (3.5). Furthermore, since
\[
\inf_{|z|<r_n} \Re h_f(z) \to \inf_{z \in E} \Re h_f(z) = -\infty
\]
and
\[
\sup_{|z|<r_n} \Re h_f(z) \to \sup_{z \in E} \Re h_f(z) = \infty,
\]
the claim (3.12) is obvious from definition (3.7).

Thus from (3.8) and (3.12) we have
\[
\log(M_{f_n}/m_{f_n}) \to 0 \quad \text{as} \quad n \to \infty,
\]
and so (3.1) is satisfied. Finally, from (3.9) and (3.5) we obtain for any integer \( N \)

\[
\sup_{n \geq N, z \in E} \frac{|\arg(f_n(z)/z)|}{\log(M_{f_n}/m_{f_n})} = \sup_{n \geq N, z \in E} \frac{|\arg(f(r_n z)/z)|}{\log\left(\max_{|z|=r_n} |f(z)|/\min_{|z|=r_n} |f(z)|\right)} \\
\geq \sup_{z \in E} \frac{|\arg(f(z)/z)|}{\log(M/m)} = \infty,
\]

so that \( \delta = 1 \) is not admissible for the sequence \( \{f_n\} \).

As our final result we consider certain sequences of convex functions and functions from \( S_b^*(\alpha) \) belonging to GS for which \( \delta = 1 \) is not admissible and we obtain finer asymptotic estimates.

**Theorem 3.3**  
(a) For any \( \Delta \in (0, \infty) \), there exists a sequence \( \{f_n\}_1^\infty \subset GS(\Delta) \) that satisfies (3.1) and

\[
\lim_{n \to \infty} \frac{\sup_{z \in E} |\arg(f_n(z)/z)|}{\log(M_{f_n}/m_{f_n})\log\log(M_{f_n}/m_{f_n})} < 0.
\]

If \( \Delta \in (0,1] \), this sequence belongs to \( S_b^*(1-\Delta) \).

(b) There exists a sequence \( \{g_n\}_1^\infty \subset GS \) of convex functions that satisfies (3.1) and (3.13).

**Proof**  
Let \( c = c(r) \) be a real positive function of \( r \in (0,1) \) and define

\[
f(z) = f(z;r) := \frac{z}{(1-rz)c(r)}, \quad f'(0) := \]

Then \( f \) is analytic in \( E \) with

\[
\sup_{z \in E} |\arg(f(z)/z)| = c(r)\log \frac{1}{1-r} \tag{3.15}
\]

Furthermore, \( \log M_f = c(r)\arcsin r \) and \( \log m_f = -c(r)\arcsin r \) so that

\[
\log(M_f/m_f) = 2c(r)\arcsin r.
\]

Hence

\[
\lim_{r \to 1} \frac{M_f}{m_f} = 1 \quad \text{if} \quad \lim_{r \to 1} c(r) = 0. \tag{3.17}
\]

In the proofs of (a) and (b) we shall take

\[
c(r) := a(r)(1-r)^b, \tag{3.18}
\]
where \( a(r) > 0, \ b > 0, \) and \( \lim_{r \to 1} a(r) = a(1) \neq 0. \) With this choice of \( c(r) \) we obtain from (3.15) and (3.16) that

\[
\lim_{r \to 1} \frac{\sup_{z \in E} |\arg(f(z)/z)|}{\log(M_f/m_f) \log(\log(M_f/m_f))} = -\log(1 - r)
\]

\[
= -\frac{1}{2(\arcsin r)} \log(2c(r) \arcsin r) \arcsin r
\]

\[
= -\frac{1}{\pi r - 1} \log(1 - r) + \log(2a(r) \arcsin r)
\]

\[
= -\frac{1}{\pi b}
\]

(a) Let \( a(r) = \Delta(1 + r)/r, \ b = 1, \) so that \( c(r) = \Delta(1 - r^2)/r, \) and the function \( f \) of (3.14) becomes

\[
f(z) = \frac{z}{(1 - rz)\Delta(1 - r^2)/r} = z +
\]

Then

\[
\frac{zf''(z)}{f(z)} - 1 = i\Delta(1 - r^2)
\]

from which it follows that

\[
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} - 1 = -\Delta(1 - r^2)\Im \left( \frac{z}{1 - rz} \right)
\]

Since \( w = z/(1 - rz) \) maps \( E \) onto the disk with center \( r/(1 - r^2) \) and radius \( 1/(1 - r^2), \) we see that

\[
\inf_{z \in E} \Re \left\{ \frac{zf'(z)}{f(z)} \right\} - 1 = -\Delta.
\]

Hence \( \Delta_s \leq \Delta, \) so that \( f \in GS(\Delta) \) and \( f \in S^b_\Delta(1 - \Delta) \) if \( \Delta \in (0,1]. \) On setting \( f_n(z) := f(z; r_n) \) with \( r_n \to 1, \) the assertion of part (a) follows from (3.17) and (3.19).

(b) Let \( a(r) = 1, \ b = 4, \) so that \( c(r) = (1 - r)^4, \) and the function \( f \) of (3.14) now becomes

\[
f(z) = \frac{z}{(1 - rz)(1 - r^2)^4} = z + \ldots
\]

We shall show that for some \( r_0 \in (0,1) \) and \( r \in (r_0,1) \) we have

\[
\Re \left\{ \frac{zf''(z)}{f'(z)} \right\} > -1, \quad z \in E;
\]

that is, we show that \( f(\cdot) = f(r; \cdot) \) is a convex mapping.

For this purpose we compute

\[
f'(z) = \frac{1 + (ic - 1)rz}{(1 - rz)^{1+ic}},
\]

\[
zf'(z) = \frac{zr(ic + 1)}{1 - rz} + \frac{zr(ic - 1)}{1 + (ic - 1)rz}
\]

(3.24)
For $z = e^{i\theta}$, we find from (3.24) that

$$
\Re \left( \frac{zf''(z)}{f'(z) + 1} \right) = \frac{\varphi(\theta)}{|1 - rz|^2 |1 + (ic - 1)rz|^2},
$$

where

$$
\varphi(\theta) = -c^3 r^3 \sin \theta + c^2 r^2 (2 + r^2 - 3r \cos \theta + 2 \sin^2 \theta) + cr [3r \sin 2\theta - (4 + 2r^2) \sin \theta] + [2r (1 - \cos \theta) + (1 - r^2)]^2.
$$

To show that $\varphi$ is positive we replace $c$ by $(1 - r)^4$ and utilize the estimate

$$
|\varphi(\theta) - [2r (1 - \cos \theta) + (1 - r^2)]^2| < 9(1 - r)^8 + 12(1 - r)^4 |\sin \theta|,
$$

$$
< 9(1 - r)^8 + (1 - r)^4 \frac{1 + 144 \sin^2 \theta}{2},
$$

$$
< \frac{3}{4} (1 - r)^4 + 300 (1 - r)^4 \sin^2 (\theta/2),
$$

where the last inequality holds for $r$ sufficiently close to 1, say $\hat{r} < r < 1$. Moreover, we can write

$$
[2r (1 - \cos \theta) + (1 - r^2)]^2 = [4r \sin^2 (\theta/2) + (1 - r^2)]^2
$$

$$
= 16r^2 \sin^4 (\theta/2) + \frac{1}{4} (1 - r)^4
$$

$$
+ (1 - r)^2 \sin^2 (\theta/2) (8r - 300 (1 - r)^2]
$$

$$
+ \frac{3}{4} (1 - r)^4 + 300 (1 - r)^4 \sin^2 (\theta/2),
$$

which on comparison with (3.27) shows that $\varphi(\theta) > 0$ for $r \in (r_0, 1)$, where $r_0 > \hat{r}$ is suitably chosen. Thus (3.23) holds. (We remark that it is also simple to establish (3.23) when $b > 4$.)

Finally we note that since $f$ is convex and bounded, then $f \in S_b^* \subset GS(1)$ and so $g_n(z) := f(z; r_n), r_n > r_0, r_n \rightarrow 1$, satisfies (3.1) and (3.13).\hfill \Box

It seems plausible (and is consistent with numerical experiments) that the sequence constructed in part (a) of the last theorem is, in a sense, asymptotically extremal for sequences from $GS(\Delta)$ satisfying (3.1). Thus we make the following equivalent conjectures.

\textbf{Conjecture A.} For any $\Delta \in (0, \infty)$ and any sequence \( \{f_n\} \in GS(\Delta) \setminus GS(0) \) satisfying (3.1) there holds

$$
\inf_{z \in E} \frac{|\arg(f_n(z)/z)|}{\log(M_{f_n}/m_{f_n}) \log \log(M_{f_n}/m_{f_n})} > -\infty
$$

\textbf{Conjecture B.} For any sequence \( \{g_n\} \) such that $g_n$ is analytic in $E$, $g_n(z) \neq 0$, $g(0) = 0$, $\alpha_n := \sup_{z \in E} \Re g_n(z) < \infty$, $\beta_n := \inf_{z \in E} \Re g_n(z) > -\infty$, $\epsilon_n := \alpha_n - \beta_n \rightarrow$
0 as \( n \to \infty \), and
\[
\min \left\{ \inf_{z \in \mathcal{E}} \text{Re}(z g'_n(z)), \sup_{z \in \mathcal{E}} \text{Re}(z g'_n(z)) \right\} < \Delta < \infty,
\]
there holds
\[
\inf_{n \geq 1, z \in \mathcal{E}} \frac{\text{Im} g_n(z)}{\varepsilon_n \log \varepsilon_n} > -\infty.
\]

**Conjecture C.** For any sequence of positive Borel measures \( \{\mu_n\}_{n=1}^{\infty} \) on \([0,2\pi]\) satisfying \( d\mu_n \neq d\theta \), \( \int_0^{2\pi} d\mu_n(\theta) = 2\pi \),
\[
\begin{align*}
\alpha_n &:= \sup_{z \in \mathcal{E}} I_n(z) < \infty, \\
\beta_n &:= \inf_{z \in \mathcal{E}} I_n(z) > -\infty, \\
I_n(z) &:= \int_0^{2\pi} \log |1 - e^{i\theta} z| d\mu_n(\theta),
\end{align*}
\]
with \( \varepsilon_n := \alpha_n - \beta_n \to 0 \) as \( n \to \infty \), there holds
\[
\inf_{n \geq 1, z \in \mathcal{E}} \frac{\int_0^{2\pi} \arg(1 - ze^{i\theta}) d\mu_n(\theta)}{\varepsilon_n \log \varepsilon_n} > -\infty.
\]

We remark that the equivalence of Conjectures A and B follows from the relation
\[
g_n(z) = \log \left( \frac{f_n(z)}{f_n(0)z} \right),
\]
while the equivalence of B and C can be seen from the representation
\[
g_n(z) = \pm \frac{\Delta}{\pi} \int_0^{2\pi} \log \frac{1}{|1 - e^{i\theta} z|} d\mu_n(\theta)
\]

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**References**