Indag. Mathem., N.S., 6 (1), 105–119 March 27, 1995

Behavior of polynomials of best H^p approximation

by M.P. Moskova[1] and E.B. Saff[2]

[1] *Institute of Mathematics, Bulgarian Academy of Sciences, Sofia 1090, Bulgaria*
[2] *Institute for Constructive Mathematics, Department of Mathematics, University of South Florida, Tampa, FL 33620, U.S.A.*

Communicated by Prof. J. Korevaar at the meeting of February 28, 1994

ABSTRACT

For best polynomial approximation, we prove that every Hardy space H^p, $p \neq 2$, for the unit disk is quite different from the Hardy space H^2, in the sense that for most functions f (with respect to category) the polynomials of best H^p approximation to f need not interpolate f and do not converge faster to f inside the unit disk than the H^p-norm error. These properties also hold for 'most' entire functions f as well as for 'most' functions f having Maclaurin series with real coefficients. Also, in the Appendix, we give a detailed proof of the little known fact of uniqueness of best H^1 polynomial approximants.

1. INTRODUCTION

Let $\mathcal{A}$ denote the algebra of functions f analytic in the open unit disk $D := \{z : |z| < 1\}$ and continuous on $\bar{D} = \{z : |z| \leq 1\}$. If Π_n is the set of polynomials in z of degree at most n, then we set

$$E_{n,p}(f) := \inf_{Q \in \Pi_n} \|f - Q\|_p = \|f - Q_{n,p}(f, \cdot)\|_p,$$

which is the error in the best H^p approximation of f out of Π_n, where

[1] The research of this author was partially supported by the Bulgarian National Foundation for Scientific Research, Grant No. MM201. This work was done while she visited the University of South Florida, Tampa.
[2] The research of this author was supported, in part, by the U.S. National Science Foundation under grant DMS 920-3659.

$$\|f\|_p = \left\{ \frac{1}{2\pi} \int_{|z|=1} |f(z)|^p |dz| \right\}^{1/p}, \qquad \|f\|_\infty = \sup_{|z|<1} |f(z)|,$$

is the norm in Hardy space H^p. It is well known that the polynomial of best H^2 approximation $Q_{n,2}(f,z)$ to $f(z)$ is just the n-th partial sum of its Maclaurin expansion and, for any $f \in H^2$,

$$\text{(1.1)} \qquad \limsup_{n\to\infty} \left\{ \max_{|z|\le r} \frac{|f(z) - Q_{n,2}(f,z)|}{E_{n,2}(f)} \right\}^{1/n} \le r, \quad 0 < r < 1.$$

Furthermore, $Q_{n,2}(f,z)$ interpolates f in the point $z = 0$ (considered of multiplicity $n+1$).

It is natural to ask if the best H^p $(1 \le p \le \infty)$ polynomial approximants have similar properties. Surprisingly, we shall show that $p = 2$ is the only value for which the error $f(z) - Q_{n,p}(f,z)$ at some point $z \in D$ is guaranteed to converge faster than $E_{n,p}(f)$ for all $f \in \mathcal{A}$. The case $p = \infty$ was considered in [ST], where it is proved that for almost all functions $f \in \mathcal{A}$ (in the sense of categories)

$$\text{(1.2)} \qquad \limsup_{n\to\infty} \min_{|z|\le 1} \frac{|f(z) - Q_{n,\infty}(f,z)|}{E_{n,\infty}} = 1,$$

i.e. the error function $f(z) - Q_{n,\infty}(f,z)$ tends to zero at the same rate as $E_{n,\infty}(f)$ and, moreover, there exists a sequence of integers $\{n_k\}$ such that $f(z) - Q_{n_k,\infty}(f,z)$ has no zeros in $\bar{D}$. A comparison of (1.1) and (1.2) shows that there is a huge difference between H^2 and H^∞. The purpose of this paper is to show that a similar distinction exists between H^2 and H^p $(p \ne 2)$ in the sense that for any $p \ne 2$, an equality similar to (1.2) holds for almost all functions. The only result in this direction up to now is given in [IS], where the authors prove that (1.1) fails for any $p \ne 2$ and, in particular, there exists a function $h_p \in \mathcal{A}$ with

$$\limsup_{n\to\infty} |h_p(0) - Q_{n,p}(h_p, 0)|^{1/n} = 1.$$

The essential ingredient in the proof of our main results is the following proposition, which will be established in Section 3.

Proposition 1.1. *For any $p \ne 2$, $1 \le p \le \infty$, there exists a function $f_p \in \mathcal{A}$ whose best constant approximant c_p in the H^p norm is not in $f_p(\bar{D})$.*

(Of course, when $p = 2$, the constant of best H^2 approximation to $f \in \mathcal{A}$ is simply $f(0)$.)

This paper is organized as follows. Section 2 contains the statements of the main results; their proofs are given in Section 3. In the Appendix we provide a detailed proof of the uniqueness of best H^1 polynomial approximants.

2. MAIN RESULTS

For $f \in \mathcal{A}$ we let $Q_n(f) = Q_{n,p}(f,z) \in \Pi_n$ denote the polynomial of best H^p

approximation to f. For $1 \le p < \infty$, the best approximant $Q_{n,p}$ is unique and is characterized by the following property:

$$(2.1) \qquad \int_{|z|=1} |(f - Q_{n,p})(z)|^{p-1}\, \overline{\operatorname{sgn}}(f - Q_{n,p})(z) P_n(z) |dz| = 0$$

for all $P_n \in \Pi_n$. These facts are well known for $1 < p < \infty$ (see, for example, [S, Chapter 4]). However, for $p = 1$, the authors could not find a convenient reference; the proof of uniqueness due to Havinson [H, Theorem 12] (available only in Russian) is formulated in an abstract setting which requires some effort to verify for H^1. Thus we provide a complete and self-contained proof for $p = 1$ in the Appendix.

We shall investigate the normalized error

$$(2.2) \qquad B_{n,p}(f, z) := \frac{f(z) - Q_{n,p}(f, z)}{E_{n,p}(f)},$$

where we set $B_{n,p}(f, z) \equiv 0$ in case $f \in \Pi_n$. Our main theorem asserts that, in general, $\{B_{n,p}(f, z)\}_{n=0}^{\infty}$ does not tend to zero at any point of the closed unit disk.

Theorem 2.1. *For any $p \ne 2$, $1 \le p < \infty$, there exists a constant $\tau_p > 0$ with the following two properties*:

$$(\text{i}) \qquad \limsup_{n\to\infty} \min_{|z|\le 1} \frac{|f(z) - Q_{n,p}(f, z)|}{E_{n,p}(f)} \le \tau_p, \quad \forall f \in \mathcal{A}.$$

(ii) *The set S_p of functions from $\mathcal{A}$ for which strict inequality holds in* (i) *is of the first category in $\mathcal{A}$ with respect to the uniform norm.*

From Theorem 2.1 we immediately get the following.

Corollary 2.2. *For any $p \ne 2$, $1 \le p < \infty$, there exist a function $f \in \mathcal{A}$ and a subsequence $\{n_k\}$ of the natural numbers such that $f(z) - Q_{n_k,p}(f, z)$ does not have zeros in $\bar{D}$ and, moreover,*

$$\lim_{k\to\infty} \min_{|z|\le 1} |B_{n_k,p}(f, z)| > 0.$$

We remark that Theorem 2.1 cannot be extended to the whole disk algebra $\mathcal{A}$. This is easily seen from the next example.

Example 2.3. The function

$$f^*(z) := \sum_{k=0}^{\infty} a_k z^{(2k+1)!!},$$

where $\sum |a_k| < \infty$ and $(2k+1)!! = (2k+1)(2k-1)\cdots 1$, has the partial sums of the defining series as its best polynomial approximants for any p, $1 \le p \le \infty$ (cf. [A] and [B]).

Since the set of functions analytic on the closed unit disk $\bar{D}$ is of the first category in $\mathcal{A}$, it follows from Theorem 2.1, that there exists a function $f \in \mathcal{A}$ that has a singularity on the unit circle and satisfies

$$\limsup_{n\to\infty} \min_{|z|\leq 1} |B_{n,p}(f,z)| > 0. \tag{2.3}$$

But the last inequality is not due to the existence of a singularity. It turns out that there exists an entire function for which (2.3) holds (see Lemma 3.9). Actually, (2.3) holds for almost all entire functions. Let us denote by $\mathcal{E}$ the metric space consisting of all entire functions equipped with the metric of locally uniform convergence

$$d(f,g) := \sum_{n=1}^{\infty} 2^{-n} \frac{\|f-g\|_{\infty,n}}{\|f-g\|_{\infty,n}+1},$$

where $\|f\|_{\infty,n} := \sup\{|f(z)| : |z| \leq n\}$. Then $\mathcal{E}$ is a complete metric space for which we shall prove the following result.

Theorem 2.4. *For any $p \neq 2$, $1 \leq p < \infty$, there exists a constant $\sigma_p > 0$ with the following two properties:*

$$\text{(i)} \qquad \limsup_{n\to\infty} \min_{|z|\leq 1} \frac{|f(z) - Q_{n,p}(f,z)|}{E_{n,p}(f)} \leq \sigma_p, \quad \forall f \in \mathcal{E}.$$

(ii) *The set of functions from $\mathcal{E}$ for which strict inequality holds in* (i) *is of the first category in $\mathcal{E}$.*

Another important set of functions which is of the first category in $\mathcal{A}$ is $\mathcal{A}^R$, which consists of those functions from $\mathcal{A}$ with all real coefficients in their Maclaurin series. $\mathcal{A}^R$ is a complete space (with the uniform norm) and the following theorem holds.

Theorem 2.5. *For any $p \neq 2$, $1 \leq p < \infty$, there exists a constant $\kappa_p > 0$ with the following two properties:*

$$\text{(i)} \qquad \limsup_{n\to\infty} \min_{|z|\leq 1} \frac{|f(z) - Q_{n,p}(f,z)|}{E_{n,p}(f)} \leq \kappa_p, \quad \forall f \in \mathcal{A}^R.$$

(ii) *The set of functions from $\mathcal{A}^R$ for which strict inequality holds in* (i) *is of the first category in $\mathcal{A}^R$.*

Theorem 2.5 asserts that for almost all functions with real coefficients (2.3) holds.

From the definitions of the numbers τ_p, σ_p and κ_p it is clear that $\sigma_p \leq \tau_p$ and $\kappa_p \leq \tau_p$, but we do not know if equality holds. The precise determination of these constants is an interesting open problem. It can be shown, for example, that for $p = 1$ we have $\tau_1 < 1$; more precisely,

$$\frac{1}{100} \leq \tau_1 \leq \frac{1}{4}.$$

3. PROOFS

The key step in our proofs is the construction of a function $f_p \in \mathcal{A}$ satisfying Proposition 1.1. This construction turns out to be quite simple when $1 \leq p < 2$ (one can take $f_p(z) = e^{az}$ for a suitable constant a). However, the case $p > 2$ requires substantially more effort. For this purpose we make use of the following lemma.

Lemma 3.1. *Let* $0 \leq s < \infty$, $s \neq 1$, $\mu(\theta)$ *an increasing continuous function in* $[0, \pi/4]$ *with* $\mu(0) = 0$, $\mu(\pi/4) = 1$ *and let*

$$G_s(\lambda) := \frac{\pi}{2\cosh\lambda} + \cos\lambda \int_0^{\pi/4} \left(\frac{\cosh\lambda s\mu(\theta)}{\cosh\lambda s} - \frac{\cosh\lambda\mu(\theta)}{\cosh\lambda} \right) d\theta.$$

Then $G_s(\lambda_s) = 0$ *for some* $\lambda_s > 0$.

Proof. Note that $G_s(0) = \pi/2 > 0$, and $G_s(\lambda)$ is a continuous real-valued function. Therefore, the lemma will be proved if we find λ so that $G_s(\lambda) < 0$. Let

$$g(s) := \frac{\cosh\alpha s}{\cosh\beta s}, \quad 0 < \alpha < \beta.$$

Then, for $s > 0$,

$$g'(s) = -\frac{\alpha\sinh(\beta-\alpha)s + (\beta-\alpha)\cosh\alpha s\sinh\beta s}{\cosh^2\beta s} < 0.$$

Therefore,

(3.1) $\quad g(s)$ is decreasing in $[0, \infty) \quad (0 < \alpha < \beta)$.

Obviously

(3.2) $$e^{-(\beta-\alpha)} < \frac{\cosh\alpha}{\cosh\beta} < 2e^{-(\beta-\alpha)} \quad \text{for } 0 < \alpha < \beta,$$

and

(3.3) $$|\Delta| > 0, \quad \text{where } \Delta := \left\{\theta \in \left(0, \frac{\pi}{4}\right) : \frac{1}{4} \leq \mu(\theta) \leq \frac{3}{4}\right\}.$$

Let us first consider the case $0 \leq s < 1$. Using (3.1) with $\lambda > 0$, $\alpha = \lambda\mu(\theta)$, and $\beta = \lambda$, (3.2) and (3.3) we get

$$\begin{aligned}
I_s(\lambda) &:= \int_0^{\pi/4} \frac{\cosh \lambda s\mu(\theta)}{\cosh \lambda s} - \frac{\cosh \lambda\mu(\theta)}{\cosh \lambda}\, d\theta \\
&\geq \int_\Delta \frac{\cosh \lambda s\mu(\theta)}{\cosh \lambda s} - \frac{\cosh \lambda\mu(\theta)}{\cosh \lambda}\, d\theta \\
&\geq \int_\Delta e^{-\lambda s(1-\mu(\theta))} - 2e^{-\lambda(1-\mu(\theta))} d\theta \\
&= \int_\Delta e^{-\lambda s(1-\mu(\theta))}\left(1 - 2e^{-\lambda(1-s)(1-\mu(\theta))}\right) d\theta \\
&\geq |\Delta| e^{-3\lambda s/4}\left(1 - 2e^{-\lambda(1-s)/4}\right).
\end{aligned}$$

Therefore,

$$I_s(\lambda) \geq \frac{1}{2}\, |\Delta| e^{-3\lambda s/4} \quad \text{for } \lambda > \frac{8\ln 2}{1-s} \qquad (s<1) \tag{3.4}$$

and so if k is a sufficiently large integer we have

$$\begin{aligned}
G_s((2k+1)\pi) &< \pi e^{-(2k+1)\pi} - \frac{1}{2}\, |\Delta| e^{-3(2k+1)\pi s/4} \\
&= e^{-(2k+1)\pi}\left(\pi - \frac{1}{2}\, |\Delta| e^{(1-3s/4)(2k+1)\pi}\right) < 0,
\end{aligned}$$

which proves the lemma in the case $s < 1$. For $s > 1$, using (3.4) with $1/s$ instead of s we have:

$$I_s(\lambda) = -I_{1/s}(\lambda s) \leq -\frac{1}{2}\, |\Delta| e^{-3\lambda/4} \quad \text{for } \lambda > \frac{8\ln 2}{s-1}, \qquad s > 1 \tag{3.5}$$

and, if k is a sufficiently large integer,

$$G_s(2\pi k) < \pi e^{-2\pi k} - \frac{1}{2}\, |\Delta| e^{-3\pi k/2} < 0,$$

which completes the proof. □

We now give a proof of Proposition 1.1.

Lemma 3.2. *For every $p \geq 1, p \neq 2$, there exists an $F \in \mathcal{A}$ satisfying*
(i) $\inf\{|F(z)| : |z| \leq 1\} > 0$;
(ii) $E_{0,p}(F) = \|F\|_p$.

Proof. Let

$$\phi(z) := \frac{1}{a} \int_0^z \frac{d\zeta}{\sqrt{1+\zeta^4}}, \quad \text{where } a := \int_0^1 \frac{d\zeta}{\sqrt{1+\zeta^4}},$$

and $F_\lambda(z) := \exp\{\lambda\phi(z)\}$. We remark that $w = \phi(z)$ maps D conformally onto the square $\{w : |\text{Re } w| < 1, |\text{Im } w| < 1\}$ in the w-plane. Obviously, $F_\lambda \in \mathcal{A}$ and satisfies (i). We will prove that if $\lambda = \lambda_{p-1}$ is the number from Lemma 3.1, then F_λ also satisfies (ii).

Clearly, for $\phi(z)$ we have

$$\phi(iz) = i\phi(z) \quad \text{and} \quad \phi(\bar{z}) = \overline{\phi(z)} \qquad \text{for } |z| \le 1, \tag{3.6}$$

and

$$\begin{cases} \phi(e^{i\theta}) = \dfrac{1}{a}\displaystyle\int_0^1 \frac{d\zeta}{\sqrt{1+\zeta^4}} + \frac{1}{a}\int_1^{e^{i\theta}} \frac{d\zeta}{\sqrt{1+\zeta^4}} \\ \qquad = 1 + \dfrac{i}{a}\displaystyle\int_0^\theta \frac{dt}{\sqrt{2\cos 2t}} = 1 + i\mu(\theta), \qquad |\theta| \le \frac{\pi}{4}, \end{cases} \tag{3.7}$$

where $\mu(\theta)$ is an odd, real-valued, continuous increasing function in $[-\pi/4, \pi/4]$, and $\mu(\pi/4) = 1$.

Let

$$\psi(\lambda, p) := \frac{1}{2\pi} \int_{|z|=1} |F_\lambda(z)|^{p-2} F_\lambda(z) |dz|.$$

Using (3.6) and (3.7) we get

$$\begin{cases} \psi(\lambda, p) \\ \quad = \displaystyle\sum_{k=0}^{3} \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} |\exp\{i^k \lambda \phi(e^{i\theta})\}|^{p-2} \exp\{i^k \lambda \phi(e^{i\theta})\} \\ \quad = \displaystyle\sum_{k=0}^{3} \frac{1}{2\pi} \int_0^{\pi/4} (|\exp\{i^k \lambda \phi(e^{i\theta})\}|^{p-2} \exp\{i^k \lambda \phi(e^{i\theta})\} \\ \qquad\qquad + |\exp\{i^k \lambda \overline{\phi(e^{i\theta})}\}|^{p-2} \exp\{i^k \lambda \overline{\phi(e^{i\theta})}\}) d\theta \\ \quad = \dfrac{1}{\pi} \displaystyle\int_0^{\pi/4} e^{\lambda(p-1)} \cos(\lambda\mu(\theta)) d\theta + \frac{1}{\pi} \int_0^{\pi/4} e^{i\lambda} \cosh(\lambda(p-1)\mu(\theta)) d\theta \\ \qquad + \dfrac{1}{\pi} \displaystyle\int_0^{\pi/4} e^{-\lambda(p-1)} \cos(\lambda\mu(\theta)) d\theta + \frac{1}{\pi} \int_0^{\pi/4} e^{-i\lambda} \cosh(\lambda(p-1)\mu(\theta)) d\theta \\ \quad = \dfrac{2}{\pi} \cosh(\lambda(p-1)) \displaystyle\int_0^{\pi/4} \cos(\lambda\mu(\theta)) d\theta \\ \qquad + \dfrac{2\cos\lambda}{\pi} \displaystyle\int_0^{\pi/4} \cosh(\lambda(p-1)\mu(\theta)) d\theta. \end{cases} \tag{3.8}$$

In the special case $p = 2$, we have from (3.8)

$$\psi(\lambda, 2) = \frac{2}{\pi} \cosh(\lambda) \int_0^{\pi/4} \cos(\lambda\mu(\theta)) d\theta + \frac{2}{\pi} \cos\lambda \int_0^{\pi/4} \cosh(\lambda\mu(\theta)) d\theta. \tag{3.9}$$

On the other hand,

$$\psi(\lambda, 2) = \frac{1}{2\pi} \int_{|z|=1} F_\lambda(z) |dz| = F_\lambda(0) = 1. \tag{3.10}$$

So from (3.9) and (3.10) it follows that

$$\frac{2}{\pi}\int_0^{\pi/4} \cos(\lambda\mu(\theta))d\theta = \frac{1}{\cosh\lambda} - \frac{2}{\pi}\frac{\cos\lambda}{\cosh\lambda}\int_0^{\pi/4} \cosh(\lambda\mu(\theta))d\theta$$

and substituting the last expression into (3.8) we get

$$\begin{aligned}\psi(\lambda, p) &= \frac{2}{\pi}\cosh(\lambda(p-1)) \\ &\quad\times\left(\frac{\pi}{2\cosh\lambda} + \cos\lambda\int_0^{\pi/4}\left(\frac{\cosh(\lambda(p-1)\mu(\theta))}{\cosh(\lambda(p-1))} - \frac{\cosh(\lambda\mu(\theta))}{\cosh\lambda}\right)d\theta\right) \\ &= \frac{2}{\pi}\cosh(\lambda(p-1))G_{p-1}(\lambda).\end{aligned}$$

Now, from Lemma 3.1, we have $\psi(\lambda_{p-1}, p) = 0$, and so from (2.1) we deduce that 0 is the constant of best H^p approximation to $F_{\lambda_{p-1}}$. □

Remark 3.3. From the construction above it is clear that $F_\lambda(\bar{z}) = \overline{F_\lambda(z)}$, and therefore the coefficients in the Maclaurin expansion of $F_\lambda(z)$ are all real.

Lemma 3.4. *For every $p \geq 1$, $p \neq 2$, there exists a polynomial $R_p(z)$ with the properties*:
(i) $\min\{|R_p(z)| : |z| \leq 1\} > 0$;
(ii) $E_{0,p}(R_p) = \|R_p\|_p = 1$.

Proof. Let $F = F_{\lambda_{p-1}}$ be the function from Lemma 3.2, and set $M := \inf\{|F(z)| : |z| \leq 1\}$. We know that $M > 0$. Let $S_N(z) \in \Pi_N$ be the polynomial of best uniform approximation to $F(z)$ on D, and d_N the constant of best H^p approximation to $S_N(z)$ so that $E_{0,p}(S_N) = \|S_N - d_N\|_p$.
For N sufficiently large, $\|S_N - F\|_\infty < M/4$ and therefore

$$\text{(3.11)} \qquad \min\{|S_N(z)| : |z| \leq 1\} > \frac{3M}{4} > 0 \quad \text{for } N \geq N_0.$$

On the other hand, we have

$$\begin{aligned}\|F\|_p \leq \|F - d_N\|_p &\leq \|F - S_N\|_p + \|S_N - d_N\|_p \leq \|F - S_N\|_p + \|S_N\|_p \\ &\leq \|F\|_p + 2\|F - S_N\|_p \leq \|F\|_p + 2\|F - S_N\|_\infty.\end{aligned}$$

Therefore $\lim_{N\to\infty}\|F - d_N\|_p = \|F\|_p$ and the uniqueness of the polynomial of best approximation implies $\lim_{N\to\infty} d_N = 0$. Consequently

$$\text{(3.12)} \qquad |d_N| < \frac{M}{4} \quad \text{for } N \geq N_1.$$

Now, from (3.11) and (3.12) it is clear that

$$R_p(z) := \frac{S_N(z) - d_N}{\|S_N - d_N\|_p},$$

where $N \geq \max\{N_0, N_1\}$, satisfies (i) and (ii). □

Remark 3.5. Since $F(z)$ has real coefficients in its Maclaurin expansion (see Remark 3.3), the same is true of the polynomial R_p.

Lemma 3.6. *Let $g_n(z) := g(z^{n+1})$, where $g \in \mathcal{A}$ is an arbitrary function. Then, for any $p \geq 1$,*

$$Q_{n,p}(g_n) \equiv Q_{0,p}(g).$$

Proof. The lemma is essentially proved in [IS, Corollary 2.2] where it is shown that

$$E_{n,p}(g_n) = E_{0,p}(g).$$

Since $\|g_n - c\|_p = \|g - c\|_p$ for any constant c, we get

$$\|g_n - Q_{0,p}(g)\|_p = E_{0,p}(g) = E_{n,p}(g_n),$$

and this completes the proof, since the polynomial of best approximation is unique. □

Lemma 3.7. *For every $p \geq 1$, $p \neq 2$, there exists a function $\Phi \in \mathcal{A}$ satisfying*

$$\limsup_{n\to\infty} \min_{|z|\leq 1} \frac{|\Phi(z) - Q_n(\Phi, z)|}{E_{n,p}(\Phi)} > 0, \tag{3.13}$$

where $Q_n(\Phi, \cdot\,) = Q_{n,p}(\Phi, \cdot\,) \in \Pi_n$ is the polynomial of best H^p approximation to Φ. ($E_{n,p}(\Phi) = \|\Phi - Q_n\|_p$)

Proof. Let $R = R_p$ be the polynomial from Lemma 3.4 and let

$$s := \deg R, \quad n_j := (2s)^j - 1, \quad \text{and} \quad A := \inf\{|R(z)| : |z| \leq 1\}.$$

For any integer n we set $R_n(z) := R(z^{n+1})$ and

$$\Phi_\alpha(z) := \sum_{j=0}^{\infty} \alpha^j R_{n_j}(z), \quad 0 < \alpha < 1. \tag{3.14}$$

We shall prove that if α is sufficiently small, then (3.13) holds for $\Phi(z) = \Phi_\alpha(z)$.

The series in (3.14) converges uniformly on $\bar{D}$ if $0 < \alpha < 1$, and therefore $\Phi_\alpha \in \mathcal{A}$. Moreover, $\|\Phi_\alpha - R\|_\infty \to 0$, as $\alpha \to 0$, which implies that

$$\exists \alpha_0 > 0 : \|\Phi_\alpha - R\|_\infty < \frac{A}{4} \quad \text{for } 0 < \alpha < \alpha_0 \tag{3.15}$$

and

$$\exists \alpha_1 > 0 : |Q_0(\Phi_\alpha)| < \frac{A}{4} \quad \text{for } 0 < \alpha < \alpha_1. \tag{3.16}$$

Now, let $\Phi(z) = \Phi_\alpha(z)$, where $0 < \alpha < \min\{\alpha_0, \alpha_1, 1\}$. From (3.15), (3.16), and the definition of the constant A, we have

$$|\Phi(z) - Q_0(\Phi)| > \frac{1}{2} A \quad \forall\, |z| \leq 1. \tag{3.17}$$

Furthermore,

$$
(3.18)\quad \begin{cases} \Phi(z) - Q_{n_k}(\Phi, z) = \sum_{j=0}^{k-1} \alpha^j R_{n_j}(z) + \sum_{j=k}^{\infty} \alpha^j R_{n_j}(z) \\ \qquad\qquad - Q_{n_k}\left(\sum_{j=0}^{k-1} \alpha^j R_{n_j} + \sum_{j=k}^{\infty} \alpha^j R_{nj}, z \right) \\ \qquad = \sum_{j=k}^{\infty} \alpha^j R_{n_j}(z) - Q_{n_k}\left(\sum_{j=k}^{\infty} \alpha^j R_{n_j}, z \right), \end{cases}
$$

since $\deg(\sum_{j=0}^{k-1} \alpha^j R_{n_j}) = s(n_{k-1} + 1) \leq n_k$. Also,

$$
\begin{aligned}
\sum_{j=k}^{\infty} \alpha^j R_{n_j}(z) &= \alpha^k \sum_{j=0}^{\infty} \alpha^j R_{n_{j+k}}(z) \\
&= \alpha^k \sum_{j=0}^{\infty} \alpha^j R_{n_j}(z^{n_k+1}) \\
&= \alpha^k \Phi(z^{n_k+1}).
\end{aligned}
$$

Substituting the last expression into (3.18) and using Lemma 3.6 we get

$$
(3.19)\quad \begin{cases} \Phi(z) - Q_{n_k}(\Phi, z) = \alpha^k(\Phi(z^{n_k+1}) - Q_{n_k}(\Phi(\zeta^{n_k+1}), z)) \\ \qquad\qquad = \alpha^k(\Phi(z^{n_k+1}) - Q_0(\Phi)). \end{cases}
$$

Finally, from (3.17) and (3.19) it follows that

$$
\begin{aligned}
\limsup_{n\to\infty} \min_{|z|\leq 1} \frac{|\Phi(z) - Q_n(\Phi, z)|}{E_{n,p}(\Phi)} &\geq \limsup_{k\to\infty} \min_{|z|\leq 1} \frac{|\Phi(z) - Q_{n_k}(\Phi, z)|}{E_{n_k,p}(\Phi)} \\
&= \limsup_{k\to\infty} \min_{|z|\leq 1} \frac{|\Phi(z^{n_k+1}) - Q_0(\Phi)|}{E_{0,p}(\Phi)} \\
&\geq \frac{A/2}{E_{0,p}(\Phi)} > 0. \qquad \square
\end{aligned}
$$

Remark 3.8. From (3.14) and Remark 3.5 it is clear that the coefficients in the Maclaurin expansion of Φ are real.

Proof of Theorem 2.1. The proof is similar to that of Theorem 1 in [ST]. Let

$$
\tau_p := \sup_{f\in\mathcal{A}} \limsup_{n\to\infty} \min_{|z|\leq 1} \frac{|f(z) - Q_n(f, z)|}{E_{n,p}(f)}.
$$

It follows from Lemma 3.7 that $\tau_p > 0$ and obviously (i) holds.
To prove (ii), we put

$$
S_{N,m}^p := \left\{ f \in \mathcal{A} : \min_{|z|\leq 1} \frac{|f(z) - Q_n(f, z)|}{E_{n,p}(f)} \leq \tau_p\left(1 - \frac{1}{m}\right),\ \forall n \geq N \right\}.
$$

Then $S_p = \bigcup_{m=1}^{\infty} \bigcup_{N=1}^{\infty} S_{N,m}^p$. It is easy to verify that the $S_{N,m}^p$ are closed (with respect to the uniform norm) subsets of $\mathcal{A}$.

If S_p is not of the first category, then for some m and N the set $S_{N,m}^p$ is not nowhere dense in $\mathcal{A}$. Therefore, there is an $f_0 \in \mathcal{A}$ and $\delta > 0$ such that the δ-neighborhood of f_0 in $\mathcal{A}$ is contained in $S^p_{N,m}$. Let P_0 be a polynomial satisfying $\|f_0 - P_0\|_\infty < \delta/2$ and let $M := \max\{N, \deg P_0\}$. For an arbitrary $f \in \mathcal{A}$ $(f \not\equiv 0)$, let $f^*(z) := P_0(z) + \delta f(z)/(2\|f\|_\infty)$. Then

$$\|f^* - f_0\|_\infty \leq \|P_0 - f_0\|_\infty + \frac{\delta}{2} < \delta;$$

hence $f^* \in S^p_{N,m} \subset S^p_{M,m}$, i.e.,

$$\min_{|z|\leq 1} \frac{|f^*(z) - Q_n(f^*, z)|}{E_{n,p}(f^*)} \leq \tau_p\left(1 - \frac{1}{m}\right), \quad \forall n \geq M. \tag{3.20}$$

On the other hand, for $n \geq M \geq \deg P_0$ we have

$$Q_n(f^*) = P_0 + \frac{\delta}{2} \frac{Q_n(f)}{2\|f\|_\infty},$$

that is,

$$\frac{|f(z) - Q_n(f, z)|}{E_{n,p}(f)} = \frac{|f^*(z) - Q_n(f^*, z)|}{E_{n,p}(f^*)}.$$

Thus, from (3.20) and the arbitrariness of $f \in \mathcal{A}$, we get $\mathcal{A} \subset S^p_{M,m}$, i.e.,

$$\sup_{f \in \mathcal{A}} \limsup_{n\to\infty} \min_{|z|\leq 1} \frac{|f(z) - Q_n(f, z)|}{E_{n,p}(f)} \leq \tau_p\left(1 - \frac{1}{m}\right),$$

which contradicts the definition of τ_p, and this proves the theorem. □

Lemma 3.9. *For any $p \neq 2$, $1 \leq p < \infty$, there exists an entire function f with*

$$\limsup_{n\to\infty} \min_{|z|\leq 1} \frac{|f(z) - Q_{n,p}(f, z)|}{E_{n,p}(f)} > 0. \tag{3.21}$$

Proof. The proof is similar to that of Lemma 3.7, so we merely sketch it. Let

$$f_\alpha(z) := \sum_{j=0}^{\infty} \varepsilon_j R_{n_j}(z) = \sum_{j=0}^{\infty} \alpha^j (j+1)^{-s(n_j+1)} R_{n_j}(z) \quad (0 < \alpha < 1),$$

where n_j and R_{n_j} are as in the proof of Lemma 3.7. Since $\deg(R_{n_j}) = s(n_j + 1)$, the series above converges uniformly on any compact subset of $\boldsymbol{C}$, i.e., $f_\alpha(z)$ is an entire function. Furthermore,

$$\begin{aligned}
&f_\alpha - Q_{n_k,p}(f_\alpha) \\
&= \varepsilon_k \left(R_{n_k} + \sum_{j=1}^{\infty} \frac{\varepsilon_{k+j}}{\varepsilon_k} R_{n_{k+j}} - Q_{n_k,p}\left(R_{n_k} + \sum_{j=1}^{\infty} \frac{\varepsilon_{k+j}}{\varepsilon_k} R_{n_{k+j}} \right)\right) \\
&= \varepsilon_k \left(R_{n_k} + \sum_{j=1}^{\infty} \frac{\varepsilon_{k+j}}{\varepsilon_k} R_{n_{k+j}} - Q_{0,p}\left(R + \sum_{j=1}^{\infty} \frac{\varepsilon_{k+j}}{\varepsilon_k} R_{n_j} \right)\right).
\end{aligned}$$

Since $\sum_{j=1}^{\infty} (\varepsilon_{k+j}/\varepsilon_k) < \alpha/(1-\alpha)$, the continuity of the operator of best approximation for α sufficiently small then yields (3.21). □

Proof of Theorem 2.4. From Lemma 3.9 we have

$$\sigma_p := \sup_{f \in \mathcal{E}} \limsup_{n \to \infty} \min_{|z| \le 1} \frac{|f(z) - Q_{n,p}(f, z)|}{E_{n,p}(f)} > 0.$$

Since the convergence with respect to the metric on $\mathcal{E}$ implies uniform convergence on $\bar{D}$, the proof of Theorem 2.1 works here. We only need to show that, for given $f_0 \in \mathcal{E}$ and $\delta > 0$, there exists a polynomial P_0 with $d(f_0, P_0) < \delta/2$. But,

$$\begin{aligned} d(f_0, P_0) &\le \sum_{n=1}^{J} 2^{-n} \frac{\|f_0 - P_0\|_{\infty,n}}{\|f_0 - P_0\|_{\infty,n} + 1} + \sum_{n=J+1}^{\infty} 2^{-n} \\ &\le \frac{\|f_0 - P_0\|_{\infty,J}}{\|f_0 - P_0\|_{\infty,J} + 1}(1 - 2^{-J}) + 2^{-J} \\ &\le \|f_0 - P_0\|_{\infty,J} + 2^{-J}, \end{aligned}$$

and so we can first choose J large enough and then select a polynomial P_0 that is a sufficiently good approximation to f_0 on $\{z : |z| \le J\}$. Finally, for $f \in \mathcal{E}$ ($f \not\equiv 0$), we set $f^* = P_0 + \delta f/(2\|f\|_{\infty,K})$ with K sufficiently large and proceed as in the proof of Theorem 2.1. □

Proof of Theorem 2.5. We just need to replace τ_p in the proof of Theorem 2.1 with κ_p, where

$$\kappa_p := \sup_{f \in \mathcal{A}^R} \limsup_{n \to \infty} \min_{|z| \le 1} \frac{|f(z) - Q_{n,p}(f, z)|}{E_{n,p}(f)}.$$

(It follows from Lemma 3.7 and Remark 3.8 that $\kappa_p > 0$.) □

APPENDIX

In this section we present a detailed and self-contained proof of the uniqueness of best H^1 polynomial approximants, a fact that also follows from a general uniqueness theorem due to Havinson [H, Theorem 12].

Theorem A.1. *For any function $\omega \in H^1 \backslash \Pi_n$ there exists a bounded function α on $|z| = 1$ with the following properties*:
(i) $\int_{|z|=1} \varphi(z)\alpha(z)|dz| = 0, \ \forall \varphi \in \Pi_n$;
(ii) $\|\omega - \varphi^*\|_1 = E_{n,1}(\omega) \Leftrightarrow \alpha(z) = \overline{\operatorname{sgn}}(\omega(z) - \varphi^*(z))$ *a.e. on* $|z| = 1$.

Proof. Let $\omega \in H^1 \backslash \Pi_n$ be an arbitrary function and set

(A.1) $\quad d := E_{n,1}(\omega)$.

Then $d > 0$ and hence there exists a bounded linear functional F on H^1 such that

(A.2) $\quad F(\omega) = d$,

(A.3) $\quad F(\varphi) = 0, \quad \forall \varphi \in \Pi_n$,

(A.4) $\quad \|F\| = 1$.

Let $\alpha(z)$ be the function with the property

$$(\text{A.5}) \qquad F(p) = \frac{1}{2\pi} \int_{|z|=1} p(z)\alpha(z)|dz| \quad \forall p \in H^1.$$

Then

$$(\text{A.6}) \qquad \|F\| = \|\alpha\|_\infty = 1,$$

where now $\|\alpha\|_\infty$ is the essential supremum taken on $|z| = 1$. It follows from (A.3) and (A.5) that for this choice of $\alpha(z)$, condition (i) holds.

To establish (ii), first let φ^* be a polynomial in Π_n of best H^1 approximation to ω (i.e., the left-hand equality in (ii) holds). Then (A.1), (A.2), (A.3), (A.5) and (A.6) yield

$$\begin{aligned} d = F(\omega) = F(\omega - \varphi^*) &= \frac{1}{2\pi} \int_{|z|=1} (\omega(z) - \varphi^*(z))\alpha(z)|dz| \\ &\leq \frac{1}{2\pi} \int_{|z|=1} |\omega(z) - \varphi^*(z)||\alpha(z)||dz| \\ &\leq \|\alpha\|_\infty \frac{1}{2\pi} \int_{|z|=1} |\omega(z) - \varphi^*(z)||dz| = d. \end{aligned}$$

Therefore, equality holds throughout and we have

$$(\text{A.7}) \qquad |\alpha(z)| = \|\alpha(z)\|_\infty = 1 \text{ a.e. on } |z| = 1,$$

and

$$(\text{A.8}) \qquad \alpha(z)(\omega(z) - \varphi^*(z)) = |\omega(z) - \varphi^*(z)| \text{ a.e. on } |z| = 1,$$

since $\omega - \varphi^* \in H^1$ is different from zero a.e. on $|z| = 1$. From (A.8) we get

$$\alpha(z) = \overline{\text{sgn}}(\omega(z) - \varphi^*(z)) \text{ a.e. on } |z| = 1,$$

and so the right-hand equality in (ii) holds.

Conversely let $\varphi^* \in \Pi_n$ be such that $\alpha(z) = \overline{\text{sgn}}(\omega(z) - \varphi^*(z))$ a.e. on $|z| = 1$. Then with (A.1), (A.3), and (A.5) we have

$$\begin{aligned} E_{n,1}(\omega) = d = F(\omega) = F(\omega - \varphi^*) &= \frac{1}{2\pi} \int_{|z|=1} (\omega(z) - \varphi^*(z))\alpha(z)|dz| \\ &= \frac{1}{2\pi} \int_{|z|=1} |\omega(z) - \varphi^*(z)||dz| = \|\omega - \varphi^*\|_1, \end{aligned}$$

i.e., φ^* is a polynomial of best approximation. □

Theorem A.2. *For any function $\omega \in H^1$, the polynomial of best H^1 approximation to ω out of Π_n is unique.*

Proof. Let $\omega \in H^1 \backslash \Pi_n$ and suppose that φ_1 and $\varphi_2 \in \Pi_n$ $(\varphi_1 \not\equiv \varphi_2)$ are polynomials of best approximation to ω, i.e.,

$$E_{n,1}(\omega) = \|\omega - \varphi_1\|_1 = \|\omega - \varphi_2\|_1.$$

Let

$$\varphi^*(z) := \varphi_1(z) - \varphi_2(z)$$

and

$$Z := \{z : |z| = 1, \varphi^*(z) = 0\};$$

and let $\alpha(z)$ be the function defined in Theorem A.1. Then

(A.9) $\quad \alpha(z)(\omega(z) - \varphi_i(z)) = |\omega(z) - \varphi_i(z)|$ a.e. on $|z| = 1, \quad (i = 1, 2)$

and

(A.10) $\quad |\alpha(z)| = 1$ a.e. on $|z| = 1$.

Hence

$$\alpha(z)\varphi^*(z) = |\omega(z) - \varphi_2(z)| - |\omega(z) - \varphi_1(z)| \text{ a.e.},$$

which means that $\alpha(z)\varphi^*(z)$ is real-valued almost everywhere on $|z| = 1$ and with (A.10) we get

(A.11) $\quad \alpha(z)\varphi^*(z) = \pm|\varphi^*(z)|$ a.e. on $|z| = 1$.

Now, (A.9) and (A.11) imply that

(A.12) $$\alpha(z) = \frac{|\omega(z) - \varphi_1(z)|}{\omega(z) - \varphi_1(z)} = \pm\frac{|\varphi^*(z)|}{\varphi^*(z)} \text{ a.e.}$$

and hence $(\omega(z) - \varphi_1(z))/\varphi^*(z)$ is real-valued almost everywhere on $|z| = 1$. Let us consider the function

$$T(z) := \frac{\omega(z) - \varphi_1(z)}{\varphi^*(z)}.$$

$T(z)$ is meromorphic in the unit disk D with a finite number of poles and is real-valued almost everywhere on the unit circle $\{z : |z| = 1\}$. Since $\omega \in H^1$, $\varphi_1, \varphi^* \in \Pi_n$, the function $T(z)$ can be represented by its Cauchy integral along any closed curve in $\bar{D}$ that is free of zeros of φ^* and encloses a domain free of zeros of φ^*. Hence, by the reflection principle, $T(z)$ has an analytic extension through any open arc that is free of the zeros of the denominator. In particular, $T(z)$ is continuous on $\{z : |z| = 1\}\backslash Z$ and, therefore, $\omega(z)$ is continuous on $\{z : |z| \le 1\}\backslash Z$.

Without loss of generality we suppose that $\omega(z) - \varphi_1(z) \neq 0$ on $\{z : |z| = 1\}\backslash Z$. If this is not the case, we can replace φ_1 with $\phi_1 = t\varphi_1 + (1-t)\varphi_2$, $0 < t < 1$. Indeed, from (A.9) and (A.10) and the continuity of $\omega(z)$ we get

$$\begin{aligned} &|\omega(z) - \phi_1(z)| \\ &\quad = t|\omega(z) - \varphi_1(z)| + (1-t)|\omega(z) - \varphi_2(z)| \neq 0 \text{ on } \{z : |z| = 1\}\backslash Z. \end{aligned}$$

Also, $\phi_1 - \varphi_2 = t(\varphi_1 - \varphi_2) = t\varphi^*$, so that the zero set of $\phi_1 - \varphi_2$ is again Z. Furthermore, without loss of generality we can suppose that

$$\alpha(z) \equiv \overline{\operatorname{sgn}}(\omega(z) - \varphi_1(z)).$$

Then $\alpha(z)$ is continuous on $\{z : |z| = 1\}\backslash Z$. But $|\varphi^*(z)|/\varphi^*(z)$ is also continuous on $\{z : |z| = 1\}\backslash Z$. Therefore, the change of the sign in (A.12) is possible only at the zeros of φ^*.

Let $\{\alpha_1, \alpha_2, \ldots, \alpha_r\} \subset Z$ be the points at which the sign in (A.12) changes. It is clear that $r \le n$ and r is even, say $r = 2p$. Define δ by

$$e^{i\delta} := \overline{\alpha_1}\,\overline{\alpha_2}\cdots\overline{\alpha_r},$$

and let

$$R(z) := ae^{-i\delta/2}[z^p(z-\beta_1)\cdots(z-\beta_s)(z-\alpha_1)^{k_1-1}\cdots(z-\alpha_r)^{k_r-1}],$$

where

$$\varphi^*(z) = a(z-\beta_1)\cdots(z-\beta_s)(z-\alpha_1)^{k_1}\cdots(z-\alpha_r)^{k_r}.$$

Then $R(z)$ is a polynomial of degree at most n and

$$\begin{aligned}\alpha(z)\,R(z) &= \pm\frac{a|\varphi^*(z)|}{\varphi^*(z)}\,\frac{e^{-i\delta/2}z^p\varphi^*(z)}{a(z-\alpha_1)\cdots(z-\alpha_r)}\\ &= \pm|\varphi^*(z)|\,\frac{e^{-i\delta/2}z^p}{(z-\alpha_1)\cdots(z-\alpha_r)}\\ &= \pm|\varphi^*(z)|u(z),\end{aligned}$$

where

$$u(z) := \frac{e^{-i\delta/2}z^p}{(z-\alpha_1)\cdots(z-\alpha_r)}.$$

It is easy to see that $\overline{u(z)} = u(z)$ for $|z| = 1$. Therefore, $u(z)$ is real-valued on the unit circle, and obviously it changes its sign at the points $\alpha_1, \alpha_2, \ldots, \alpha_r$. Therefore, $\alpha(z)\,R(z)$ is real-valued and does not change its sign on $\{z : |z| = 1\}$. Hence

$$\int_{|z|=1} \alpha(z)\,R(z)|dz| \neq 0,$$

which contradicts Theorem A.1. □

REFERENCES

[A] Al'per, S.Ja. – Best approximation of functions represented by lacunary Fourier and Taylor series. Izv. Akad. Nauk SSSR, Ser. Mat. **27**, 747–760 (1963).

[B] Bernstein, S.N. – On periodic functions, for which the best convergent series (of trigonometric polynomials) is the Fourier series. Trans. Leningrad Industr. Inst., Sect. Phys. and Math. **10**, 3–8 (1936). See also Collected Works, vol. **2**, Izdat. Akad. Nauk SSSR, Moscow, 178–183 (1954).

[H] Havinson, S.Ia. – On uniqueness of functions of best approximation in the metric of the space L_1. Izv. Akad. Nauk USSR, Ser. Mat. **22**, 243–270 (1958).

[IS] Ivanov, K.G. and E.B. Saff – Nongeometric convergence of best L_p $(p \neq 2)$ polynomial approximants. Proc. Amer. Math. Soc. **110**, 377–382 (1990).

[S] Shapiro, H.S. – Topics in Approximation Theory. Lecture Notes in Math. Vol. **187**, Springer-Verlag, Berlin (1971).

[ST] Saff, E.B. and V. Totik – Behavior of polynomials of best uniform approximation. Trans. Amer. Math. Soc. **316**, 567–593 (1989).