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HYPERPOLYNOMIAL APPROXIMATION OF SOLUTIONS OF NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS

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Consider the integro-differential equation

$$(*) \qquad U(x) \equiv x' + A(t, x) + \int_a^t F(t, s, x(s))ds = T(t), t \in [a, b]$$

subject to the initial condition

$$(**) \qquad x(a) = h .$$

Then a problem in approximation theory is whether a solution $x(t)$ of $((*), (**))$ can be approximated, uniformly on $[a, b]$, by a sequence of polynomials P_n, which satisfy $(**)$ and minimize the expression $\|T(\cdot) - U(P_n)\|$, where $\|\cdot\|$ is a certain norm. It is shown here that such a sequence of minimizing polynomials, or, more generally, hyperpolynomials, exists with respect to the L_p-norm $(1 < p \leqq \infty)$ and converges to $x(t)$, uniformly on $[a, b]$, under the mere assumption of existence and uniqueness of $x(t)$.

The results of this paper are intimately related to those of Stein [11], who studied the approximation of solutions of scalar linear integro-differential equations of the form

$$(1) \qquad W(x) \equiv L(x) - \int_a^b h(t, s)x(s)ds = f(t) ,$$

$(L(x) \equiv x^{(m)}(t) + f_1(t)x^{(m-1)}(t) + \cdots + f_m(t)x(t))$ subject to the two-point boundary conditions:

$$(2) \qquad W_i(x) \equiv A_i(x) + B_i(x) + \int_a^b V_i(t)x(t)dt = 0 , \quad i = 1, 2, \cdots, m$$

where $A_i(u) \equiv \sum_{k=1}^m a_{ik}u^{(k-1)}(a)$, $B_i(u) \equiv \sum_{k=1}^m b_{ik}u^{(k-1)}(b)$. Namely, he showed that under certain condition on L, h, f, if $x(t)$ is the unique solution of (1), which satisfies the linearly independent boundary conditions (2), then for every $n \geqq 2m - 1$ there exists a unique polynomial p_n of degree at most n, which satisfies (2) and best approximates the solution of (1) with respect to the L_p-norm $(1 \leqq p < \infty)$. He then considered the convergence of the sequences $\{p_n^{(k)}\}$, $k = 1, 2, \cdots, m - 1$ to the solution $x(t)$ and its derivatives up to the order $m - 1$ respectively. Extension of these results were also made for trigonometric polynomials, or linear combinations of orthonormal functions. The present paper extends the results of Stein and has points of contact with the rest of the papers in the references.

1. Preliminaries. Let $R = (-\infty, +\infty)$. For the system ((*), (**)) we assume the following: $A(t, u)$ is an m-vector of functions defined and continuous on $[a, b]XR^m$. $F(t, s, u)$ is an m-vector of functions defined and continuous on the set $S \equiv \{(t, s, u) \in [a, b]X[a, b]XR^m; s \leqq t\}$. $T(t)$ is an m-vector of functions defined and continuous on $[a, b]$.

Let B_k, $k = 1, m$, be the Banach space of all k-vectors of continuous functions on $[a, b]$ with norm

$$||f||_{B_k} = \sup_{t \in [a,b]} ||f(t)|| ,$$

where, for a vector $u \in R^k$, $||u|| = \max_{1 \leq i \leq k} |u_i|$. By B'_k we denote the Banach space of all functions $f \in B_k$ which are continuously differentiable on $[a, b]$. The norm now is

$$||f||_{B'_k} = \max_{i=0,1} \{||f^{(i)}||_{B_k}\} .$$

A sequence $\{g_n\}$ of functions in B'_1 is said to be linearly independent if every finite number of the g_n's is linearly independent on $[a, b]$. A linearly independent sequence $\{g_n\}$ is said to be a *d-sequence* if the set of all finite linear combinations of the g_n's is dense in B'_1. For each $i = 1, 2, \cdots, m$ let $\{g_{n,i}\}_{n=1}^{\infty}$ be a fixed d-sequence in B'_1. We assume without loss of generality that $g_{1,i}(a) \neq 0$, $i = 1, 2, \cdots, m$. By a *hyperpolynomial* of degree at most j we mean a function p of the form

$$p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{bmatrix} = \begin{bmatrix} c_{1,1}g_{1,1} + c_{2,1}g_{2,1} + \cdots + c_{j,1}g_{j,1} \\ c_{1,2}g_{1,2} + c_{2,2}g_{2,2} + \cdots + c_{j,2}g_{j,2} \\ \vdots \qquad \vdots \qquad \vdots \\ c_{1,m}g_{1,m} + c_{2,m}g_{2,m} + \cdots + c_{j,m}g_{j,m} \end{bmatrix} .$$

By Π_n we denote the set of all hyperpolynomials of degree at most n which satisfy the initial condition (**). For a function $f \in B_m$ we put

$$||f||_p = \left[\int_a^b ||f(t)||^p dt\right]^{1/p} , \quad 1 \leqq p < +\infty .$$

We also make use of the symbol $||f||_\infty$ instead of $||f||_{B_m}$.

2. Main results.

THEOREM 1. *Let $1 < p \leqq \infty$ and suppose that the system ((*), (**)) has a unique*[1] *solution $x(t)$ defined on $[a, b]$. Then for each n suffi-*

[1] Uniqueness means that any solution of ((*), (**)) which is defined on a subinterval $[a, c]$ of $[a, b]$ must coincide with $x(t)$.

ciently large there exists a hyperpolynomial $Q_n \in \Pi_n$ *such that*

$$\| T - U(Q_n) \|_p = \inf_{P \in \Pi_n} \| T - U(P) \|_p \,. \tag{3}$$

Furthermore, the sequence $Q_n(t)$ *converges uniformly to* $x(t)$ *on* $[a, b]$. *For the case* $p = \infty$ *we have, in addition, that the sequence* $Q'_n(t)$ *converges uniformly to* $x'(t)$ *on* $[a, b]$.

The proof requires the following lemmas:

LEMMA 1. *The set of all hyperpolynomials is dense in* B'_m.

Proof. Obvious.

LEMMA 2. *Let* $f \in B'_m$ *satisfy* (**). *Then there exists a sequence of hyperpolynomials* $p_n \in \Pi_n$, $n = 1, 2, \cdots$, *such that*

$$\lim_{n \to \infty} \| f - p_n \|_{B'_m} = 0 \,. \tag{4}$$

Proof. By Lemma 1 there exists a sequence $\{q_n\}$ of hyperpolynomials such that

$$\lim_{n \to \infty} \| f - q_n \|_{B'_m} = 0 \,. \tag{5}$$

We can (and do) assume that each q_n is of degree at most n, respectively, where $n = 1, 2, \cdots$.

Put $d_n \equiv h - q_n(a)$ and let $d_{n,i}$ be the ith component of d_n. Set

$$s_n(t) \equiv \begin{bmatrix} c_{n,1} g_{1,1}(t) \\ c_{n,2} g_{1,2}(t) \\ \vdots \\ c_{n,m} g_{1,m}(t) \end{bmatrix},$$

where $c_{n,i} \equiv d_{n,i}/g_{1,i}(a)$. Since

$$\| d_n \| = \| h - q_n(a) \| = \| f(a) - q_n(a) \| \longrightarrow 0 \text{ as } n \longrightarrow \infty \,,$$

it follows that

$$\lim_{n \to \infty} c_{n,i} = 0, \quad \text{for each} \quad i = 1, 2, \cdots, m \,.$$

Hence

$$\| s_n \|_{B'_m} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty \,. \tag{6}$$

Now define $p_n(t) \equiv q_n(t) + s_n(t)$. Then

$$p_n(a) = q_n(a) + s_n(a) = q_n(a) + d_n = h \,,$$

and so $p_n \in \Pi_n$ for each $n = 1, 2, \cdots$. From (5) and (6) it follows that

$$\lim_{n\to\infty} \| f - p_n \|_{B'_m} = 0 .$$

LEMMA 3. *Let*

$$\mu_{n,p} \equiv \inf_{P \in \Pi_n} \| T - U(P) \|_p .$$

Then $\mu_{n,p} \to 0$ *as* $n \to \infty$.

Proof. It suffices to show that $\mu_{n,\infty} \to 0$ as $n \to \infty$. By Lemma 2 there exists a sequence $p_n \in \Pi_n$, $n = 1, 2, \cdots$, such that

$$\lim_{n\to\infty} \| x - p_n \|_{B'_m} = 0 .$$

Since $x(t)$ satisfies (*) we deduce that

$$(7) \quad \mu_{n,\infty} \leqq \| T - U(p_n) \|_\infty \leqq \| x' - p'_n \|_\infty + \| A(\cdot, x) - A(\cdot, p_n) \|_\infty + (b - a) \max_{a \leqq s \leqq t \leqq b} \| F(t, s, x(s)) - F(t, s, p_n(s)) \| .$$

Obviously $\| x'_n - p'_n \|_\infty \leqq \| x_n - p_n \|_{B'_m} \to 0$ as $n \to \infty$. Also from the uniform convergence of the p_n to x and the continuity of the functions A and F it follows that the last two terms in the right-hand member of (7) tend to zero as $n \to \infty$. This proves Lemma 3.

LEMMA 4. *If* $P_n \in \Pi_n$ *is a sequence of hyperpolynomials such that*

$$(8) \quad \lim_{n\to\infty} \| T - U(P_n) \|_p = 0 , \qquad 1 < p \leqq \infty ,$$

then the $P_n(t)$ *converge uniformly to* $x(t)$ *on* $[a, b]$. *For the case* $p = \infty$ *we have, in addition, that the derivatives* $P'_n(t)$ *converge uniformly to* $x'(t)$ *on* $[a, b]$.

Proof. The proof is similar, but not identical, to that of [2, Thm. 3, p. 17]. We shall sketch the argument for the real line only.

Let M be a constant such that $|x(t)| < M$ for all $t \in [a, b]$. Note that $|h| = |x(a)| < M$. Set $\mathscr{R} \equiv [a, b] X [-M, M]$. Since the norms $\| U(P_n) \|_p$ are uniformly bounded, and the functions $A(t, u)$ and $F(t, s, u)$ are continuous, there exist constants K_1 and K_2 such that

$$\int_a^b | U(P_n)(t) - A(t, u) |^p dt \leqq K_1^p, \; u \in [-M, M] ,$$

$$| F(t, s, u) | \leqq K_2 , \qquad a \leqq s \leqq t \leqq b, \; u \in [-M, M] .$$

Let $K \equiv K_1 + K_2(b - a)^{1+1/p}$, and consider the curves C_1: $u = h + K(t - a)^{1/q}$, C_2: $u = h - K(t - a)^{1/q}$, where q satisfies the equation $1/p + 1/q = 1$. Let t_i^*, $a < t_i^* \leqq b$, $i = 1, 2$, be the abscissa of the second point of intersection of the curve C_i with the boundary of the rectangle $\mathscr{R}$. Put $t^* \equiv \min(t_1^*, t_2^*)$. We shall show that for each n there holds

$$|P_n(t)| \leqq M,\ t \in [a, t^*] . \tag{9}$$

Let t_n be the abscissa of the first point to the right of a at which the graph of $P_n(t)$ intersects the boundary of $\mathscr{R}$. Integrating the equation

$$P_n'(t) = U(P_n)(t) - A(t, P_n(t)) - \int_a^t F(t, s, P_n(s))ds \tag{10}$$

from a to t_n, we deduce that

$$\begin{aligned} |P_n(t_n) - h| &\leqq \int_a^{t_n} |U(P_n)(t) - A(t, P_n(t))|dt + \int_a^{t_n}\int_a^t |F(t, s, P_n(s))|dsdt \\ &\leqq \left[\int_a^{t_n} |U(P_n)(t) - A(t, P_n(t))|^p dt\right]^{1/p} (t_n - a)^{1/q} \\ &\quad + K_2(b - a)(t_n - a) \\ &\leqq K_1(t_n - a)^{1/q} + K_2(b - a)^{1+1/p}(t_n - a)^{1/q} = K(t_n - a)^{1/q} . \end{aligned}$$

Hence the point $(t_n, P_n(t_n))$ lies between the curves C_1 and C_2. Thus $t_n \geqq t^*$, which proves (9).

It also follows from integrating the equation (10) that the sequence $P_n(t)$ is equicontinuous on $[a, t^*]$. Therefore, by Ascoli's Theorem, each subsequence of the $P_n(t)$ possesses a subsequence which converges uniformly on $[a, t^*]$. Suppose that $y(t)$ is the uniform limit on $[a, t^*]$ of the subsequence $P_k(t)$. From (8) and Hölder's inequality it follows that

$$\lim_{k\to\infty} \int_a^t U(P_k)(\tau)d\tau = \int_a^t T(\tau)d\tau , \quad t \in [a, b] . \tag{11}$$

Taking the limit as $k \to \infty$ in the equation

$$P_k(t) - h = \int_a^t U(P_k)(\tau)d\tau - \int_a^t A(\tau, P_k(\tau))d\tau - \int_a^t\int_a^\tau F(\tau, s, P_k(s))dsd\tau ,$$

we deduce from (11) and the continuity of the functions A and F that

$$y(t) - h = \int_a^t T(\tau)d\tau - \int_a^t A(\tau, y(\tau))d\tau - \int_a^t\int_a^\tau F(\tau, s, y(s))dsd\tau ,$$

for $t \in [a, t^*]$. Thus $y(t)$ satisfies the system ((*), (**)) on $[a, t^*]$ and so must equal $x(t)$ on this interval. Since $y(t)$ was an arbitrarily chosen limit function, the original sequence $P_n(t)$ must converge to $x(t)$ uniformly on $[a, t^*]$.

Considering the fact that the proof given above carries over under the more general hypothesis that the initial values of the $P_n(t)$ converge to the corresponding initial value of $x(t)$, one can show, as in the proof of [2, Thm. 3, p. 17], that the sequence $P_n(t)$ converges to $x(t)$ uniformly on $[a, b]$.

For the case $p = \infty$ it follows immediately from equation (10) that $\lim_{n\to\infty} P'_n(t) = x'(t)$ uniformly on $[a, b]$.

Proof of Theorem 1. It is clear from Lemmas 3 and 4 that if the minimizing hyperpolynomials Q_n exist, then they have the asserted convergence properties.

We first show that if Q_k does not exist, then there is a hyperpolynomial $P_k \in \Pi_k$ such that

$$\|T - U(P_k)\|_p < \mu_{k,p} + 1/k \,, \tag{12}$$

and

$$\|P_k\|_\infty > k \,. \tag{13}$$

If this were not the case, there exists a sequence of hyperpolynomials $\pi_j \in \Pi_k$ such that

$$\|T - U(\pi_j)\|_p \longrightarrow \mu_{k,p} \quad \text{as} \quad j \longrightarrow \infty \,, \tag{14}$$

and

$$\|\pi_j\|_\infty \leqq k \,, \quad \forall j \,.$$

It is not difficult to show that the set $\{\pi \in \Pi_k \mid \|\pi\|_\infty \leqq k\}$ is compact in the B'_m norm. Hence there is a subsequence of the π_j which converges in the B'_m norm to a hyperpolynomial $\pi_0 \in \Pi_k$. From (14) and the continuity of the functions A and F it follows that

$$\|T - U(\pi_0)\|_p = \mu_{k,p} \,,$$

which is a contradiction.

Now suppose that there is an increasing sequence of positive integers k such that Q_k does not exist. Then there is a sequence of hyperpolynomials $P_k \in \Pi_k$ which satisfy (12) and (13). For this sequence we have

$$\|T - U(P_k)\|_p \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty \,, \tag{15}$$

and

$$\|P_k\|_\infty \longrightarrow \infty \quad \text{as} \quad k \longrightarrow \infty \,.$$

But from (15) and Lemma 4 we also have $\|P_k\|_\infty \to \|x\|_\infty$ as $k \to \infty$, which is a contradiction.

Hence Q_n exists for n sufficiently large. This completes the proof of Theorem 1.

To prove the existence and convergence of best L_1 approximating hyperpolynomials we impose Lipschitz conditions on the functions A, F.

THEOREM 2. *Suppose that*

$$\|A(t, u) - A(t, v)\| \leqq \lambda_1 \|u - v\|, \ (t, u, v) \in [a, b] X R^m X R^m ,$$
$$\|F(t, s, u) - F(t, s, v)\| \leqq \lambda_2 \|u - v\|, \ (t, s, u, v) \in S X R^m ,$$

where λ_1, λ_2 *are fixed positive constants. Let the system* ((*), (**)) *have the unique solution* $x(t)$ *on* $[a, b]$. *Then for each* n *sufficiently large there exists a hyperpolynomial* $Q_n \in \Pi_n$ *such that*

$$\| T - U(Q_n) \|_1 = \inf_{P \in \Pi_n} \| T - U(P) \|_1 .$$

Furthermore, the sequence $Q_n(t)$ *converges uniformly to* $x(t)$ *on* $[a, b]$.

The proof relies on the following analogue of Lemma 4:

LEMMA 5. *If* $P_n \in \Pi_n$ *is a sequence of hyperpolynomials such that* $\lim_{n \to \infty} \| T - U(P_n) \|_1 = 0$, *then the* $P_n(t)$ *converge uniformly to* $x(t)$ *on* $[a, b]$.

Proof. Clearly,

$$\begin{aligned} \|x(t) - P_n(t)\| &\leqq \int_a^t \| T(\tau) - U(P_n)(\tau) \| d\tau + \int_a^t \|A(\tau, x(\tau)) - A(\tau, P_n(\tau))\| d\tau \\ &\quad + \int_a^t \int_a^\tau \| F(\tau, s, x(s)) - F(\tau, s, P_n(s)) \| ds d\tau \\ &\leqq \| T - U(P_n) \|_1 + \lambda_1 \int_a^t \| x(\tau) - P_n(\tau) \| d\tau \\ &\quad + \lambda_2 (b - a) \int_a^t \| x(\tau) - P_n(\tau) \| d\tau . \end{aligned}$$

From Gronwall's inequality we deduce that

$$\|x(t) - P_n(t)\| \leqq \| T - U(P_n) \|_1 \exp [(\lambda_1 + \lambda_2(b - a))(b - a)] .$$

Thus $\|x - P_n\|_\infty \to 0$ as $n \to \infty$.

Proof of Theorem 2. It follows from Lemmas 3 and 5 that if the minimizing hyperpolynomials exist, then they converge uniformly to $x(t)$ on $[a, b]$. To establish existence one argues as in the proof of Theorem 1.

REMARKS. Let A, F satisfy the conditions of Theorem 2 and, for

$1 \leqq p < \infty$, let $Q_n \in \Pi_n$ denote L_p-norm-minimizing hyperpolynomials. Concerning the degree of convergence of the Q_n to x it can be shown, by use of Hölder's inequality and Gronwall's inequality, that

$$\|x - Q_n\|_\infty \leqq \mu_{n,p}(b - a)^{(p-1)/p} \exp\left[(\lambda_1 + \lambda_2(b - a))(b - a)\right] .$$

Also if the functions $T(t) - U(Q_n)(t)$ satisfy a Lipschitz condition on $[a, b]$ uniformly w.r.t. n, the sequence $Q'_n(t)$ converges uniformly to $x'(t)$ on $[a, b]$. The proof of this fact follows from Theorem 5 in [13].

The results of this paper can be extended to integro-differential equations with Fredholm integrals of the form

$$W(x) = x' + A(t, x) + \int_a^b F(t, s, x(s))ds = T(t) .$$

It would be of interest to obtain similar results for equations of the type (*) under linearly independent boundary conditions of the form:

$$Bx(a) + Cx(b) + \int_a^b V(t)x(t)dt = h ,$$

where B, C are constant $m \times m$ matrices and V is a continuous $m \times m$ matrix-valued function on $[a, b]$.

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