

L_p EXTENSIONS OF GONCHAR'S INEQUALITY FOR RATIONAL FUNCTIONS

UDC 517.5

A. L. LEVIN AND E. B. SAFF

ABSTRACT. Given a condenser (E, F) in the complex plane, let $C(E, F)$ denote its capacity and let $\mu^* = \mu_E^* - \mu_F^*$ be the (signed) equilibrium distribution for (E, F) . Given a finite positive measure μ on $E \cup F$, let $G(\mu'_E) = \exp(\int \log(d\mu/d\mu_E^*) d\mu_E^*)$ and $G(\mu'_F) = \exp(\int \log(d\mu/d\mu_F^*) d\mu_F^*)$. We show that for $0 < p, q < \infty$ and for any rational function r_n of order n

$$(*) \quad \|r_n\|_{L_p(d\mu, E)} \|1/r_n\|_{L_q(d\mu, F)} \geq e^{-n/C(E, F)} G^{1/p}(\mu'_E) G^{1/q}(\mu'_F),$$

which extends a classical result due to A. A. Gonchar. For a symmetric condenser we also obtain a sharp lower bound for $\|r_n - \lambda\|_{L_p(d\mu, E \cup F)}$, where $\lambda = \lambda(z)$ is equal to 0 on E and 1 on F . The question of exactness of $(*)$ and the relation to certain n -widths are also discussed.

Bibliography: 16 titles.

§1. INTRODUCTION

Let E and F be disjoint closed sets of positive logarithmic capacity in the extended complex plane $\bar{\mathbb{C}}$. The pair (E, F) is called a *condenser*. Put

$$(1.1) \quad \sigma_n(E, F) := \sup_{\{r_n\}} \frac{\min_F |r_n|}{\max_E |r_n|},$$

where $\{r_n\}$ denotes that the supremum is taken over all rational functions $r_n (\neq 0)$ of order $\leq n$. This quantity was introduced by A. A. Gonchar in [Go1] and [Go2]. He proved (cf. [Go3]) that

$$\sigma_n(E, F) \leq e^{n/C(E, F)} \quad n = 0, 1, 2,$$

and that

$$(1.3) \quad \liminf_{n \rightarrow \infty} [\sigma_n(E, F)]^{1/n} \geq e^{1/C(E, F)},$$

where $C(E, F)$ denotes the capacity of the condenser (E, F) . This capacity can be defined in several different ways. The one adopted by Gonchar enabled him to prove (1.2) under the additional assumption that one of the sets, say E , lies in a connected component of $\bar{\mathbb{C}} \setminus F$. Yet, his method can be modified to deal with the general case (cf. [Go3], Remark 1). Widom [W1] gave a very short proof of (1.2) for an arbitrary condenser. He used an alternative definition of $C(E, F)$ as the Green capacity of E relative to the open set $\bar{\mathbb{C}} \setminus F$. Yet another definition was utilized by Saff and Totik ([ST], Chapter 7), who established a weighted analogue of (1.2). We

1991 *Mathematics Subject Classification*. Primary 30A10, 30C85; Secondary 31A15.

The second author's research was supported, in part, by NSF grants DMS-881-4026 and DMS-891-2423.

also mention that T. Ganelius [Ga2] has shown that, under some further assumptions on E, F , inequality (1.3) can be improved to

$$(1.4) \quad \sigma_n(E, F) \geq \text{const} \cdot e^{n/C(E, F)}, \quad |n = 1, 2, \dots$$

Returning to the definition of σ_n , observe that it can be rewritten as

$$(1.5) \quad \sigma_n^{-1}(E, F) = \inf_{\{r_n\}} \{ \|r_n\|_{L_\infty(E)} \|1/r_n\|_{L_\infty(F)} \}.$$

It thus makes sense to introduce the quantity

$$(1.6) \quad \delta_n(E, F) = \delta_n(E, F; \|\cdot\|_E, \|\cdot\|_F) := \inf_{\{r_n\}} \{ \|r_n\|_E \|1/r_n\|_F \}$$

where $\|\cdot\|_E, \|\cdot\|_F$ are given norms on E and F , respectively. Natural choices for these norms are L_p -norms (with respect to some measure) or the L_∞ -norm (with some weight function). For such norms, we will establish the analogue of (1.2). In particular, if μ_E^*, μ_F^* are the probability measures on E, F respectively, such that $\mu^* := \mu_E^* - \mu_F^*$ is the equilibrium distribution on (E, F) (we define this in the next section), then for any $0 < p, q < \infty$ and for any rational function r_n of order $\leq n$, the following inequality holds:

$$(1.7) \quad \|r_n\|_{L_p(E, d\mu_E^*)} \|1/r_n\|_{L_q(F, d\mu_F^*)} \geq e^{-n/C(E, F)}, \quad n = 0, 1, \dots$$

Note that since μ_E^*, μ_F^* are unit measures, inequality (1.7) implies (1.2).

The estimates (1.2), (1.3) have immediate applications to approximation problems. Let $\lambda(z)$ be the function that is equal to 0 on E and 1 on F , and set

$$\rho_n(E, F) := \inf_{\{r_n\}} \|\lambda - r_n\|_{L_\infty(E \cup F)}.$$

It is not difficult to show (see [Go3]) that

$$(1.8) \quad \frac{1}{\sigma_n^{1/2} + 1} \leq \rho_n \leq \frac{1}{\sigma_n^{1/2} - 1}, \quad n = 0, 1, \dots,$$

and consequently, since $\sigma_n \rightarrow \infty$, we have

$$\rho_n \sim \sigma_n^{-1/2} = \delta_n^{1/2}(E, F; \|\cdot\|_{L_\infty(E)}, \|\cdot\|_{L_\infty(F)}), \quad \text{as } n \rightarrow \infty.$$

Unfortunately, there is no obvious relation between ρ_n and δ_n for norms other than the sup-norm. For the general L_p -norm case, we are able to prove the lower bound for ρ_n only for a symmetric condenser. Our method also applies to the case when E and F have a point in common. As an illustration, we prove that for $0 < p < \infty$ and for any r_n of order $\leq n$, the following inequality holds:

$$\left\{ \int_{-1}^0 |r_n(x)|^p dx + \int_0^1 |1 - r_n(x)|^p dx \right\}^{1/p} \geq c_p n^{1/2p} e^{-\pi\sqrt{n/p}}.$$

This inequality was proved by Vyacheslavov [V] for *real-valued* r_n and for $p \geq 1$.

Another related matter is the Kolmogorov n -width. Let Ω be a domain in $\bar{\mathbb{C}}$ and let $A_\infty(\Omega)$ denote the collection of all functions f that are analytic in Ω and satisfy $|f| \leq 1$ there. Let K be a compact set in Ω . Given a measure μ on K , the Kolmogorov n -width of $A_\infty(\Omega)$ with respect to the norm $\|\cdot\|_{L_p(K; d\mu)}$, $0 < p < \infty$, is defined by

$$(1.10) \quad d_n(A_\infty(\Omega), \|\cdot\|_{L_p(K; d\mu)}) := \inf_{X_n} \sup_{f \in A_\infty(\Omega)} \inf_{g \in X_n} \|f - g\|_{L_p(K; d\mu)},$$

where X_n runs through all n -dimensional subspaces of $L_p(K; d\mu)$. Using the method of Fisher and Micchelli [FM], we can show that

$$(1.11) \quad d_n(A_\infty(\Omega), \|\cdot\|_{L_p(K; d\mu)}) \leq \inf_{\{r_n\}} \{ \|r_n\|_{L_p(K; d\mu)} \|1/r_n\|_{L_\infty(\partial\Omega)} \}.$$

If Ω is a disk, the equality holds in (1.11). In this case, the results of O. Parfenov in [P1] and [P2] can be applied to get the strong asymptotics for $\delta_n(K, \partial\Omega)$.

§2. SOME BASIC FACTS

A convenient (for our purpose) definition of condenser capacity was given by T. Bagby in [B]. Throughout we assume that E, F are disjoint closed subsets of \bar{C} , each having positive logarithmic capacity.

(a) Let μ_E and μ_F be Borel probability measures on E and F , respectively, and assume that both measures have finite logarithmic energy. Let $\mu = \mu_E - \mu_F$ denote the corresponding signed measure on $E \cup F$. For any such μ , it is known (cf. [H], Theorem 16.4.2) that the integral

$$I(\mu) := \iint \log \frac{1}{|z-t|} d\mu(z) d\mu(t)$$

exists and is positive. Let

$$V := \inf_{\mu} I(\mu).$$

It turns out that $0 < V < \infty$ since E and F both have positive logarithmic capacity. The condenser capacity $C(E, F)$ is then defined by

$$(2.2) \quad C(E, F) := V^{-1}$$

(some authors use e^{-V} instead of V^{-1} to define $C(E, F)$).

(b) The infimum in (2.1) is attained for the unique signed measure $\mu^* = \mu_E^* - \mu_F^*$, which is called the *equilibrium distribution* for (E, F) . The corresponding *equilibrium potential* is then defined by

$$(2.3) \quad u^*(z) := \int \log \frac{1}{|z-t|} d\mu^*(t).$$

It is known that u^* exists and is finite for any $z \in \bar{C}$. Moreover (cf. [B], Theorem 1), there exist finite constants V_E, V_F such that

$$(2.4) \quad V_E \geq 0, \quad V_F \leq 0;$$

$$V_E - V_F = V$$

$$V_F \leq u^*(z) \leq V_E \quad \text{for all } z \in \bar{C};$$

$$u^*(z) = V_E \quad \text{quasi-everywhere on } E,$$

$$(2.8) \quad u^*(z) = V_F \quad \text{quasi-everywhere on } F,$$

where quasi-everywhere (q.e.) means that the property holds except for a set of zero capacity. We also mention that μ^* is concentrated on $\partial E \cup \partial F$. Thus, u^* is harmonic in the complement of $\partial E \cup \partial F$. It then follows from (2.7) that $u^*(z) = V_E$ on each component of $\bar{C} \setminus \partial E$ that does not contain points of F . (An analogous statement concerning components of $\bar{C} \setminus \partial F$ follows from (2.8).)

In particular, if $\infty \notin \partial E$ and if the unbounded component of $\bar{C} \setminus \partial E$ does not intersect F , then $u^*(\infty) = 0$ (see (2.3) and note: $\int \log |z|^{-1} d\mu^*(t) = (\log |z|^{-1}) \int d\mu^* = 0$) and therefore, $V_E = 0$.

(c) A condenser (E, F) is called *symmetric* if E lies in the (open) upper half-plane and F is the reflection of E in the real axis. For such condensers, the equilibrium distribution μ^* is also symmetric, that is (cf. [B], Lemma 9), if $K \subset E$ is any Borel set and \bar{K} is its reflection in the real axis, then

$$(2.9) \quad \mu_E^*(K) = \mu_F^*(\bar{K}).$$

It then follows from (2.3) that $u^*(\bar{z}) = -u^*(z)$ and therefore (by (2.7), (2.8))

$$(2.10) \quad V_E = -V_F = 1/(2C(E, F)).$$

(d) Given a (positive) measure μ on $E \cup F$, we denote by μ_E, μ_F its restrictions on E, F , respectively. Let $\mu_E = \mu_E^s + \mu_E^a$ be the canonical decomposition of μ_E into the singular and the absolutely continuous parts with respect to μ_E^* . We denote by μ'_E the Radon-Nikodým derivative $d\mu_E^a/d\mu_E^*$ of μ_E^a with respect to μ_E^* . Let

$$(2.11) \quad G(\mu'_E) := \exp \left(\int (\log \mu'_E) d\mu_E^* \right)$$

be the geometric mean of μ'_E . Similarly, we define μ'_F and $G(\mu'_F)$. If both $G(\mu'_E)$ and $G(\mu'_F)$ are positive, we say that the measure μ satisfies the *Szegő condition* with respect to μ^* .

§3. GONCHAR'S INEQUALITY GENERALIZED

Theorem 3.1. *Let (E, F) be a condenser and let μ be a finite (positive) Borel measure on $E \cup F$. Then, for any $0 < p, q < \infty$ and for any rational function r_n of order $\leq n$, we have*

$$(3.1) \quad \|r_n\|_{L_p(d\mu_E)} \|1/r_n\|_{L_q(d\mu_F)} \geq e^{-n/C(E, F)} G^{1/p}(\mu'_E) G^{1/q}(\mu'_F),$$

where μ_E, μ_F denote the restrictions of μ to E, F respectively.

Proof. Let $\mu_E = \mu_E^s + \mu_E^a$. Since μ_E is positive, so is μ_E^s . Hence,

$$\int |r_n|^p d\mu_E \geq \int |r_n|^p \mu'_E d\mu_E^*.$$

Assuming $\int (\log \mu'_E) d\mu_E^* > -\infty$ (otherwise (3.1) is obvious) and applying Jensen's inequality, we obtain

$$\frac{1}{p} \log \int |r_n|^p \mu'_E d\mu_E^* \geq \int \log |r_n| d\mu_E^* + \frac{1}{p} \log G(\mu'_E)$$

and similarly,

$$\frac{1}{q} \log \int |1/r_n|^q \mu'_F d\mu_F^* \geq - \int \log |r_n| d\mu_F^* + \frac{1}{q} \log G(\mu'_F),$$

where we used the fact that μ_E^* and μ_F^* are probability measures. Hence,

$$\log \{ \|r_n\|_{L_p(d\mu_E)} \|1/r_n\|_{L_q(d\mu_F)} \} \geq \int \log |r_n| d\mu^* + \frac{1}{p} \log G(\mu'_E) + \frac{1}{q} \log G(\mu'_F).$$

It remains to show that

$$\int \log |r_n| d\mu^* \geq -n/C(E, F).$$

Write

$$(3.3) \quad r_n(z) = A \prod_{i=1}^k (z - \alpha_i) / \prod_{i=1}^{\ell} (z - \beta_i),$$

$\max(k, \ell) \leq n$ Since $\int d\mu^* = 0$, we obtain (see (2.3))

$$\int \log |r_n| d\mu^* = \sum_{i=1}^{\ell} u^*(\beta_i) - \sum_{i=1}^k u^*(\alpha_i).$$

Applying (2.6) and (2.4), we then obtain

$$(3.4) \quad \int \log |r_n| d\mu^* \geq \ell V_F - k V_E \geq -n(V_E - V_F)$$

and (3.2) follows. \square

Remark 1. One can consider the measures $d\mu_E = w_E^p d\mu_E^*$, $d\mu_F = w_F^q d\mu_F^*$, where $w_E \in L_p(d\mu_E^*)$, $w_F \in L_q(d\mu_F^*)$. We then obtain

$$(3.5) \quad \|r_n w_E\|_{L_p(d\mu_E^*)} \|(1/r_n) w_F\|_{L_q(d\mu_F^*)} \geq e^{-n/C(E, F)} G(w_E) G(w_F).$$

If $w_E \in L_{\infty}(d\mu_E^*)$ and/or $w_F \in L_{\infty}(d\mu_F^*)$, then (3.5) holds also for $p = \infty$ and/or $q = \infty$. Furthermore, inequality (3.2) shows that (3.5) holds also for $p = 0$ and/or $q = 0$, where $\|f\|_{L_0}$ means $\exp(\int \log |f|)$.

Remark 2. The inequality (3.1) can be strengthened in the spirit of Remark 1 in [Go3]. For example, let $\infty \notin \partial E$ and assume that the unbounded component of $\bar{C} \setminus \partial E$ does not intersect F . Let D be the union of all components of $\bar{C} \setminus \partial E$ that do not intersect F . Then (see the end of subsection 2(b)) $V_E = 0$ and $u^*(z) = 0$ on D . Thus, the proof of Theorem 3.1 shows that for any rational function r , we have:

$$(3.6) \quad \|r\|_{L_p(d\mu_E)} \|1/r\|_{L_q(d\mu_F)} \geq e^{-m/C(E, F)} G^{1/p}(\mu'_E) G^{1/q}(\mu'_F),$$

where m is the number of poles of $r(z)$ that are not in D . In particular, (3.6) holds for $r = P/Q_m$, where Q_m is a polynomial of degree m and P is an arbitrary polynomial.

Remark 3. Let (E, F) be a symmetric condenser (with E in the upper half-plane) and let P_n be a polynomial of degree $k \leq n$. Then (3.4) and (2.10) give

$$\int \log |P_n| d\mu^* \geq -k V_E \geq -n V_E = -n/(2C(E, F)).$$

Since the equilibrium distribution μ^* is symmetric (see (2.9)) we may rewrite this in the form:

$$(3.7) \quad \int_E \log \left| \frac{P_n(z)}{P_n(\bar{z})} \right| d\mu_E^* \geq \frac{-n}{2C(E, F)}.$$

This inequality remains valid if we replace \int_E by \int_A , where A is any Borel subset of E . Indeed, write $P_n = Q_m R_{n-m}$, where the zeros of Q_m lie in the open upper half-plane and those of R_{n-m} lie in the closed lower half-plane. Then we have, for any z with $\text{Im } z \geq 0$,

$$\left| \frac{Q_m(z)}{Q_m(\bar{z})} \right| \leq 1 \quad \text{and} \quad \left| \frac{P_n(z)}{P_n(\bar{z})} \right| \geq \left| \frac{Q_m(z)}{Q_m(\bar{z})} \right|$$

Hence,

$$\begin{aligned} \int_A \log \left| \frac{P_n(z)}{P_n(\bar{z})} \right| d\mu_E^* &\geq \int_A \log \left| \frac{Q_m(z)}{Q_m(\bar{z})} \right| d\mu_E^* \\ &\geq \int_E \log \left| \frac{Q_m(z)}{Q_m(\bar{z})} \right| d\mu_E^* \geq \frac{-m}{2C(E, F)}. \end{aligned}$$

$m \leq n$, we obtain

$$(3.9) \quad \int_A \log \left| \frac{P_n(z)}{P_n(\bar{z})} \right| d\mu_E^* \geq \frac{-n}{2C(E, F)}, \quad \text{for } A \subseteq E.$$

This can be viewed as a kind of Newman's inequality.

Finally, we discuss the sharpness of (3.1). Assuming that μ satisfies the Szegő condition with respect to μ^* we obtain from (3.1):

$$\liminf_{n \rightarrow \infty} \delta_n^{1/n}(E, F; \|\cdot\|_{L_p(d\mu_E)}, \|\cdot\|_{L_q(d\mu_F)}) \geq e^{-1/C(E, F)}$$

where δ_n is defined in (1.6). The matching upper bound for $\limsup \delta_n^{1/n}$ follows from (1.3). Hence, the exponent in (3.1) is sharp.

The following example shows that in certain geometric settings the factor $G^{1/p}(\mu'_E)G^{1/q}(\mu'_F)$ is the best possible.

Let E be the circle $|z| = \rho < 1$ and let F be the unit circle $|z| = 1$. In this case, $e^{-1/C(E, F)} = \rho$ and $d\mu_E^*(\rho e^{i\theta}) = d\mu_F^*(e^{i\theta}) = (1/2\pi)d\theta$. Let $d\mu_E, d\mu_F$ be arbitrary Borel probability measures on E and on F , respectively. By a result of Parfenov [P1] (see also [LS]), given $0 < p, q < \infty$ there exist sequences $\{B_n^{(1)}\}, \{B_n^{(2)}\}$ of Blaschke products of respective orders n such that

$$\|B_n^{(1)}(z)\|_{L_p(d\mu_E)} \sim \rho^n G^{1/p}(\mu'_E) \quad \text{and} \quad \|B_n^{(2)}(\rho/z)\|_{L_q(d\mu_F)} \sim \rho^n G^{1/q}(\mu'_F),$$

where $a_n \sim b_n$ means $a_n/b_n \rightarrow 1$ as $n \rightarrow \infty$.

Define the sequence $\{r_n\}$ of rational functions of order $\leq n$, by

$$r_n(z) := B_{n/2}^{(1)}(z)/B_{n/2}^{(2)}(\rho/z), \quad \text{if } n \text{ is even,}$$

$$r_n(z) := z r_{n-1}(z), \quad \text{if } n \text{ is odd.}$$

It is then obvious that

$$\|r_n\|_{L_p(d\mu_E)} \|1/r_n\|_{L_q(d\mu_F)} \sim \rho^n G^{1/p}(\mu'_E) G^{1/q}(\mu'_F) \quad \text{as } n \rightarrow \infty.$$

§4. APPROXIMATION OF FUNCTIONS CONSTANT ON TWO DISJOINT SETS

Given a condenser (E, F) , let λ denote the function that equals 0 on E and 1 on F . Given $0 < p < \infty$ and given a finite (positive) measure μ on $E \cup F$, set

$$\rho_n(E, F, \|\cdot\|_{L_p(d\mu)}) := \inf_{\{r_n\}} \|\lambda - r_n\|_{L_p(d\mu)}.$$

Gonchar's result (1.3) and the estimate (1.8) imply that

$$\limsup_{n \rightarrow \infty} \rho_n^{1/n}(E, F, \|\cdot\|_{L_p(d\mu)}) \leq e^{-1/2C(E, F)}.$$

The lower bound (for a symmetric condenser) is given by

Theorem 4.1. *Let (E, F) be a symmetric condenser (with E in the upper half-plane) and assume that the finite (positive) measure μ satisfies the Szegő condition with respect to the equilibrium distribution μ^* for (E, F) . Then, for $0 < p < \infty$, we have*

$$(4.3) \quad \rho_n(E, F, \|\cdot\|_{L_p(d\mu)}) \geq (1 + o(1))[G(\mu'_E)G(\mu'_F)]^{1/2p} e^{-n/2C(E, F)},$$

where μ_E, μ_F denote the restrictions of μ to E, F , respectively.

Proof. For $n = 1, 2, \dots$, let r_n be a rational function that realizes the infimum in (4.1). It then follows from (4.2) that

$$(4.4) \quad \int |r_n|^p \mu'_E d\mu_E^* + \int |1 - r_n|^p \mu'_F d\mu_F^* \leq \rho_n^p \leq c\eta^n,$$

where $0 < \eta < 1$ and $c > 0$ are independent of n . Set

$$(4.5) \quad E_n := \{z \in E : |r_n(z)| > 1 - 2^{-1/p}\}, \quad F_n := \{z \in F : |1 - r_n(z)| > 1 - 2^{-1/p}\}.$$

Then we have by (4.4) that

$$(4.6) \quad \int_{E_n} \mu'_E d\mu_E^* + \int_{F_n} \mu'_F d\mu_F^* \leq 2c\eta^n / (2^{1/p} - 1)^p.$$

Since μ satisfies the Szegő condition, (4.6) implies that (as we will show later)

$$(4.7) \quad \mu_E^*(E_n) + \mu_F^*(F_n) = o(1/n).$$

Let $A_n := E_n \cup \tilde{F}_n$ (where \sim denotes the reflection of a set about the real axis). Since μ^* is symmetric, we obtain

$$(4.8) \quad \alpha_n := \mu_E^*(A_n) = \mu_F^*(\tilde{A}_n) = o(1/n).$$

By the definition of E_n, F_n and A_n , we obtain (see (4.4)):

$$\rho_n^p \geq \frac{1}{2} \int_{E \setminus A_n} \left| \frac{r_n}{1 - r_n} \right|^p \mu'_E d\mu_E^* + \frac{1}{2} \int_{F \setminus \tilde{A}_n} \left| \frac{1 - r_n}{r_n} \right|^p \mu'_F d\mu_F^*$$

and therefore (by the concavity of log):

$$(4.9) \quad p \log \rho_n \geq \frac{1}{2} \log \int_{E \setminus A_n} \left| \frac{r_n}{1 - r_n} \right|^p \mu'_E d\mu_E^* + \frac{1}{2} \log \int_{F \setminus \tilde{A}_n} \left| \frac{1 - r_n}{r_n} \right|^p \mu'_F d\mu_F^*.$$

For the first integral, we have, by Jensen's inequality (see also (4.8)),

$$\begin{aligned} & \log \int_{E \setminus A_n} \left| \frac{r_n}{1 - r_n} \right|^p \mu'_E d\mu_E^* \\ & \geq p \int_{E \setminus A_n} \log \left| \frac{r_n}{1 - r_n} \right| \frac{d\mu_E^*}{1 - \alpha_n} + \frac{1}{1 - \alpha_n} \int_{E \setminus A_n} \log \mu'_E d\mu_E^* + \log(1 - \alpha_n) \\ & = p \int_{E \setminus A_n} \log \left| \frac{r_n}{1 - r_n} \right| \frac{d\mu_E^*}{1 - \alpha_n} + \log G(\mu'_E) + o(1). \end{aligned}$$

(Here we only used the fact that $\alpha_n = o(1)$.) A similar inequality holds for the second integral in (4.9). We thus obtain

$$(4.10) \quad \begin{aligned} \log \rho_n & \geq o(1) + \frac{1}{2p} \log[G(\mu'_E)G(\mu'_F)] \\ & + \frac{1}{2(1 - \alpha_n)} \left\{ \int_{E \setminus A_n} \log \left| \frac{r_n}{1 - r_n} \right| d\mu_E^* + \int_{F \setminus \tilde{A}_n} \log \left| \frac{1 - r_n}{r_n} \right| d\mu_F^* \right\} \end{aligned}$$

By the symmetry, the expression in the braces can be put into the form

$$\int_{E \setminus A_n} \log \left| \frac{r_n(z)}{1 - r_n(z)} \frac{1 - r_n(\bar{z})}{r_n(\bar{z})} \right| d\mu_E^*.$$

Since this last integrand is of the form $\log |P_{2n}(z)/P_{2n}(\bar{z})|$, where P_{2n} is a polynomial of degree $\leq 2n$, it follows from (3.9) that the last term in (4.10) satisfies

$$\frac{1}{2(1 - \alpha_n)} \{ \cdot \} \geq \frac{1}{2(1 - \alpha_n)} \frac{-2n}{2C(E, F)} = -\frac{n}{2C(E, F)} + o(1)$$

(since $\alpha_n = o(1/n)$ by (4.8)), and the proof is complete provided we establish (4.7). Actually we shall prove the following slightly more general statement:

Let σ be a finite (positive) measure on a compact set K . Let $f \in L_1(K; d\sigma)$ and assume also that $\log f \in L_1(K; d\sigma)$. Let $\{K_n\}$ be a sequence of subsets of K such that $\sigma(K_n) > 0$ and $\varepsilon_n := \int_{K_n} f d\sigma \rightarrow 0$ as $n \rightarrow \infty$. Then $\sigma(K_n) = o(1/\log(1/\varepsilon_n))$.

Indeed, set $a_n := \int_{K_n} \log f d\sigma$. Since $\log f \in L_1$, we have $a_n \geq -a$ for some $a > 0$ independent of n . Hence,

$$(4.11) \quad \begin{aligned} \log \varepsilon_n &= \log \int_{K_n} f d\sigma \geq \frac{1}{\sigma(K_n)} \int_{K_n} \log f d\sigma + \log \sigma(K_n) \\ &\geq \frac{-a}{\sigma(K_n)} + \log \sigma(K_n). \end{aligned}$$

Since $\varepsilon_n \rightarrow 0$, it follows from (4.11) that $\sigma(K_n) \rightarrow 0$ as $n \rightarrow \infty$. By the absolute continuity of integral, we then obtain

$$a_n = \int_{K_n} \log f d\sigma \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Returning to inequality (4.11), we see that

$$\log \varepsilon_n \geq \frac{a_n}{\sigma(K_n)} + \log \sigma(K_n), \quad a_n \rightarrow 0$$

and the result follows. \square

Remark. We assumed that the condenser (E, F) is symmetric about the real axis. Given a condenser (E', F') that is symmetric about some straight line or about some circle, we may use a suitable Möbius transformation to map (E', F') onto the condenser (E, F) considered above. This transformation preserves rational functions of given order and transfers (in a natural way) the equilibrium distribution on (E', F') to that on (E, F) . Thus, Theorem 4.1 is valid for any symmetric condenser.

§5. APPROXIMATION OF THE UNIT STEP FUNCTION ON $[-1, 1]$

In this section, we prove the estimate (1.9). Let

$$(5.1) \quad \Delta_n^p := \inf_{\{r_n\}} \left\{ \int_{-1}^0 |r_n(x)|^p dx + \int_0^1 |1 - r_n(x)|^p dx \right\}$$

and let r_n^* denote a function that realizes the infimum. For any $0 < \varepsilon < 1$, we have

$$(5.2) \quad \Delta_n^p > \int_{-1}^{-\varepsilon} |r_n^*|^p dx + \int_{\varepsilon}^1 |1 - r_n^*|^p dx.$$

It is well known that the equilibrium distribution for the condenser

$$(E, F) = ([-1, -\varepsilon], [\varepsilon, 1])$$

is given by

$$d\mu_E^*(-x) = d\mu_F^*(x) = \frac{dx}{K'(\varepsilon)\sqrt{(x^2 - \varepsilon^2)(1 - x^2)}}, \quad \varepsilon \leq x \leq 1,$$

and that $1/C(E, F) = 2\pi K(\varepsilon)/K'(\varepsilon)$, where $K(\varepsilon)$, $K'(\varepsilon)$ denote the complete elliptic integrals for moduli ε and $\sqrt{1 - \varepsilon^2}$, respectively. We also have, as $\varepsilon \rightarrow 0$,

$$K(\varepsilon) = \pi/2 + o(1), \quad K'(\varepsilon) = \log(1/\varepsilon) + O(1).$$

Using these facts we could proceed as in §4 by estimating the right-hand side of (5.2) from below and choosing ε in an "optimal" way. But it will be simpler to utilize the measure $dx/(x \log(1/\varepsilon))$ (a unit measure on $[\varepsilon, 1]$) instead of $d\mu^*(x)_E$ and to apply the classical Newman's inequality (cf. [N])

$$(5.3) \quad \int_a^b \log \left| \frac{P_n(x)}{P_n(-x)} \right| \frac{dx}{x} \geq -\frac{\pi^2 n}{2}, \quad 0 < a < b.$$

Notice that (5.3) implies (in the same way as (3.7) implies (3.8)) that

$$(5.4) \quad \int_K \log \left| \frac{P_n(x)}{P_n(-x)} \right| \frac{dx}{x} \geq -\frac{\pi^2 n}{2}, \quad K \subset (0, \infty).$$

Returning to (5.2), let us introduce the exceptional set

$$A_n := \{x \in [\varepsilon, 1] : |1 - r_n^*(x)|^p > \frac{1}{2} \text{ or } |r_n^*(-x)|^p > \frac{1}{2}\}.$$

From (5.2), we see that $\text{meas}(A_n) := \int_{A_n} dx \leq 2\Delta_n^p$. Set (compare with (4.8))

$$\alpha_n := \int_{A_n} \frac{dx}{x \log(1/\varepsilon)}$$

and observe that

$$\alpha_n \leq \int_\varepsilon^{\varepsilon+2\Delta_n^p} \frac{dx}{x \log(1/\varepsilon)} = \log \left(1 + \frac{2\Delta_n^p}{\varepsilon} \right) / \log \frac{1}{\varepsilon}$$

Proceeding as in §4 (with $dx/(x \log(1/\varepsilon))$ instead of $d\mu_E^*$ and with $d\mu_E x \log(1/\varepsilon) dx$) we obtain

$$(6) \quad \begin{aligned} \log \Delta_n &\geq \frac{1}{2(1 - \alpha_n)} \int_{[\varepsilon, 1] \setminus A_n} \log \left| \frac{P_{2n}(x)}{P_{2n}(-x)} \right| \frac{dx}{x \log(1/\varepsilon)} \\ &\quad + \frac{1}{p(1 - \alpha_n)} \int_{[\varepsilon, 1] \setminus A_n} \log \left(x \log \frac{1}{\varepsilon} \right) \frac{dx}{x \log(1/\varepsilon)} + \frac{1}{p} \log(1 - \alpha_n). \end{aligned}$$

Applying (5.4) and observing that

$$\int_{[\varepsilon, 1] \setminus A_n} \log x \frac{dx}{x} \geq \int_\varepsilon^1 \frac{\log x}{x} dx = -\frac{1}{2} \log^2 \frac{1}{\varepsilon},$$

we get from (5.6) that

$$\log \Delta_n \geq -\frac{1}{2(1 - \alpha_n)} \left\{ \frac{\pi^2 n}{\log(1/\varepsilon)} + \frac{1}{p} \log \frac{1}{\varepsilon} \right\} + \frac{1}{p} \log \log \frac{1}{\varepsilon} + \frac{1}{p} \log(1 - \alpha_n)$$

Now we choose

$$\varepsilon := \Delta_n^p.$$

We then see from (5.5) that

$$\alpha_n \leq c / \log(1/\Delta_n).$$

Using (5.8) and (5.9), we obtain from (5.7) that

$$(5.10) \quad \log \Delta_n \geq -\frac{1}{2} \left\{ \frac{\pi^2 n}{p \log(1/\Delta_n)} + \log \frac{1}{\Delta_n} \right\} - c_1 \frac{n}{\log^2(1/\Delta_n)} \\ + \frac{1}{p} \log \left(p \log \frac{1}{\Delta_n} \right) + O(1)$$

To obtain the desired lower bound, define τ_n by

$$(5.11) \quad \Delta_n =: \tau_n n^{1/2p} e^{-\pi\sqrt{n/p}}$$

and assume that $\tau_n \rightarrow 0$ as $n \rightarrow \infty$. This will lead to a contradiction. Indeed, (5.11) implies that

$$\log \frac{1}{\Delta_n} \geq \pi \sqrt{\frac{n}{p}} - \frac{1}{2p} \log n \quad (\text{for } n \text{ large}).$$

Hence (5.10) gives (after subtracting $\frac{1}{2} \log \Delta_n$ from both sides)

$$\frac{1}{2} \log \Delta_n \geq -\frac{1}{2} \pi \sqrt{\frac{n}{p}} - \frac{1}{4p} \log n + \frac{1}{2p} \log n + O(1).$$

Replacing here Δ_n by (5.11), we obtain

$$\frac{1}{2} \log \tau_n \geq O(1),$$

which contradicts the assumption $\tau_n \rightarrow 0$.

§6. ESTIMATES FOR n -WIDTHS

We first prove the estimate (1.1).

Theorem 6.1. *Let Ω be a domain in $\bar{\mathbb{C}}$ and let $A_\infty(\Omega)$ denote the collection of all functions f that are analytic in Ω and satisfy $|f| \leq 1$ there. Let K be a compact set in Ω and let μ be a finite (positive) measure on K . Then, for $0 < p < \infty$, we have*

$$(6.1) \quad d_n(A_\infty(\Omega), \|\cdot\|_{L_p(K; d\mu)}) \leq \inf_{\{r_n\}} \|r_n\|_{L_p(K; d\mu)} \|1/r_n\|_{L_\infty(\partial\Omega)},$$

where the infimum is taken over all rational functions of order $\leq n$.

Proof. Assume first that Ω is bounded by a finite number of disjoint analytic simple closed curves. Let $g(z, \xi)$ be the Green function for Ω with singularity at ξ . Then the conjugate function $\tilde{g}(z, \xi)$ is multiple-valued, but the function

$$\varphi(z, \xi) := \frac{\partial}{\partial z} (g(z, \xi) + i\tilde{g}(z, \xi))$$

is analytic and single-valued in Ω . The following facts are well known:

- (a) $\varphi(z, \xi)$ has a simple pole at ξ with the residue 1.
- (b) $\varphi(z, \xi)$ is continuous in both arguments.
- (c)

$$\frac{1}{2\pi} \int_{\partial\Omega} |\varphi(z, \xi)| |dz| = \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial g(z, \xi)}{\partial \nu_z} |dz| = 1$$

(ν_z denotes the inward normal to $\partial\Omega$ at $z \in \partial\Omega$).

In order to prove (6.1) it suffices to consider rational functions with simple zeros, and we may assume that they do not lie on $\partial\Omega$. Let $R(z)$ be such a function and

let z_1, \dots, z_m ($m \leq n$) denote those zeros of R that belong to Ω . For any f that is analytic in $\bar{\Omega}$, we have by (a), (b) that

$$\frac{1}{2\pi i} \int_{\partial\Omega} \frac{R(\xi)}{R(z)} f(z) \varphi(z, \xi) dz = f(\xi) + R(\xi) \sum_{i=1}^m \frac{f(z_i) \varphi(z_i, \xi)}{R'(z_i)}$$

If $|f| \leq 1$ in $\bar{\Omega}$, then by (c) we get

$$\left| \frac{1}{2\pi i} \int_{\partial\Omega} \frac{R(\xi)}{R(z)} f(z) \varphi(z, \xi) dz \right| \leq |R(\xi)| \cdot \|1/R\|_{L_\infty(\partial\Omega)} \frac{1}{2\pi} \int_{\partial\Omega} |\varphi(z, \xi)| |dz| = |R(\xi)| \cdot \|1/R\|_{L_\infty(\partial\Omega)}.$$

Thus, (6.2) gives that the distance in $L_p(K; d\mu)$ from $f(\xi)$ to span $\{R(\xi)\varphi(z_i, \xi)\}_{i=1}^m$ does not exceed $\|R\|_{L_p(K; d\mu)} \|1/R\|_{L_\infty(\partial\Omega)}$. Since any $f \in A_\infty(\Omega)$ can be approximated (uniformly on K) by those $f \in A_\infty(\Omega)$ that are analytic in $\bar{\Omega}$, (6.1) follows.

It remains to remove the restriction we put on Ω . This is done in a standard way (see the proof of Theorem 4 in [FM]). \square

Assume now that Ω is the unit disk $\Delta := \{z : |z| < 1\}$. For this case, Fisher and Micchelli proved ([FM], Theorem 1) that

$$(6.3) \quad d_n(A_\infty(\Delta), L_p(K; d\mu)) = \inf_{\{B_n\}} \|B_n\|_{L_p(K; d\mu)},$$

where the infimum is taken over all Blaschke products of order $\leq n$. Since $\|1/B_n\|_{L_\infty(\partial\Omega)} = 1$, we obtain from (6.1), (6.3) that

$$(6.4) \quad d_n(A_\infty(\Delta), L_p(K; d\mu)) = \inf_{\{r_n\}} \|r_n\|_{L_p(K; d\mu)} \|1/r_n\|_{L_\infty(\partial\Omega)}.$$

This equality provides another example of the exactness of the generalized Gonchar inequality. Let $p = 2$ and assume that K is a simple closed analytic curve. Let $d\mu = w(z)|dz|$, where $w(z)$ is a continuous positive function on K and $|dz|$ is the arc length. Let $A_2(\Delta)$ denote the unit ball of the Hardy space $H^2(\Delta)$.

For this case, Parfenov found the strong asymptotics for the n -width. Although stated in different terms, the result reads (see [P2])

$$(6.5) \quad d_n(A_2(\Delta), L_2(K; d\mu)) \sim e^{-n/C(K, \partial\Delta)} G(\mu')^{1/2},$$

where μ' denotes, as before, the derivative $d\mu/d\mu^*$. Since $A_2(\Delta) \supset A_\infty(\Delta)$, we have

$$d_n(A_\infty(\Delta), L_2(K; d\mu)) \leq d_n(A_2(\Delta), L_2(K; d\mu)).$$

This inequality, together with (3.1) (for $p = 2, q = \infty$) and (6.4) implies

$$\inf_{\{r_n\}} \|r_n\|_{L_p(K; d\mu)} \|1/r_n\|_{L_\infty(\partial D)} \sim e^{-n/C(K, \partial D)} G(\mu')^{1/2}.$$

BIBLIOGRAPHY

- [B] T. Bagby, *The modulus of a plane condenser*, J. Math. and Mech. 17 (1967), 315–329.
- [FM] S. D. Fisher and C. A. Micchelli, *The n -widths of sets of analytic functions*, Duke Math. J. 47 (1980), 789–801.
- [Ga1] T. H. Ganelius, *Rational approximation in the complex plane and on the line*, Ann. Acad. Sci. Fenn. Ser. A I Math. 2 (1976), 129–145.
- [Ga2] ———, *Some extremal functions and approximation*, Fourier Analysis and Approximation Theory (Proc. Conf., Budapest, 1976), Colloq. Math. Soc. János Bolyai, vol. 19, North-Holland, Amsterdam, 1978, pp. 371–381.

- [Go1] A. A. Gonchar, *Estimates of growth of rational functions and some of their applications*, Mat. Sb. **72** (115) (1967), 489–503; English transl., Math. USSR Sb. **1** (1967), 445–456.
- [Go2] —, *On a generalized analytic continuation*, Mat. Sb. **76** (118) (1968), 135–146; English transl., Math. USSR Sb. **5** (1968), 129–140.
- [Go3] —, *Zolotarev problems connected with rational functions*, Mat. Sb. **78** (1969), 640–654; English transl., Math. USSR Sb. **7** (1969), 623–635.
- [H] E. Hille, *Analytic function theory*, vol. 2, Ginn and Company, Boston, MA, 1962.
- [LS] A. L. Levin and E. B. Saff, *Szegő type asymptotics for minimal Blaschke products*, Proc. US-USSR Approximation Theory Conf. 1990 (to appear).
- [N] D. J. Newman, *Rational approximation to $|x|$* , Michigan Math. J. **11** (1964), 11–14.
- [P1] O. G. Parfenov, *Widths of a class of analytic functions*, Mat. Sb. **117** (159) (1982), 279–285; English transl., Math. USSR Sb. **45** (1983), 283–289.
- [P2] —, *The singular numbers of imbedding operators for certain classes of analytic and harmonic functions*, Problemy Mat. Anal., vyp. 9, Izdat. Leningrad. Univ., Leningrad, 1984, pp. 56–66; English transl., J. Soviet Math., **35** (1986), 2193–2200.
- [ST] E. B. Saff and V. Totik, *Logarithmic potentials with external fields*, Springer-Verlag, Berlin (to appear).
- [V] N. S. Vyacheslavov, *On the least deviations of the function $\operatorname{sgn} x$ and its primitives from rational functions in the L_p -metrics, $0 < p \leq \infty$* , Mat. Sb. **103** (145) (1977), 24–36; English transl., Math. USSR Sb. **32** (1977), 19–31.
- [W1] H. Widom, *An inequality for rational functions*, Proc. Amer. Math. Soc. **24** (1970), 415–416.
- [W2] —, *Rational approximation and n -dimensional diameter*, J. Approx. Theory **5** (1972), 343–361.

DEPARTMENT OF MATHEMATICS, OPEN UNIVERSITY, MAX ROWE EDUCATIONAL CENTER, 16 KLAUSNER STREET, P.O.B. 39328, RAMAT-AVIV, TEL-AVIV 61392, ISRAEL

INSTITUTE FOR CONSTRUCTIVE MATHEMATICS, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH FLORIDA, TAMPA, FLORIDA 33620, USA

Received 12/JUNE/91

Translated by E. B. SAFF