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ASYMPTOTIC DISTRIBUTION OF POLES AND ZEROS OF BEST RATIONAL APPROXIMANTS TO x^α ON $[0,1]$

E. B. SAFF

Institute for Constructive Mathematics, Department of Mathematics,
University of South Florida
Tampa, Florida 33620, U.S.A.

H. STAHL

TFH–Berlin /FB2, Luxemburgerstr. 10, 13353 Berlin 65, Germany

Abstract. Let $r_n^* \in \mathcal{R}_{nn}$ be the best rational approximant to $f(x) = x^\alpha$, $1 > \alpha > 0$, on $[0,1]$ in the uniform norm. It is well known that all poles and zeros of r_n^* lie on the negative axis $\mathbb{R}_{<0}$. In the present paper we investigate the asymptotic distribution of these poles and zeros as $n \to \infty$. In addition we determine the asymptotic distribution of the extreme points of the error function $e_n = f - r_n^*$ on $[0,1]$, and survey related convergence results.

1. Introduction. Let $\mathcal{P}_n$ denote the set of all polynomials of degree at most $n \in \mathbb{N}$ with real coefficients, $\mathcal{R}_{mn}$ the set $\{p/q \mid p \in \mathcal{P}_m,\ q \not\equiv 0\}$, $m, n \in \mathbb{N}$, of rational functions. The *best rational approximant* $r_{mn}^* = r_{mn}^*(f, [0,1]; \cdot) \in \mathcal{R}_{mn}$ to the function f on the set $[0,1]$ together with the *minimal approximation error* $E_{mn} = E_{mn}(f, [0,1])$ is defined by

$$E_{mn}(f,[0,1]) := \|f - r_{mn}^*\|_{[0,1]} = \inf_{r \in \mathcal{R}_{mn}} \|f - r\|_{[0,1]}, \tag{1.1}$$

where $\|\cdot\|_{[0,1]}$ denotes the sup-norm on $[0,1]$. It is well known that the best approximant $r_n^* := r_{nn}^*$ to the function $f(x) = x^\alpha$ on $[0,1]$ exists and is unique (see [Me], 9.1 and 9.2, or [Ri], 5.1).

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The approximation of the function x^α on $[0,1]$ or equivalently of $|x|^\alpha$ on $[-1,1]$, $\alpha>0$, by polynomials or rational functions is a model problem for the approximation of more general classes of functions. Much effort has been invested in studying the convergence speed of the sequences $\{E_{n0}(|x|^\alpha,[-1,1])\}$ and $\{E_{nn}(|x|^\alpha,[-1,1])\}$ as $n\to\infty$. In the polynomial case a major was the publication of [Be1] and [Be2] by S. Bernstein in 1913 and 1938, and in the rational case the publication of [Ne] by D. J. Newman in 1964. The papers [Be1] and [Ne] deal with polynomial and rational best approximation of $|x|$ an $[-1,1]$, which is the most important special case of the more general problem of approximating $|x|^\alpha$ on $[-1,1]$ for $\alpha\in\mathbb{R}_+\setminus 2\mathbb{N}$ or x^α on $[0,1]$ for $\alpha\in\mathbb{R}_+\setminus\mathbb{N}$.

Since $f_\alpha(x)=|x|^\alpha$ is an even function, it is an immediate consequence of the uniqueness of best rational approximants that all best approximants r_n^*, $n\in\mathbb{N}$, on $[-1,1]$, are also even functions, and therefore we have

$$r^*_{2m+i,2m+j}=r^*_{2m,2m}=r^*_{2m}\qquad\text{for } m\in\mathbb{N} \text{ and } i,j\in\{0,1\}. \tag{1.2}$$

Replacing x^2 by x in both functions $f_{2\alpha}$ and r^*_{2m} gives us the identity

$$r^*_{2n}(f_{2\alpha},[-1,1];x)=r^*_n(f_\alpha,[0,1];x^2)\qquad\text{for all } n\in\mathbb{N}. \tag{1.3}$$

Hence, the approximation of $|x|^{2\alpha}$ on $[-1,1]$ and x^α on $[0,1]$, $\alpha>0$, poses equivalent problems.

Newman's path breaking result in [Ne] has been improved and extended. Especially, it has been extended to the rational approximation of x^α and $|x|^\alpha$. Important contributions can be found in [FrSz], [Bu1–2], [Go1–3], [Vy1–3], and [Ga]. Surveys are contained in [Vy3] and [St2]. The best result known presently for the rational approximation of $|x|$ on $[-1,1]$ is

THEOREM 1 ([St1]). *We have*

$$E_{nn}(|x|,[-1,1])=8e^{-\pi\sqrt{n}}(1+o(1))\qquad \textit{as } n\to\infty, \tag{1.4}$$

where $o(\cdot)$ denotes Landau's little oh.

For the more general problem of approximating x^α on $[0,1]$ the proof of the following theorem has just been announced:

THEOREM 2 ([St3]). *For $\alpha>0$ we have*

$$E_{nn}(x^\alpha,[0,1])=4^{\alpha+1}|\sin\pi\alpha|e^{-2\pi\sqrt{\alpha n}}(1+o(1))\qquad \textit{as } n\to\infty. \tag{1.5}$$

Remark. With identity (1.3) we get as a corollary to Theorem 2 the error formula

$$E_{nn}(|x|^\alpha,[-1,1])=4^{\alpha/2+1}\left|\sin\pi\frac{\alpha}{2}\right|e^{-\pi\sqrt{\alpha n}}(1+o(1))\qquad\text{as } n\to\infty \tag{1.6}$$

for $\alpha>0$. Choosing $\alpha=1$ shows that (1.4) is a special case of (1.6). Independently of [St3] formula (1.5) has been conjectured in [VC] on the basis of high precision calculations of the limits $\lim_{n\to\infty}e^{2\pi\sqrt{\alpha n}}E_{nn}(x^\alpha,[0,1])$ for the values $\alpha=1/8,1/4,3/8,5/8,3/4,7/8$.

The doubly infinite array $\{r^*_{m,n}\}_{m,n=0}^\infty$ of all best approximants to x^α on $[0,1]$ is called the *Walsh table* of x^α. The sequence $\{r^*_n\}$ is its diagonal. In [SaSt1] and [SaSt2] the convergence behavior of off-diagonal ray sequences (i.e. sequences $\{r^*_{mn}\}$ that satisfy $m/n \to c \neq 1$) has been investigated. These are a bridge between diagonal sequences of rational approximants and polynomial approximants. Consequently, their convergence behavior shows a mixture of characteristics of booth types of approximants.

In the present paper we prove results about the asymptotic distribution of poles and zeros of the (diagonal) approximants r^*_n as $n \to \infty$, and results about the asymptotic distribution of the extreme points of the error function

$$e_n(z) := z^\alpha - r^*_n(z) \tag{1.7}$$

on $[0,1]$ as $n \to \infty$.

Precise knowledge (or a good guess) about the distribution of poles, zeros, and extreme points was basic for the investigation of convergence of r^*_n in [Ne], [Vy1], [Vy2], and [Ga]. The understanding of these distributions gives insight into the nature of the convergence process, and it can be hoped that it may be helpful in finding new strategies for the investigation of the convergence behavior of rational approximants to more general classes of functions than the family x^α, $\alpha > 0$.

In the present paper we shall prove results only under the restriction $0 < \alpha < 1$ in case of rational approximation of x^α on $[0,1]$, and correspondingly $0 < \alpha < 2$ in case of the approximation of $|x|^\alpha$ on $[-1,1]$.

All results will be formulated in the next section and immediate consequences will also be proved there. The two main theorems will be proved in Section 3.

2. Results. In the first lemma we assemble known results about the location of poles, zeros, and extreme points for $n \in \mathbb{N}$ fixed. These results can be found in [SaSt2], Lemma 1.5 and Theorem 1.7.

LEMMA 2.1. *Let* $0 < \alpha < 1$.

(a) *The best rational approximant* $r^*_n = r^*_{nn}(x^\alpha, [0,1]; \cdot)$ *is of exact numerator and denominator degree* n.

(b) *All* n *zeros* $\zeta_{1n}, \ldots, \zeta_{nn}$ *and poles* $\pi_{1n}, \ldots, \pi_{nn}$ *of* r^*_n *lie on the negative half-axis* $\mathbb{R}_{<0}$ *and are interlacing; i.e., with an appropriate numbering we have*

$$0 > \zeta_{1n} > \pi_{1n} > \zeta_{2,n} > \pi_{2,n} > \ldots > \zeta_{nn} > \pi_{nn} > -\infty. \tag{2.1}$$

(c) *The error function* (1.7) *has exactly* $2n+2$ *extreme points* $\eta_{1n}, \ldots, \eta_{2n+2,n}$ *on* $[0,1]$, *and with an appropriate numbering we have*

$$0 = \eta_{1n} < \eta_{2,n} < \ldots < \eta_{2n+2,n} = 1, \tag{2.2}$$

$$\eta^\alpha_{jn} - r^*_n(\eta_{jn}) = (-1)^j E_{nn}(x^\alpha, [0,1]), \quad j = 1, \ldots, 2n+2. \tag{2.3}$$

Remark. If $\alpha > 1$, then part (b) of Lemma 2.1 may no longer be true. It cannot be excluded that $[\alpha]$ ($[\alpha] \in \mathbb{N}$ and $\alpha - 1 < [\alpha] \leq \alpha$) zeros and poles of r^*_n lie outside of $\mathbb{R}_{<0}$.

If we consider best rational approximants $r_n^* = r_n^*(|x|^\alpha, [-1,1]; \cdot)$ to $|x|^\alpha$ on $[-1,1]$ instead of approximants to x^α on $[0,1]$, then we get results similar to the of Lemma 2.1. However, now poles and zeros all lie on the imaginary axis.

LEMMA 2.2. *Let $0 < \alpha < 2$ and $n \in \mathbb{N}$ even.*

(a) *The best rational approximant $r_n^* = r_n^*(|x|^\alpha, [-1,1]; \cdot)$ is of exact numerator and denominator degree n.*

(b) *Half the zeros $\{\zeta_{jn}\}$ and half the poles $\{\pi_{jn}\}$ lie on the positive imaginary axis $i\mathbb{R}_{>0}$ and the other half on the negative imaginary axis. With an appropriate numbering we have*

$$(2.4)\qquad 0 < \frac{1}{i}\zeta_{1n} < \frac{1}{i}\pi_{1n} < \frac{1}{i}\zeta_{2,n} < \frac{1}{i}\pi_{2,n} < \dots < \frac{1}{i}\zeta_{n/2,n} < \frac{1}{i}\pi_{n/2,n} < \infty,$$

$$(2.5)\qquad \zeta_{jn} = -\zeta_{j-n/2,n}, \quad \pi_{jn} = -\pi_{j-n/2,n} \quad \text{for } j = n/2+1, \dots, n.$$

(c) *The error function (1.7) has exactly $2n+3$ extreme points $\eta_{1n}, \dots, \eta_{2n+3,n}$ on $[-1,1]$, and with an appropriate numbering we have*

$$\text{;)}\qquad \begin{gathered} 0 = \eta_{n+2,n} < \eta_{n+1,n} < \dots < \eta_{1,n} = 1, \\ \eta_{jn} = -\eta_{2n+4-j,n} \quad \text{for } j = n+3, \dots, 2n+3, \end{gathered}$$

and

$$|\eta_{jn}|^\alpha - r_n^*(\eta_{jn}) = (-1)^{j+1} E_{nn}(|x|^\alpha, [- \;\; 1]) \quad \text{for } j = 1, \quad , 2n+3$$

Proof. Lemma 2.2 is an immediate consequence of Lemma 2.1 if we substitute the independent variable z by z^2 in each of the functions r_n^*, $f_\alpha(x) = x^\alpha$, and e_n.

In the next two theorems we state the main results of the paper. We first consider the asymptotic distribution of poles and zeros. as $n \to \infty$.

THEOREM 3. *Let $0 < \alpha < 1$, and let $\{\zeta_{jn}\}_{j=1}^n$ and $\{\pi_{jn}\}_{j=1}^n$ be the set of zeros and poles, respectively, of the best rational approximant $r_n^* = r_n^*(x^\alpha, [0,1])$. Then for any interval $[c,d] \subseteq \mathbb{R}_{<0}$ with $-\infty \le c \le d < 0$ we have*

$$(2.8)\qquad \lim_{n\to\infty} \frac{1}{\sqrt{n}} \operatorname{card}\{\zeta_{jn} \in [c,d]\} = \frac{\sqrt{\alpha}}{\pi} \int_{|d|}^{|c|} \frac{dt}{t\sqrt{1+t}},$$

and

$$(\text{---})\qquad \lim_{n\to\infty} \frac{1}{\sqrt{n}} \operatorname{card}\{\pi_{jn} \in [c,d]\} = \frac{\sqrt{\alpha}}{\pi} \int_{|d|}^{|c|} \frac{dt}{t\sqrt{1+t}}.$$

Remark. (1) From (2.8) and (2.9) we learn that poles and zeros are asymptotically dense in $\mathbb{R}_{<0}$. The number of poles or zeros on any given closed subinterval $[c,d] \subseteq \mathbb{R}_{<0}$ grows like $\sqrt{n}$ as $n \to \infty$. Since the total number of poles or zeros is n, the result proves that almost all poles and zeros tend to the origin as $n \to \infty$.

(2) It is vary probable that Theorem 3 holds for all $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$. However, a proof will become more complicated since when $\alpha > 1$ there are poles and zeros of

r_n^* outside of $\mathbb{R}$, which demand special attention. It seems worth mentioning that the asymptotic distributions in (2.28) and (2.29) are independent of α except for the factor $\sqrt{\alpha}$.

If we proceed in the same way as in the transition from Lemma 2.1 to Lemma 2.2, then as a corollary we can deduce the asymptotic distribution of poles and zeros of best approximants to $|x|^\alpha$ on $[-1,1]$ from Theorem 3.

COROLLARY. *Let $n \in \mathbb{N}$ be even, $0 < \alpha < 2$, and let $\{\zeta_{jn}\}_{j=1}^n$ and $\{\pi_{jn}\}_{j=1}^n$ be the set of zeros and poles, respectively, of the approximant $r_n^* = r_n^*(|x|^\alpha, [-1,1]; \cdot)$. Then for any interval $[ic, id] \subseteq \mathbb{R}\backslash\{0\}$, i.e. for $0 < c \leq d \leq \infty$ or $-\infty \leq c \leq d < 0$ we have*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \operatorname{card}\{\zeta_{jn} \in [ic, id]\} = \frac{\sqrt{\alpha}}{\pi} \int_a^b \frac{dt}{t\sqrt{1+t^2}}$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \operatorname{card}\{\pi_{jn} \in [ic, id]\} = \frac{\sqrt{\alpha}}{\pi} \int_a^b \frac{dt}{t\sqrt{1+t^2}} \tag{2.11}$$

with $a := \sqrt{\min(|c|,|d|)}$ and $b := \sqrt{\max(|c|,|d|)}$

Proof. From identity (1.3) it follows that $\zeta_{j,2n}$ is a zero of $r_{2n}^*(|x|^{2\alpha}, [-1,1]; \cdot)$ if and only if $\zeta_{j,2n}^2$ is a zero of $r_n^*(x^\alpha, [0,1]; \cdot)$. Note that the zeros of $r_{2n}^*(|x|^{2\alpha}, [-1,1]; \cdot)$ are mapped pairwise onto the zeros of $r_n^*(x^\alpha, [0,1])$. If the mapping $z \mapsto \sqrt{z}$ is applied to all variables in (2.8) and (2.9), then one arrives at (2.10) and (2.11). ∎

Next, we turn to the investigation of the asymptotic distribution of extreme points. Again, we start with a result for best rational approximants to x^α on $[0,1]$, and deduce from that the corresponding result for approximants to $|x|^\alpha$ on $[-1,1]$ as a corollary.

THEOREM 4. *Let $0 < \alpha < 1$, and let $\{\eta_{jn}\}_{j=1}^{2n+2}$ be the extreme points of the error function $e_n(x) = f_\alpha(x) - r_n^*(f_\alpha, [0,1]; x)$, $f_\alpha(x) = x^\alpha$, on $[0,1]$. Then for any interval $[c,d] \subseteq (0,1]$ with $0 < c \leq d \leq 1$ we have*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \operatorname{card}\{\eta_{jn} \in [c,d]\} = \frac{2\sqrt{\alpha}}{\pi} \int_c^d \frac{dt}{t\sqrt{1-t}}. \tag{2.12}$$

Remarks. (1) Formula (2.12) shows that the extreme points are asymptotically dense in $[0,1]$, and futher that almost all extreme points tend to the origin as $n \to \infty$.

(2) As with Theorem 3, it seems that Theorem 4 holds for all $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$, but this will not be proved here.

From Theorem 4 we can deduce the following analogous result for the approximants $r_n^* = r_n(|x|^\alpha, [-1,1]; \cdot)$ to $|x|^\alpha$ on $[-1,1]$.

COROLLARY. *Let $n \in \mathbb{N}$ be even, $0 < \alpha < 2$, and let $\{\eta_{jn}\}_{j=1}^{2n+3}$ be the set of extreme points of the error function $e_n(x) = |x|^\alpha = r_n^*(|x|^\alpha, [-1,1]; \cdot)$ on $[-1,1]$. Then for any interval $[c,d] \subseteq [-1,1] \setminus \{0\}$; i.e., for $0 < c \leq d \leq 1$ or $-1 \leq c \leq d < 0$ we have*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \operatorname{card}\{\eta_{jn} \in [c,d]\} = \frac{2\sqrt{\alpha}}{\pi} \int^{b} \frac{dt}{t\sqrt{1-t^2}}, \tag{2.13}$$

with $a := \sqrt{\min(|c|,|d|)}$ and $b := \sqrt{\max(|c|,|d|)}$.

Remark. It has been stated in part (c) of Lemmas 2.1 and 2.2 that we have exactly $2n+2$ extreme points in the case of the approximation of x^α on $[0,1]$, and $2n+3$ extreme points in the case of the approximation of $|x|^\alpha$ on $[-1,1]$.

Theorems 3 and 4 give information about the asymptotic density of zeros, poles, and extreme points. These results are not precise enough for determining the position of individual zeros, poles, or extreme points. In [St4] asymptotic formulae have been proved that give the location of zeros, poles, and extreme points in the special case of the approximant $r_n^*(|x|, [-1,1])$ with such a degree of precision that the position of individual objects can be distinguished. We give an example of this type of result. In [St4], Theorem 2.2, it has been shown that if

$$F_n(y) := \frac{n+1}{2} - \frac{1}{\pi} \int_y^\infty \left[\frac{\sqrt{n}}{t\sqrt{1}} + \frac{1}{\pi t} \log \frac{t}{1+\sqrt{1+t^2}} \right] dt, \tag{2.14}$$

then

$$\frac{iF_n(j_n - 1/2)}{\zeta_{j_n n}} = 1 + o\left(\frac{1}{\sqrt{n}}\right) \quad \text{and} \quad \frac{iF_n(j_n)}{\pi_{j_n n}} \qquad)$$

for all sequences of indices $\{j_n \in \{1, \ldots, n/2\} \mid n \in 2\mathbb{N}\}$ that satisfy

$$\frac{n}{2} - j_n = \mathcal{O}(\sqrt{n}) \quad \text{as } n \to \infty. \tag{2.16}$$

It can be shown that condition (2.16) is equivalent to

$$1/\zeta_{j_n n} = \mathcal{O}(1) \quad \text{or} \quad 1/\pi_{j_n n} = \mathcal{O}(1) \quad \text{as } n \to \infty, \tag{2.17}$$

where $\mathcal{O}(\cdot)$ denotes Landau's big oh. Since the error terms are smaller than $1/\sqrt{n}$ and the number of zeros and poles on a given interval grows like $\sqrt{n}$, it follows that the asymptotic relations in (2.15) are precise enough to determine individual objects. It is immediate that for the case $\alpha = 1$ the estimates (2.15) imply the asymptotic relations (2.10) and (2.11).

The proof of (2.15) is deeper and much more complicated than that of Theorems 3 or 4, which can be given by purely potential-theoretic considerations. However, the results of Theorems 3 and 4 are valid for approximants to x^α and $|x|^\alpha$, while the results in [St4] could only be proved for the approximation of $|x|$

on $[-1,1]$ or $\sqrt{x}$ on $[0,1]$. So far it is not clear how the results of [St4] can be extended to a more general class of functions.

3. Proofs of Theorems 3 and 4. The two proofs will be prepared by three lemmas. We start by studying as an auxiliary function

$$f_n(z) := \frac{z^\alpha - r_n^*(z)}{z^\alpha + r_n^*(z)} \quad \text{for } z \in \mathbb{C} \setminus \mathbb{R}_-, \tag{3.1}$$

where $r_n^* = r_n^*(x^\alpha, [0,1]; \cdot)$, $0 < \alpha < 1$, $n \in \mathbb{N}$, and z^α is the principal branch with $z^\alpha > 0$ for $z > 0$. On $\mathbb{R}_- := \mathbb{R}_{\leq 0}$ the function f_n has different boundary values for continuation from above and below $\mathbb{R}_-$. These values will be denoted by $f_n(z + i0)$ and $f_n(z - i0)$, $z \in \mathbb{R}_-$.

LEMMA 3.1. (a) *f_n is analytic in $\mathbb{C} \setminus \mathbb{R}_-$.*

(b) *f_n has exactly $2n+1$ zeros $z_{1n}, \ldots, z_{2n+1,n}$ in $\mathbb{C} \setminus \mathbb{R}_-$. These zeros are all contained in $(0,1)$, and we have*

$$\eta_{jn} < z_{jn} < \eta_{j+1,n}, \qquad j = 1, \ldots, 2n+1, \tag{3.2}$$

where η_{jn} are the extreme points introduced in (2.2).

(c) *The boundary values of f_n on $\mathbb{R}_-$ satisfy the restrictions*

$$m := \min\left(\tan\frac{\pi}{2}\alpha, \cot\frac{\pi}{2}\alpha\right) \leq |f_n(z \pm i0)| \leq \max\left(\tan\frac{\pi}{2}\alpha, \cot\frac{\pi}{2}\alpha\right) =: M$$

for $z \in \mathbb{R}_-$.

Proof. It is a consequence of the alternating signs in (2.3) of Lemma 2.1 that f_n has at least $2n+1$ zeros z_{jn}, $j = 1, \ldots, 2n+1$, satisfying (3.2). Since $f_n(z_{jn}) = 0$ implies $z_{jn}^\alpha = r_n^*(z_{jn})$, we learn from the existence of these zeros that the rational function $r_n^* \in \mathcal{R}_{nn}$ interpolates z^α at the $2n+1$ points z_{jn}, and therefore r_n^* is determined by this interpolation property. As a consequence we can apply formulae from the theory of rational interpolation.

Set $r_n^* = p_n/q_n$, $p_n, q_n \in \mathcal{P}_n$. From the theory of rational interpolants (multipoint Padé approximants) to Markov or Stieltjes functions (see [StTo], Lemma 6.1.2) we know that the interpolation error $e_n(z) = z^\alpha - r_n^*(z)$ can be represented as

$$e_n(z) = \frac{\omega_n(z)}{2\pi i q_n(z)^2} \oint_C \frac{q_n(\zeta)^2 \zeta^\alpha \zeta}{\omega_n(\zeta)(\zeta - z)} = \frac{\sin \pi\alpha}{\pi} \frac{\omega_n(z)}{q_n(z)^2} \int_{-\infty}^{0} \frac{q_n(x)^2 |x|^\alpha dx}{\omega_n(x)(x - z)}$$

$$\omega_n(z) := \prod_{i=1}^{2n+1} z - z_{jn} \tag{3.5}$$

and C is a closed integration contour in $\mathbb{C}\setminus\mathbb{R}_-$ surrounding z and all interpolation points z_{jn}, $j=1,\dots,2n+1$. Since q_n and ω_n are real polynomials, and $\omega_n(x)\neq 0$ for all $x\in\mathbb{R}_-$, it is easy to verify that

$$\frac{\sin\pi\alpha}{\pi}\int_{-\infty}^{0}\frac{q_n(x)^2|x|^\alpha dx}{\omega_n(x)(x-z)}\neq 0 \quad \text{for all } z\in\mathbb{C}\setminus\mathbb{R}_-. \tag{3.6}$$

Hence, part (b) of the lemma follows from (3.4).

Next, we consider f_n on $\mathbb{R}_-$. We have

$$f_n(0)=-1, \qquad f_n(\infty)=1, \tag{3.7}$$

and all zeros ζ_{jn} and poles π_{jn} of r_n^* are characterized by the property that

$$f_n(\zeta_{jn}\pm i0)=1, \qquad f_n(\pi_{jn}\pm i0)=-1 \quad \text{for } j=1,\dots,n. \tag{3.8}$$

On $\mathbb{R}_-$ the function f_n can be written as

$$f_n(z\pm i0)=\frac{1-e^{\mp i\pi\alpha}r_n^*(z)|z|^{-\alpha}}{1+e^{\mp i\pi\alpha}r_n^*(z)|z|^{-\alpha}}, \qquad z\in\mathbb{R}_-. \tag{3.9}$$

The values of the function $r_n^*(z)/|z|^\alpha$ run through the whole real axis $\mathbb{R}$ if z in moved from 0 to $\pi_{1,n}$ or from $\pi_{j,n}$ to $\pi_{j+1,n}$, $j=1,\dots,n-1$. By elementary trigonometric calculations it can be verified that the mapping

$$x\mapsto\frac{1-e^{-i\alpha\pi}x}{1+e^{-i\alpha\pi}x}, \qquad x\in\mathbb{R}, \tag{3.10}$$

maps $\mathbb{R}$ onto the circle

$$K_\alpha:=\left\{z\in\mathbb{C}\;\middle|\;|z+i\cot\alpha\pi|=\frac{1}{\sin\pi\alpha}\right\} \tag{3.11}$$

Hence, it follows from (3.9) that

$$f_n(z+i0)\in K_\alpha \qquad \text{for all } z\in\mathbb{R}_-. \tag{3.12}$$

Moreover, together with (3.7) and (3.8) we deduce that $\arg f_n(z+i0)$ grows by $(2n+1)\pi$ if z runs through $\mathbb{R}_-+i0$ from $-\infty$ to 0.

For the boundary values of f_n on $\mathbb{R}_-$ from below, i.e. on $\mathbb{R}_- - i0$, we have

$$f_n(z-i0)\in\overline{K}_\alpha:=\{z\mid\bar z\in K_\alpha\} \qquad \text{for all } z\in\mathbb{R}_-, \tag{3.13}$$

and $\arg f_n(z-i0)$ grows again by $(2n+1)\pi$ if z runs through $\mathbb{R}_- - i0$ from 0 to $-\infty$.

Since $\tan\frac{\pi}{2}\alpha$ and $-\cot\frac{\pi}{2}\alpha$ are extremal ordinates of the circle K_α, part (c) follows from (3.12) and (3.13).

Since $\arg f_n$ grows by $(2n+1)2\pi$ along the whole boundary of $\mathbb{C}\setminus\mathbb{R}_-$, and since f_n has exactly $2n+1$ zeros in $\mathbb{C}\setminus\mathbb{R}_-$, it follows from the argument principle that f_n has no poles in $\mathbb{C}\setminus\mathbb{R}_-$. This proves part (a) of the lemma. ∎

In the assertions of the next lemma we summarize (3.12), (3.13), and the immediate conclusions made in this connection.

LEMMA 3.2. *We have*

$$f_n(z+i0) \in K_\alpha \quad \text{and} \quad f_n(z-i0) \in \overline{K}_\alpha \quad \text{for all } z \in \mathbb{R}_-. \tag{3.14}$$

The function $\arg f_n(z+i0)$ *is increasing and* $\arg f_n(z-i0)$ *is decreasing for* $z \in \mathbb{R}_-$, *and we have*

$$\begin{aligned} &\arg f_n(0+i0) - \arg f_n(-\infty+i0) \\ &\qquad = \arg f_n(-\infty - i0) - \arg f_n(0-i0) = (2n+1)\pi. \end{aligned} \tag{3.15}$$

Let the function p be harmonic in $\mathbb{C} \setminus \mathbb{R}_-$, continuous in $\overline{\mathbb{C}} \setminus \{0\}$, and let it have the boundary values

$$p(z) = \begin{cases} \pi & \text{for } z \in (0,1], \\ 0 & \text{for } z \in \mathbb{R}_-. \end{cases}$$

The function p is uniquely determined by these properties.

LEMMA 3.3. *For the function* p *we have the representations*

$$p(z) = \int \log \frac{1}{|z-x|} d(\nu - \widehat{\nu})(x) = \int g_{\mathbb{C}\setminus\mathbb{R}_-}(z,x)\, d\nu(x) \tag{3.17}$$

where ν *and* $\widehat{\nu}$ *are positive measures defined by*

$$\frac{d\nu(x)}{dx} = \frac{1}{\pi x\sqrt{1-x}} \qquad x \in (0,1], \tag{3.18}$$

$$\frac{d\widehat{\nu}(x)}{dx} = \frac{-1}{\pi x\sqrt{1-x}} \qquad x \in \mathbb{R}_-, \tag{3.19}$$

where $g_{\mathbb{C}\setminus\mathbb{R}_-}(z,x)$ *is the Green function with pole at* x *for* $\mathbb{C}\setminus\mathbb{R}_-$ (for the definition see [StTo], Appendix V).

P r o o f. Consider the function

$$f(z) := \frac{1}{\pi} \int_{-\infty}^{1} \log\left(\frac{1}{z-x}\right) \frac{dx}{x\sqrt{1-x}} \qquad z \in \mathbb{C} \setminus \mathbb{R}_-, \tag{3.20}$$

where the integral at $x = 0$ is understood as Cauchy principal value, the square root is assumed to be positive, and the logarithm is defined in $\mathbb{C} \setminus \mathbb{R}_-$. Differentiating (3.20) yields

$$\begin{aligned} f'(z) &= \frac{1}{\pi} \int_{-\infty}^{1} \frac{dx}{(x-z)x\sqrt{1-x}} \\ &= \frac{-1}{i\pi} \int_{-\infty}^{1} \frac{dx}{(x-z)x\sqrt{x-1}}, \qquad z \in \mathbb{C} \setminus \mathbb{R}_- \end{aligned} \tag{3.21}$$

where in the last integral we have assumed that $\sqrt{x-1} > 0$ for $x > 1$. Since by this convention the square root is of different sign on both sides of $(-\infty, 1]$, we can duplicate the integration path. Because of the different signs of the square root, we can also pass from the Cauchy principal value to an integration path along two small halfcircles around $x = 0$. In this way we see that

$$f'(z) = \frac{-1}{2\pi i} \oint_C \frac{d\zeta}{(\zeta - z)\zeta\sqrt{\zeta - 1}}, \tag{3.22}$$

where C is a closed positively oriented integration path in $\mathbb{C} \setminus \mathbb{R}_-$ surrounding z. Cauchy's integral formula then yields that

$$f'(z) = \frac{-1}{z\sqrt{z-1}}, \qquad z \in \mathbb{C} \setminus \mathbb{R}_- \tag{3.23}$$

From (3.20) and (3.23) we deduce that

$$f(z) = \int_{-\infty}^{z} f'(\zeta)\, d\zeta = -\int_{-\infty}^{z} \frac{d\zeta}{\zeta\sqrt{\zeta - 1}} \tag{3.24}$$

for any integration path form $-\infty$ to z in $\overline{\mathbb{C}} \setminus \mathbb{R}_-$. As a consequence we see that

$$\operatorname{Re} f(z \pm i0) = 0 \qquad \text{for } z \in \mathbb{R}_{<0}. \tag{3.25}$$

Integrating along the halfcircle $S = \{z = \varepsilon e^{i(\pi - t)} \mid 0 \le t \le \pi\}$, $\varepsilon > 0$, yields

$$\int_S \frac{d\zeta}{\zeta\sqrt{\zeta - 1}} = -\int_0^{\pi} \frac{i\varepsilon e^{i(\pi-t)} dt}{\varepsilon e^{i(\pi-t)}\sqrt{\varepsilon e^{i(\pi-t)} - 1}} \to -\pi \qquad \text{as } \varepsilon \to 0. \tag{3.26}$$

From (3.24) it follows that $\operatorname{Re} f$ is constant on $(0, 1]$ for the same reason as it was constant on $\mathbb{R}_{<0}$. It then follows from (3.25) and (3.26) that

$$\operatorname{Re} f(z) = \pi \qquad \text{for } z \in (0, 1]. \tag{3.27}$$

Note that in (3.24) the integration runs in the opposite direction of that in (3.26). From (3.25) and (3.27) we deduce that $p(z) = \operatorname{Re} f(z)$, which together with (3.20) proves the first equality in (3.17).

The second equality follows from considering the difference

$$d(z) := \int \log \frac{1}{|z - x|} d(\nu - \widehat{\nu})(x) - \int g_{\mathbb{C} \setminus \mathbb{R}_-}(z, x) d\nu(x) \tag{3.28}$$

The function d is harmonic in $\mathbb{C} \setminus \mathbb{R}_-$ and continuous on $\mathbb{C} \setminus \{0\}$. Since $d(z) = 0$ for $z \in \mathbb{R}_-$, it follows that $d \equiv 0$, which proves the second equality in (3.17). ∎

Proof of Theorem 4. In this proof potential-theoretic tools play a fundamental role. It follows from Lemma 3.1 that

$$q_n(z) := \frac{1}{2\sqrt{\alpha n}} \log \frac{1}{|f_n(z)|} \tag{3.29}$$

is superharmonic in $\mathbb{C} \setminus \mathbb{R}_-$, where f_n is the function defined in (3.1). Because of (3.3), q_n is bounded on $\mathbb{R}_-$.

We define a positive measure μ_n by

$$\mu_n \quad \frac{1}{2\sqrt{\alpha n}} \sum_{j=1}^{2n+1} \delta_{z_{jn}} \tag{3.30}$$

where δ_z is the Dirac measure at the point $z \in \mathbb{C}$, and the z_{jn} are the zeros of f_n mentioned in (3.2) of Lemma 3.1. The Green potential associated with μ_n is defined as

$$g_n(z) := g(\mu_n; z) := \int g_{\mathbb{C}\setminus\mathbb{R}_-}(z, x)\,d\mu_n(x), \tag{3.31}$$

where $g_{\mathbb{C}\setminus\mathbb{R}_-}(z, x)$ is the Green function in $\mathbb{C} \setminus \mathbb{R}_-$ with logarithmic pole at $x \in \mathbb{C}\setminus\mathbb{R}_-$. Since the Green function is identically zero on $\mathbb{R}_-$, we also have $g_n(z) = 0$ for all $z \in \mathbb{R}_-$. From the estimate of f_n on $\mathbb{R}_-$ in (3.3) and the fact that q_n and g_n have the same logarithmic singularities in $\mathbb{C} \setminus \mathbb{R}_-$, we deduce that

$$\begin{aligned}\frac{1}{2\sqrt{\alpha n}}\log\frac{1}{M} + g_n(z) &\le q_n(z)\\ &\le g_n(z) + \frac{1}{2\sqrt{\alpha n}}\log\frac{1}{m} \qquad \text{for all } z \in \mathbb{C}\end{aligned} \tag{3.32}$$

As a consequence we have

$$\begin{aligned}|q_n(z) - g_n(z)| &\le \frac{1}{2\sqrt{\alpha n}}\max\left(\log\frac{1}{m}, -\log\frac{1}{M}\right)\\ &= \frac{1}{2\sqrt{\alpha n}}\log M \qquad \text{for all } z \in \mathbb{C}.\end{aligned} \tag{3.33}$$

In (3.33) we have used the fact that $1/m = M$; the constants m and M have been defined in (3.3). The estimate (3.33) shows that the sequences $\{q_n\}$ and $\{g_n\}$ have identical limits if a limit exists.

In the sequel we denote the approximation error $E_{nn}(x^\alpha, [0, 1])$ by ε_n. From definition (3.1) of f_n we deduce that

$$|f_n(z)| = \frac{|e_n(z)|}{|2z^\alpha - e_n(z)|} \le \frac{\varepsilon_n}{|2z^\alpha - \varepsilon_n|} \qquad \text{for } z \in [(\varepsilon_n/2)^{1/\alpha}, 1]$$

From Theorem 2 we know that

$$\varepsilon_n = 4^{1+\alpha}(\sin\alpha\pi)e^{-2\pi\sqrt{\alpha n}}(1 + o(1)) \qquad \text{as } n \quad \infty \tag{3.35}$$

Inserting (3.35) into (3.34) yields with (3.29) that

$$\liminf_{n\to\infty} q_n(z) \ge \liminf_{n\to\infty}\frac{1}{2\sqrt{\alpha n}}[\log|2z^\alpha - \varepsilon_n| - \log\varepsilon_n] \ge \pi \tag{3.36}$$

$$|f_n(\eta_{jn})| = \frac{\varepsilon_n}{|2\eta_{jn}^\alpha - (-1)^j\varepsilon_n|} \ge \frac{\varepsilon_n}{2 + \varepsilon_n} \ge \frac{2\varepsilon_n}{\ }$$

for all $n \in \mathbb{N}$ and $j \in \{1 \quad , 2n+2\}$. With (3.29) and (3.35) this implies that

$$\limsup_{n\to\infty} q_n(\eta_{j_n n}) \le \pi$$

for any sequence $\{\eta_{j_n n} \mid j_n \in \{1, \dots, 2n+2\},\ n \in \mathbb{N}\}$.

From (3.33) we learn that the limits (3.36) and (3.38) also hold true if q_n is substituted by $g_n = g(\mu_n; \cdot)$. We will determine the limit distribution of the sequence $\{\mu_n\}$ by studding the convergence behavior of the sequence $\{g_n\}$. In the analysis the limits (3.36) and (3.38) will play a major role. However, there are two difficulties: (i) the total mass of the measure μ_n tends to infinity as $n \to \infty$, and (ii) the support of μ_n touches the boundary point 0 of $\mathbb{C} \setminus \mathbb{R}_-$ as $n \to \infty$. In a certain sense both phenomena have an opposite effect and compensate mutually. But the situation demands a careful analysis.

First, we show that all restrictions $\mu_n|_{[a,1]}$, $0 < a \le 1$, are bounded as $n \to \infty$. Let a, $0 < a < 1$, be fixed, and let $\widehat{\mu}_n$ be the balayage measure resulting from sweeping μ_n out of the domain $\mathbb{C} \setminus (\mathbb{R}_- \cup [a,1])$ onto $\mathbb{R}_- \cup [a,1]$ (for the definition of balayage and a summary of its properties see Appendix VII in [StTo]). Let $\widehat{\mu}_n^a$ and $\check{\mu}_n^a$ be the restrictions of $\widehat{\mu}_n$ onto $[a,1]$ and $\mathbb{R}_-$, respectively. Thus, we have $\widehat{\mu}_n = \widehat{\mu}_n^a + \check{\mu}_n^a$. Since $g_{\mathbb{C}\setminus\mathbb{R}_-}(z,x) = 0$ for all $z \in \mathbb{R}_-$, it follows from the properties of balayage that

$$g(\widehat{\mu}_n^a; z) = g(\mu_n; z) \qquad \text{for all } z \in [a,1], \tag{3.39}$$

where $g(\widehat{\mu}_n^a; \cdot)$ denotes the Green potential of the measure $\widehat{\mu}_n^a$ as defined in (3.31). From (2.2) in Lemma 2.1, we know that $\eta_{2n+2,n} = 1$ is an extreme point of the error function e_n for all $n \in \mathbb{N}$. Therefore, we can deduce from (3.33), (3.38), and (3.39) that

$$\limsup_{n\to\infty} g(\widehat{\mu}_n^a; 1) \le \pi. \tag{3.40}$$

Since $\operatorname{supp}(\widehat{\mu}_n^a) \subseteq [a,1]$, the boundedness of the sequence of Green potentials $\{g(\widehat{\mu}_n^a; \cdot)\}$ at the interior point 1 of the domain $\mathbb{C} \setminus \mathbb{R}_-$ implies that the $\{\|\widehat{\mu}_n^a\|\}$ is bounded as $n \to \infty$. From the definition of balayage we know that $\widehat{\mu}_n^a \ge \mu_n|_{[a,1]}$. Hence, there exists a constant $c_0 = c_0(a) < \infty$ with

$$\mu_n([a,1]) \le c_0 \qquad \text{for } n \in \mathbb{N}. \tag{3.41}$$

Since the sequence $\{\mu_n\}$ is bounded on each interval $[a,1]$, $0 < a \le 1$, it follows from Helly's Theorem (the weak compactness of the unit ball of positive measures) that any infinite sequence $\{\mu_n\}_{n\in N}$, $N \subseteq \mathbb{N}$, contains an infinite subsequence, denoted again by $\{\mu_n\}_{n\in N}$, which is weakly convergent in $\mathbb{C}\setminus\mathbb{R}_-$, i.e. there exists a measure μ with support $\operatorname{supp}(\mu) \subseteq [0,1]$ and

$$\mu_n \xrightarrow{*} \mu \qquad \text{as } n \to \infty,\ n \in N. \tag{3.42}$$

Here, $\xrightarrow{*}$ denotes the weak convergence of measures in $\mathbb{C}\setminus\mathbb{R}_-$, i.e. for each function f continuous and having compact support in $\mathbb{C} \setminus \mathbb{R}_-$ we have $\int d\mu_n \to f d\mu$ as $n \to \infty$, $n \in N$.

In order to understand the convergence behavior of the sequence $\{g_n = g(\mu_n;\cdot)\}$ we split the measure μ_n in two parts. Let $a \in (0,1)$ be a point satisfying $\mu(\{a\}) = 0$, and set

$$g_n^a := g(\mu_n|_{[a,1]};\cdot) \quad \text{and} \quad g_n^{ac} := g(\mu_n|_{[0,a)};\cdot), \; n = 1,2,\ldots \tag{3.43}$$

From (3.42) it follows that

$$\mu_n|_{[a,1]} \xrightarrow{*} \mu|_{[a,1]} \qquad \text{as } n \to \infty, \; n \in N, \tag{3.44}$$

and from (3.41) we know that $\mu([a,1])$ is finite. Since all Green potential are nonnegative, it follows from (3.39), (3.40), and (3.43) that

$$\limsup_{n\to\infty, n\in N} g_n^{ac}(1) \leq \pi. \tag{3.45}$$

The Green potential $g_n^{ac} = g(\mu_n|_{[0,a)};\cdot)$ is harmonic and nonnegative in $\mathbb{C} \setminus (\mathbb{R}_- \cup [0,a])$. From (3.45) together with Harnack's inequality (see Appendix III of [StTo]), the boundedness of the harmonic conjugate. and Montel's Theorem it follows that there exists an infinite subsequence of N, which we continue to denote by N, such that the limit

$$\lim_{n\to\infty, n\in N} g_n^{ac}(z) =: g^{ac}(z) \tag{3.46}$$

holds locally uniformly in $\mathbb{C} \setminus (\mathbb{R}_- \cup [0,a])$. The function g^{ac} is harmonic in $\mathbb{C} \setminus \mathbb{R}_- \cup [0,a])$.

The convergence of the sequence $\{g_n^a\}$ is determined by (3.44). We have

$$\lim_{n\to\infty, n\in N} g_n^a(z) = g^a(z) := g(\mu|_{[a,1]};z) \tag{3.47}$$

for z locally uniformly in $\mathbb{C} \setminus (\mathbb{R}_- \cup [a,1])$. From the Lower Envelope Theorem for potentials it follows that

$$\liminf_{n\to\infty, n\in N} g_n^a(z) = g^a(z) \qquad \text{for qu. e. } z \in [a,1] \tag{3.48}$$

(cf. Appendix III of [StTo]), and for the principle of descent (cf. Appendix III of [StTo]) if follows that for any sequence $x_n \to x_0 \in [a,1]$ as $n \to \infty$, $n \in N$ we have

$$\liminf_{n\to\infty, n\in N} g_n^a(x_n) \geq g^a(x_0). \tag{3.49}$$

(The Lower Envelope Theorem and the principle of descent hold not only for logarithmic potentials, but also for Green potentials since the Green function can be represented by a logarithmic potential (cf. Appendix V of [StTo])).

It is possible to select a sequence $a_m \to 0$ with $1 > a_m > 0$ and $\mu(\{a\}) = 0$. For each $[a_m, 1]$ there exists an infinite subsequence N_m with $N_{m+1} \subseteq N_m$ and the limits (3.46)–(3.49) hold. If one choose a diagonal sequence from $(N_1, N_2, \ldots)$ and denotes this sequence by N, then the limits (3.46)–(3.49) hold for this sequence N and for all subintervals $[a_m, 1] \subseteq (0,1]$. This proves that the limit

$$\lim_{n\to\infty, n\in N} g_n(z) =: g(z) \tag{3.50}$$

holds locally uniformly in $\mathbb{C} \setminus (\mathbb{R}_- \cup [0,1])$, and if on $[a_m, 1]$ the function g is defined as $g := g^{a_m} + g^{a_m c}$ with (3.46) and (3.47), then the limit

$$\liminf_{n\to\infty, n\in N} g_n(z) = g(z) \tag{3.51}$$

holds for quasi every z in $(0,1]$, and for $x_n \to x_0 \in (0,1]$ as $n \to \infty$, $n \in N$, we have

$$\liminf_{n\to\infty, n\in N} g_n(x_n) \ge g(x_0). \tag{3.52}$$

The function g is harmonic in $\mathbb{C} \setminus (\mathbb{R}_- \cup [0,1])$, superharmonic in $\mathbb{C} \setminus \mathbb{R}_-$, and we have $g(z) = 0$ for $z \in \mathbb{R}_{<0}$.

From (3.52) together with (3.38), which also hold for g_n because of (3.33), it follows that

$$g(z) \le \pi \quad \text{for all } z \in \bigcap_{m=1}^{\infty} \overline{\bigcup_{n\ge m} \{\eta_{1n}, \dots, \eta_{2n+n,n}\}} \setminus \{0\}. \tag{3.53}$$

The point 0 had to be excluded, since (3.38) is not available for the boundary point 0. Since we know from (3.2) that between two adjacent extreme points η_{jn} and $\eta_{j+1.n}$ there is a zero z_{jn} of e_n, it follows from (3.42) that

$$\operatorname{supp}(\mu) \subseteq \bigcap_{m=1}^{\infty} \overline{\bigcup_{n\ge m} \{\eta_{1n}, \quad , \eta_{2n+n.n}\}} \tag{3.54}$$

From the limit (3.51) and the asymptotic estimate (3.36), which also holds for g_n because of (3.33), it follows that $g(z) \ge \pi$ for quasi every $z \in [0,1]$. The function g is superharmonic in $\mathbb{C}\setminus\mathbb{R}_-$ and therefore continuous in the fine topology (for the definition see Appendix III of [StTo]). Set $E := \{z \in (0,1] \mid g(z) < \pi\}$, then E is a Borel set of capacity zero. Hence, in the fine topology all points of E are isolated, and from the continuity of g it follows that $E \ne \emptyset$, and consequently that

$$g(z) \ge \pi \quad \text{for all } z \in (0,1]. \tag{3.55}$$

In the next step of the proof we show that the inequality (3.53) holds everywhere on $(0,1]$. Set $S := \{z \in \mathbb{C} \setminus \mathbb{R}_- \mid g(z) > \pi\}$. Since g is superharmonic in $\mathbb{C} \setminus \mathbb{R}_-$, the set S is open. If $S = \emptyset$, then (3.53) is proved for all $z \in (0,1]$. We assume that

$$S \ne \emptyset. \tag{3.56}$$

From (3.53) and (3.54) we know that $g(z) \le \pi$ on $\operatorname{supp}(\mu) \setminus \{0\}$. Therefore, $S \ne \emptyset$ implies that $0 \in \overline{S}$. The superharmonicity of g in $\mathbb{C} \setminus \mathbb{R}_-$ and the minimum principle imply that the set S is simply connected. Since S is symmetric with respect to $\mathbb{R}$ it further follows that

$$S \cap [0,1] = (0,a) \quad \text{for some } a \in (0.1]. \tag{3.57}$$

From $g(z) > \pi$ for all $z \in (0,a)$ and (3.53) we deduce that the set $\{\eta_{jn} \mid j \in \{1,\dots,2n_2\},\ n \in N\}$ has no limit points in $(0,a)$. Hence, we can select a sequence of indices $\{f_n\}_{n\in N}$ with

$$\eta_{j_n n} \to 0 \quad \text{and} \quad \eta_{j_n+1,n} \to a \quad \text{as } n \to \infty,\ n \in N. \tag{3.58}$$

Note that $a \in \operatorname{supp}(\mu)$ or $a = 1$ because of the superharmonicity of g in $\mathbb{C} \setminus \mathbb{R}_-$. The sequence N is the same as that in the definition of g in (3.50).

Before we continue with the main investigation, we have to introduce an auxiliary function. Let D denote the domain $\mathbb{C} \setminus (\mathbb{R}_- \cup \{|z| \le 1\})$ and let h be the function harmonic in D with boundary values

$$h(z) \le \begin{cases} 1 & \text{for } |z| = 1, z \neq -1, \\ 0 & \text{for } z \in (-\infty,-1). \end{cases} \tag{3.59}$$

By elementary considerations one can verify that

$$h(z) \le \frac{c}{\sqrt{|z|}} \qquad \text{for } z \in \mathbb{C} \setminus \{0\} \tag{3.60}$$

with an appropriate constant $c < \infty$. It is not difficult to verify that the Green function $g_{\mathbb{C}\setminus\mathbb{R}_-}(z,x)$, $x \in \mathbb{R}_+$, behaves monotonically on half-circles around the origin. We have $g_{\mathbb{C}\setminus\mathbb{R}_-}(re^{it},x) \le g_{\mathbb{C}\setminus\mathbb{R}_-}(re^{it'},x)$ for $0 \le t' \le t \le \pi$ and $x \in \mathbb{R}_+$. As a consequence we have

$$g(\mu_n; re^{it}) \le g(\mu_n; re^{it'}) \qquad \text{for } 0 \le t' \le t \le \pi, r > 0, \tag{3.61}$$

and a corresponding behavior for $0 \ge t' \ge t \ge \pi$.

We continue with the main investigation: Let $g_n = g(\mu_n;\cdot)$ be broken down into

$$g_n(z) = \tilde{g}_n(z) + g(\mu_n|_{(\eta_{j_n n},1]}; z), \tag{3.62}$$

which is similar to the decomposition used in (3.43). Because of the non-negativity of Green potentials we have

$$0 \le \tilde{g}_n(z) \le g_n(z) \le g_n(\eta_{j_n,n}) h(z/\eta_{j_n n}) \qquad \text{for } |z| \ge \eta_{j_n n}. \tag{3.63}$$

The last inequality in (3.63) follows from (3.61) and (3.59) together with the maximum principle for harmonic functions.

From (3.29), (3.35), (3.33), (3.60), and (3.63) we then deduce that for $z \in \mathbb{C} \setminus \mathbb{R}_-$ we have

$$\begin{aligned} \tilde{g}_n(z) &\le \frac{1}{2\sqrt{an}} \left(\log \frac{1}{M} + \log \frac{2}{5} + \log \varepsilon_n \right) \frac{c\sqrt{\eta_{j_n n}}}{\sqrt{|z|}} \\ &= \mathcal{O}(\sqrt{\eta_{j_n n}}) = o(1) \qquad \text{as } n \to \infty,\ n \in N \end{aligned} \tag{3.64}$$

This implies that

$$\lim_{n\to\infty, n\in N} \tilde{g}_n(z) = 0 \tag{3.65}$$

locally uniformly in $\mathbb{C} \setminus \mathbb{R}_-$

Because of (3.2) in each open interval $(\eta_{j_n n}, \eta_{j_n+1,n})$ there can be at most one of the zeros used in the definition (3.30) of the measure μ_n. Its contribution to the weak limit (3.42) is negligible. It follows therefore from (3.58) that the limit measure μ has no mass in the open interval $(0, a)$. From (3.42) we deduce that

$$\mu_n|_{(\eta_{j_n n},1]} \xrightarrow{*} \mu|_{[a,1]} \quad \text{as } n \to \infty,\ n \in N, \tag{3.66}$$

and we have $\mu|_{(0,1]} = \mu|_{[a,1]}$. From (3.62), (3.65), and (3.66) it then follows that

$$\lim_{n\to\infty, n\in N} g_n(z) = \lim_{n\to\infty, n\in N} g(\mu_n|_{[a/2,1]}; z) = g(z) \tag{3.67}$$

locally uniformly in $\mathbb{C} \setminus (\mathbb{R}_- \cup [0,1])$. The last limit in (3.67) holds even locally uniformly in $\mathbb{C} \setminus (\mathbb{R}_- \cup [a,1])$. From (3.67), (3.53), (3.54), and the first maximum principle (see Appendix III of [StTo]) it then follows that

$$g(z) \le \sup_{x\in \operatorname{supp}(\mu)\cap[a,1]} g_n(\mu_n|_{[a/2,1]}; z) \le \pi \quad \text{for all } z \in \mathbb{C} \setminus \{0\}. \tag{3.68}$$

These inequalities contradict assumption (3.56), which implies that $g(z) > \pi$ on $S \cap [0,1]$. Consequently, we have proved that $S = \emptyset$ and

$$g(z) \le \pi \quad \text{for all } z \in (0,1]. \tag{3.69}$$

Since $g(z) = 0$ for $z \in \mathbb{R}_{<0}$, we learn from the inequalities (3.55) and (3.69) that g is identical with the function p introduced in (3.16). Hence, the measure μ is also identical with the measure ν described in (3.18) of Lemma 3.3. This proves that

$$\frac{d\mu(x)}{dx} = \frac{1}{\pi x\sqrt{1-x}}, \quad x \in (0,1]. \tag{3.70}$$

The description of μ in (3.70) is independent of the special selection of the subsequences $N \subseteq \mathbb{N}$ that have been used at several steps of the analysis. We can therefore conclude that the limit (3.42) holds not only for N, but also for the full sequence $\mathbb{N}$. From (3.2), (3.30), (3.42), and (3.70) it then follows that for $0 < c \le d \le 1$ we have

$$\frac{1}{\sqrt{n}} \operatorname{card}\{\eta_{j_n} \in [c,d]\} - 2\sqrt{\alpha}\mu([c,d]) = \frac{2\sqrt{\alpha}}{\pi} \int_c^d \frac{dt}{t\sqrt{1-t}}$$

which proves Theorem 4.

Proof of Theorem 3. While in the proof of Theorem 4 the measure μ in (3.42) was of central importance, now the measure $\widehat{\mu}$ derived from μ by balayage will play a principal role. The measure $\widehat{\mu}$ describes the asymptotic distribution of zeros ζ_{jn} and poles π_{jn} of the approximants r_n^*.

Let $\widehat{\mu}_n$ and $\widehat{\mu}$ be the measures that result from balayage of the measures μ_n and μ out of the domain $\mathbb{C} \setminus \mathbb{R}_-$ onto $\mathbb{R}_-$ (for the definition of balayage see Appendix VII of [StTo]). The measures μ_n have been defined in (3.30) and μ in

(3.42). It follows from (3.42) that

$$\widehat{\mu}_n \xrightarrow{*} \widehat{\mu} \quad \text{as } n \to \infty,\ n \in N. \tag{3.72}$$

(The subsequence $N \subseteq \mathbb{N}$ is the same as that in (3.42).) The measures $\widehat{\mu}_n$ and $\widehat{\mu}$ are positive and $\operatorname{supp}(\widehat{\mu}), \operatorname{supp}(\widehat{\mu}_n) \subseteq \mathbb{R}_-$, $n = 1, 2, \ldots$

Let q_n^* be the harmonic conjugate of q_n in $\mathbb{C} \setminus [-\infty, 1]$ with $q_n^*(\infty + i0) = 0$. The function q_n has been defined in (3.29). Since $f_n(x) > 0$ for $x > 1$, we have

$$q_n^*(x) = 0 \quad \text{for } x \in (1, \infty]. \tag{3.73}$$

By g_n^* we denote the conjugate function of the Green potential $g_n = g(\mu_n : \cdot)$ with $g_n^*(\infty + i0) = 0$. The function g_n was introduced in (3.31). Again we have

$$g_n^*(x) = 0 \quad \text{for } x \in (1, \infty]. \tag{3.73}$$

It follows from (3.29) that

$$q_n^*(z) = \frac{-1}{2\sqrt{\alpha n}} \arg f_n(z).$$

From Lemma 3.2 we know that $q_n^*(x+i0)$ is decreasing and $q_n^*(x-i0)$ is increasing for $-\infty \leq x \leq 0$. In the same way we learn from the definition (3.31) of the Green potential g_n that $g_n^*(x+i0)$ is decreasing and $g_n^*(x-i0)$ increasing for $-\infty \leq x \leq 0$.

Since $\widehat{\mu}_n$ is the measure generated by balayage of μ_n out of the domain $\mathbb{C}\setminus\mathbb{R}_-$, it follows from (3.31) that

$$g_n(z) = \int g_{\mathbb{C}\setminus\mathbb{R}_-}(z, x)\,d\mu_n(x) \qquad \int \log \frac{1}{|z-x|}\,d(\mu_n - \widehat{\mu}_n)(x) \tag{3.76}$$

The harmonic conjugate of $\log(1/|z-x|)$ is $-\arg(z-x)$, and therefore we have

$$g_n^*(z) = \int \arg(z-x)\,d(\widehat{\mu}_n - \mu_n)(x) + \text{const.} \qquad \text{for } z \in \mathbb{C} \setminus \mathbb{R}_-. \tag{3.77}$$

We assume that $\arg(\cdot)$ is defined in $\mathbb{C}\setminus\mathbb{R}_-$. It follows from (3.74) that the constant in (3.77) has a value such that the boundary values of g_n^* on $\mathbb{R}_-$ are given by

$$g_n^*(x + i0) = \pi\widehat{\mu}_n([-\infty, x]) \qquad \text{for } x \in \mathbb{R}_-. \tag{3.78}$$

Since we have seen in (3.70) that μ is identical to the measure ν in Lemma 3.3, it follows from (3.19) that

$$\frac{d\widehat{\mu}(x)}{dx} = \frac{-1}{\pi x\sqrt{1-x}} \qquad x \in \mathbb{R}_{<0}. \tag{3.79}$$

From (3.78) we know that the function $g_n^*(x + i0)/\pi$, $x \in \mathbb{R}_-$. is the distribution function of the measure $\widehat{\mu}_n$. Therefore we have

$$\lim_{n\to\infty, n\in N} g_n^*(x + i0) \qquad g^*(x + i0) = -\int_{-\infty}^{x} \frac{dt}{t\sqrt{1-t}} \tag{3.80}$$

for almost all $x \in \mathbb{R}_-$.

We now return to the functions q_n^* considered in (3.73) and (3.75). Since the functions $q_n^*(x + i0)$ are monotonic for $x \in \mathbb{R}_-$, it follows from Helly's selection theorem that there exists a subsequence of N, which we continue to denote by N, such that

$$\lim_{n\to\infty, n\in N} q_n^*(x+i0) =: q^*(x+i0) \qquad \text{for almost } x \in \mathbb{R}_-$$

The difference $g_n^* - q_n^*$ is the harmonic conjugate to $g_n - q_n$ satisfying $(g_n^* - q_n^*)(x) = 0$ for $x \in [1, \infty]$ because of (3.73) and (3.74). It follows from (3.33) and the Schwarz representation formula for conjugate functions that

$$\lim_{n\to\infty, n\in N} (g_n^* - q_n^*)(z) = 0 \qquad \text{locally uniformly in } \mathbb{C} \setminus \mathbb{R}_-. \tag{3.82}$$

The function $(g_n^* - q_n^*)(z)$ is harmonic in $\mathbb{C} \setminus \mathbb{R}_-$ and solves the Dirichlet problem in $\mathbb{C} \setminus \mathbb{R}_-$ for the boundary function $(g_n^* - q_n^*)(x \pm i0)$, $x \in \mathbb{R}_{<0}$. Taking the monotonicity of $g_n^*(x \pm i0)$ and $q_n^*(x \pm i0)$ on $\mathbb{R}_-$ into consideration, it follows from (3.82) that

$$\lim_{n\to\infty, n\in N} (g_n^* - q_n^*)(x \pm i0) = 0 \qquad \text{for all } x \in \mathbb{R}_- \tag{3.83}$$

From (3.80), (3.81 and (3.82) it follows that

$$q_n^*(x+i0) \to \int_{-\infty}^{x} \frac{dt}{t\sqrt{1-t}} \qquad \text{as } n \to \infty,\ n \in N, \text{ and } x \in \mathbb{R}_-$$

Since the right-hand side of (3.84) is independent of the set selection of the subsequence $N \subseteq \mathbb{N}$, the limit (3.84) holds for the full sequence $\mathbb{N}$.

From Lemma 3.2, (3.8), and (3.50) we know that for two adjacent zeros ζ_{jn} and $\zeta_{j+1,n}$ of r_n^* we have

$$|q_n^*(\zeta_{j+1,n} + i0) - q_n^*(\zeta_{jn} + i0)| = \frac{2\pi}{2\sqrt{\alpha n}} = \frac{\pi}{\quad} \tag{3.85}$$

For arbitrary $-\infty \le c \le d < 0$ it therefore follows that

$$\left| \operatorname{card}\{\zeta_{jn} \in [c,d]\} - \frac{\sqrt{\alpha n}}{\pi} |q_n^*(c+i0) - q_n^*(d+i0)| \right| \le 2,$$

and consequently it follows from (3.84) that

(3.87)

$$\frac{1}{\sqrt{n}} \operatorname{card}\{\zeta_{jn} \in [c,d]\} \to \frac{\sqrt{\alpha}}{\pi} \left| \int_c^d \frac{dt}{t\sqrt{1-t}} \right| \quad \frac{\sqrt{\alpha}}{\pi} \int_{|d|}^{|c|} \frac{dt}{t\sqrt{1+t}} \qquad \text{as } n \to \infty$$

which proves the first assertion of Theorem 3. The second assertion follows in exactly the same way if we start instead of (3.85) from

$$|q_n^*(\pi_{j+1,n} + i0) - q_n^*(\pi_{jn} + i0)| = \frac{2\pi}{2\sqrt{\alpha n}} = \frac{\pi}{\sqrt{\alpha n}} \tag{3.88}$$

which again follows from Lemma 3.2, (3.8), and (3.75). This completes the proof of Theorem 3. ■

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