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CONSTRUCTIVE APPROXIMATION

Optimal Ray Sequences of Rational Functions Connected with the Zolotarev Problem

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Abstract. Given two compact disjoint subsets E_1, E_2 of the complex plane, the third problem of Zolotarev concerns estimates for the ratio

$$\sup_{z\in E_1}|r(z)|/\inf_{z\in E_2}|r(z)|,$$

where r is a rational function of degree n. We consider, more generally, the infimum Z_{mn} of such ratios taken over the class of all rational functions r with numerator degree m and denominator degree n. For any "ray sequence" of integers (m, n); that is, $m/n \to \lambda$, $m + n \to \infty$, we show that $Z_{mn}^{1/(m+n)}$ approaches a limit $L(\lambda)$ that can be described in terms of the solution to a certain minimum energy problem with respect to the logarithmic potential. For example, we prove that $L(\lambda) = \exp(-F(\tau))$, where $\tau = \lambda/(\lambda + 1)$ and $F(\tau)$ is a concave function on $[0, 1]$ and we give a formula for $F(\tau)$ in terms of the equilibrium measures for $E_1^* \cup E_2^*$ and the condenser (E_1^*, E_2^*), where E_1^*, E_2^* are suitable subsets of E_1, E_2. Of particular interest is the choice for λ that yields the smallest value for $L(\lambda)$. In the case when E_1, E_2 are real intervals, we provide for this purpose a simple algorithm for directly computing $F(\tau)$ and for the determination of near optimal rational functions r_{mn}. Furthermore, we discuss applications of our results to the approximation of the characteristic function and to the generalized alternating direction iteration method for solving Sylvester's equation.

1. Introduction

Let E_1, E_2 be disjoint closed sets (of positive logarithmic capacity) in the extended complex plane $\bar{\mathbf{C}}$. Given a pair (m, n) of nonnegative integers we denote by $\mathbf{R}_{mn}$ the class of all rational functions $r = p/q$, where $p \in \mathbf{P}_m$, $q \in \mathbf{P}_n$ are polynomials of degrees at most m, n, respectively. We set

$$Z_{mn} = Z_{mn}(E_1, E_2) := \inf_{r\in\mathbf{R}_{mn}} \left(\frac{\sup_{z\in E_1}|r(z)|}{\inf_{z\in E_2}|r(z)|}\right). \tag{1.1}$$

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If E_1 (resp. E_2) is unbounded, we require $m \leq n$ (resp. $m \geq n$). We call $Z_{mn}(E_1, E_2)$ the (m, n)th *Zolotarev number* for E_1, E_2. Obviously,

$$Z_{mn}(E_1, E_2) = Z_{nm}(E_2, E_1). \tag{1.2}$$

The case $m = n$ is classical. It was first considered by E. Zolotarev and was settled completely by A. A. Gonchar, who showed in [Go] that

$$\lim_{n\to\infty} Z_{nn}^{1/n}(E_1, E_2) = \exp\left(\frac{-1}{\operatorname{cap}(E_1, E_2)}\right), \tag{1.3}$$

where $\operatorname{cap}(E_1, E_2)$ denotes the capacity of the condenser (E_1, E_2). The cases $m = 0$ or $n = 0$ are also well known. By the Bernstein–Walsh lemma, we have, for any polynomial $p \in \mathbf{P}_m$,

$$|p_m(z)| \leq \|p_m\|_{E_1} \exp(m g_{D_\infty(E_1)}(z; \infty)), \tag{1.4}$$

where $\|\cdot\|$ denotes the sup-norm, and $g_{D_\infty(E_1)}(z; \infty)$ is the Green function with pole at ∞ for the unbounded component $D_\infty(E_1)$ of $\bar{\mathbf{C}}\backslash E_1$. It readily follows from (1.4) that

$$Z_{m0}^{1/m}(E_1, E_2) \geq \exp\left(-\min_{z\in E_2} g_{D_\infty(E_1)}(z; \infty)\right). \tag{1.5}$$

Since the Fekete polynomials $F_m = F_m(z; E_1)$ for E_1 satisfy

$$\lim_{m\to\infty}\left(\frac{F_m(z; E_1)}{\|F_m\|_{E_1}}\right)^{1/m} = \exp(g_{D_\infty(E_1)}(z; \infty))$$

uniformly on compact subsets of $\mathbf{C}\backslash E_1$, we obtain

$$\lim_{m\to\infty} Z_{m0}^{1/m}(E_1, E_2) = \exp\left(-\min_{z\in E_2} g_{D_\infty(E_1)}(z; \infty)\right). \tag{1.6}$$

(If E_2 contains points in any bounded component of $\mathbf{C}\backslash E_1$, then $Z_{m0} = 1$, by the maximum principle, which agrees with (1.6), since $g_{D_\infty(E_1)}(z; \infty) \equiv 0$ there). Similarly (recall (1.2)),

$$\lim_{n\to\infty} Z_{0n}^{1/n}(E_1, E_2) = \exp\left(-\min_{z\in E_1} g_{D_\infty(E_2)}(z; \infty)\right). \tag{1.7}$$

It is not hard to show that both the limits in (1.6), (1.7) are larger than the limit in (1.3). However, it is more natural to compare them with $\lim_{n\to\infty} Z_{nn}^{1/2n}$ rather than with $\lim_{n\to\infty} Z_{nn}^{1/n}$. This is because the class $\mathbf{R}_{nn}$ depends on $2n + 1$ free parameters while the classes $\mathbf{R}_{n0}$ and $\mathbf{R}_{0n}$ each depend on $n + 1$ parameters.

In this paper we consider, more generally, ray sequences of integers (m, n) such that $m/n \to \lambda$, $m + n \to \infty$, and we prove (see Theorem 6.1) that

$$\lim_{m/n\to\lambda} Z_{m,m}^{1/(m+n)} = \exp(-F(\tau)), \tag{1.8}$$

where $\tau = \lambda/(\lambda + 1)$ and $F(\tau)$ is a quantity that arises in the solution to the minimum energy problem (with respect to the logarithmic potential) for placing a positive charge of amount τ on E_1 and a negative charge of amount $1 - \tau$ on E_2. (This electrostatics problem is discussed in Section 3.) We prove in Theorem 3.2 that $F(\tau)$ is a positive concave function on $(0, 1)$ and provide in Corollary 4.2 a simple formula for $F(\tau)$ in terms of the equilibrium measure for the conductor $E_1^* \cup E_2^*$ and the equilibrium measure for the condenser (E_1^*, E_2^*), where E_1^*, E_2^* are suitable subsets of E_1, E_2, respectively.

In Section 5 we provide two simple examples for which $F(\tau)$ can be explicitly obtained. For general configurations, we describe in Section 7 three numerical methods for determining rational functions $r_{mn}(z)$ that satisfy

$$\lim_{\substack{m/n \to \lambda \\ m+n \to \infty}} \left\{ \frac{\sup_{E_1} |r_{mn}|}{\inf_{E_2} |r_{mn}|} \right\}^{1/(m+n)} = \exp(-F(\tau)). \tag{1.9}$$

Thereby we obtain approximations for $F(\tau)$. These numerical techniques are generalizations of methods due to Fekete, Fejér, Walsh, and Bagby.

In the important case when E_1, E_2 are intervals of the real axis, we provide in Section 8 a simple *direct* algorithm for computing $F(\tau)$ that involves elliptic integrals. Thus the optimal value of λ, i.e., the value for which the right-hand side of (1.8) is least, can easily be determined as illustrated in Example 8.3. Furthermore, we give *explicit* formulas for the zeros and poles of rational functions r_{mn} that satisfy (1.9).

Finally, in Section 9, we provide two applications of our results. The first concerns the rate of best uniform approximation to the signum function

$$\chi(z) := \begin{cases} 0, & z \in E_1, \\ 1, & z \in E_2, \end{cases}$$

by ray sequences $r_{m,n}$, where $m/n \to \lambda$, $0 < \lambda \leq 1$. The second application deals with the choice of optimal parameters in the generalized alternating direction iteration (ADI) method for solving Sylvester's equation

$$AX - XB = C,$$

where A, B, C are given matrices.

2. Potential Theoretic Preliminaries

Given a signed Borel measure μ on $\mathbf{C}$, its *logarithmic energy* is defined by

$$I(\mu) := \iint \log \frac{1}{|z - t|} \, d\mu(z) \, d\mu(t), \tag{2.1}$$

and its *logarithmic potential* is given by

$$U^{\mu}(z) := \int \log \frac{1}{|z - t|} \, d\mu(t).$$

Let E_1, E_2 be disjoint compact subsets of $\mathbf{C}$, both of positive logarithmic capacity, and let m_1, m_2 be given positive numbers. We denote by $\mathcal{M} = \mathcal{M}(E_1, E_2, m_1, m_2)$ the set of all signed measures $\mu = \mu_1 - \mu_2$, where, for $i = 1, 2$, μ_i is a positive Borel measure on E_i of total mass $\|\mu_i\| = m_i$.

For the function (which we regard as an "external field")

$$Q(z) := a \log \frac{1}{|z - c|},$$

assume that one of the following conditions holds:

(i) $a = 0$,
(ii) $a > 0$, $c \in E_1$,
(iii) $a < 0$, $c \in E_2$,

and consider the minimal energy problem

$$V := \inf_{\mu \in \mathcal{M}} \left\{ I(\mu) + 2 \int Q \, d\mu \right\}.$$

The following theorems are special cases of results proved by Saff and Totik [ST] for a class of "admissible" weight functions of the form $\exp(-Q)$.

Theorem 2.1. *Let E_1, E_2, $\mathcal{M}$, V, and Q be as above. Then:*

(a) *V is finite.*
(b) *There is a unique $\mu^* = \mu_1^* - \mu_2^* \in \mathcal{M}$ such that*

$$V = I(\mu^*) + 2 \int Q \, d\mu^*.$$

(c) *μ^* has finite logarithmic energy and both Q and U^{μ^*} are bounded on the support of μ^*. Hence, U^{μ^*} is bounded on compact subsets of $\mathbf{C}$.*
(d) *Constants F_1, F_2 exist such that*

$$U^{\mu^*}(z) + Q(z) \le F_1 \qquad \text{on } \operatorname{supp}(\mu_1^*),$$

$$U^{\mu^*}(z) + Q(z) \ge F_1 \qquad \text{q.e. on } E_1,$$

$$U^{\mu^*}(z) + Q(z) \ge -F_2 \qquad \text{on } \operatorname{supp}(\mu_2^*),$$

$$U^{\mu^*}(z) + Q(z) \le -F_2 \qquad \text{q.e. on } E_2$$

where q.e. (quasi-everywhere) means neglecting sets of zero logarithmic capacity.

Furthermore,

$$V = m_1 F_1 + m_2 F_2 + \int Q \, d\mu^*.$$

Theorem 2.2. *Let $\sigma = \sigma_1 - \sigma_2$ be any signed measure that satisfies $\|\sigma_i\| = m_i$, $i = 1, 2$, and such that not every point of the complement of $E_1 \cup E_2$ belongs to* $\operatorname{supp}(\sigma)$. *Then*

$$\underset{z \in E_1}{\text{“inf”}}(U^{\sigma}(z) + Q(z)) \quad \underset{z \in E_2}{\text{“sup”}}(U^{\sigma}(z) + Q(z)) \leq F_1 + F_2,$$

with equality for $\sigma = \mu^*$ (cf. Theorem 2.1).

Here and throughout “inf” and “sup” means, respectively, the inf and sup neglecting sets of zero capacity.

Remark. In Theorem 2.2 we do not require that $\operatorname{supp}(\sigma_i) \subset E_i$, $i = 1, 2$. We also note that while Theorem 2.2 is proved in [ST] for $m_1 = m_2 = 1$, exactly the same proof applies for any m_1, m_2.

Theorem 2.3. *Suppose that $\sigma = \sigma_1 - \sigma_2 \in \mathcal{M}$ has finite energy and that*

$$U^{\sigma}(z) + Q(z) = \tilde{F}_1 := \underset{z \in E_1}{\text{“inf”}}(U^{\sigma}_{(z)} + Q(z)) \qquad \text{q.e. on } \operatorname{supp}(\sigma_1)$$

and

$$U^{\sigma}(z) + Q(z) \qquad \tilde{F}_2 := \underset{z \in E_2}{\text{“sup”}}(U^{\sigma}(z) + Q(z)) \qquad \text{q.e. on } \operatorname{supp}(\sigma_2).$$

Then σ μ^* (and, of course, $\tilde{F}_i$ F_i for $= 1, 2$).

Theorem 2.4. *Assume additionally that E_1, E_2 are regular; that is, every component of the complement of $E_1 \cup E_2$ is regular with respect to the Dirichlet problem. Then U^{μ^*} is continuous in $\mathbf{C}$ and relations* (2.6), (2.8) *hold everywhere on the corresponding sets. Consequently,*

$$U^{\mu^*} + Q = F_1 \qquad \text{everywhere on } \operatorname{supp}(\mu_1^*),$$
$$U^{\mu^*} + Q = -F_2 \qquad \text{everywhere on } \operatorname{supp}(\mu_2^*). \tag{2.11}$$

Remark. In view of our special form of Q, Theorem 2.4 follows immediately from Theorems VIII.2.1 and I.5.1 in [ST].

3. The Energy Problem

Let E_1, E_2 be closed (not necessarily bounded) subsets of $\mathbf{C}$ that are a positive distance apart (on the Riemann sphere) and assume that both sets have positive logarithmic capacity. Given $0 < \tau < 1$, let $\mathcal{M}_\tau$ denote the set of all signed measures $\mu = \mu_1 - \mu_2$, where $\mu_i \geq 0$, $\operatorname{supp}(\mu_i) \subset E_i$, $i = 1, 2$, and $\|\mu_1\| = \tau$, $\|\mu_2\| = 1 - \tau$. Set

$$V_\tau := \inf_{\mu \in \mathcal{M}_\tau} I(\mu), \tag{3.1}$$

where $I(\mu)$ is defined in (2.1).

For this extremal problem, we first describe analogues of the results of Section 2.

Theorem 3.1. *Let E_1, E_2, $\mathcal{M}_\tau$, and V_τ be as above, and assume that one of the following conditions holds:*

(a) *E_1 and E_2 are bounded.*
(b) *E_1 is unbounded and $\tau \le \frac{1}{2}$.*
(c) *E_2 is unbounded and $\tau \ge \frac{1}{2}$.*

Then the conclusions of Theorems 2.1–2.4 hold with $Q \equiv 0$.

In particular, there is a unique measure $\mu^* = \mu^*(\tau) \in \mathcal{M}_\tau$ satisfying $V_\tau = I(\mu^*)$ and constants $F_1 = F_1(\tau)$, $F_2 = F_2(\tau)$ exist such that

$$U^{\mu^*} \le F_1 \quad \text{on } \operatorname{supp}(\mu_1^*), \qquad U^{\mu^*} \ge F_1 \quad \text{q.e. on } E_1, \tag{3.2}$$

$$U^{\mu^*} \ge -F_2 \quad \text{on } \operatorname{supp}(\mu_2^*), \qquad U^{\mu^*} \le -F_2 \quad \text{q.e. on } E_2, \tag{3.3}$$

where $\mu^* = \mu_1^* - \mu_2^*$. Consequently,

$$U^{\mu^*} = F_1 \qquad \text{q.e. on } \operatorname{supp}(\mu_1^*),$$
$$U^{\mu^*} = -F_2 \qquad \text{q.e. on } \operatorname{supp}(\mu_2^*).$$

Furthermore, for any signed measure $\sigma = \sigma_1 - \sigma_2$ with $\|\sigma_1\| = \tau$, $\|\sigma_2\| = 1 - \tau$, and such that not every point of the complement of $E_1 \cup E_2$ belongs to $\operatorname{supp}(\sigma)$,

$$\text{“}\inf_{z \in E_1}\text{”}\; U^\sigma(z) - \text{“}\sup_{z \in E_2}\text{”}\; U^\sigma(z) \le F_1 + F_2 =: F$$

holds. Moreover, for cases (b) *and* (c), $\operatorname{supp}(\mu^*)$ *is compact unless $\tau = \frac{1}{2}$.*

Proof. Case (a) requires no proof. Also, since cases (b) and (c) are symmetric, we consider only (b).

Pick $c \notin E_1 \cup E_2$. Then the mapping $z - c = (\zeta - c)^{-1}$ transforms $E_1 \cup \{\infty\}$ and E_2 into disjoint compact sets $\tilde{E}_1$, $\tilde{E}_2$, where $c \in \tilde{E}_1$. The measures μ_1, μ_2 are transformed in a natural way into measures $\tilde{\mu}_1$, $\tilde{\mu}_2$ on $\tilde{E}_1$, $\tilde{E}_2$, respectively. Furthermore, the energy integral $I(\mu)$ becomes

$$\iint \log\left(\frac{|\zeta - c|\,|s - c|}{|\zeta - s|}\right) d\tilde{\mu}(\zeta)\, d\tilde{\mu}(s) = I(\tilde{\mu}) + 2(2\tau - 1) \int \log|\zeta - c|\, d\tilde{\mu}(\zeta)$$
$$= I(\tilde{\mu}) + 2 \int Q\, d\tilde{\mu},$$

where

$$Q(\zeta) := (1 - 2\tau) \log \frac{1}{|\zeta - c|}.$$

Notice that since $0 < \tau \le \frac{1}{2}$, Q satisfies condition (i) or (ii) of Section 2. Thus, by Theorem 2.1, a unique extremal (equilibrium) measure $\tilde{\mu}^*$ exists for the transformed problem, and so its corresponding measure μ^* in the z-plane satisfies $V_\tau = I(\mu^*)$.

Next observe, from Theorem 2.1(c), that $Q(\zeta)$ is bounded on $\operatorname{supp}(\tilde{\mu}^*)$. Thus, if $0 < \tau < \frac{1}{2}$, we have $\tilde{\mu}^*(B) = 0$ for some neighborhood B of c, which means that μ^* has compact support in this case. We also note that the potential $U^{\mu^*}(z)$ is transformed to

$$\int \log\left(\frac{|\zeta - c|\,|s - c|}{|\zeta - s|}\right) d\tilde{\mu}^*(s) = U^{\tilde{\mu}^*}(\zeta) - U^{\tilde{\mu}^*}(c) + Q(\zeta), \tag{3.5}$$

and we observe that $U^{\tilde{\mu}^*}(c)$ is finite (if $c \notin \operatorname{supp}(\tilde{\mu}^*)$ this is obvious; if $c \in \operatorname{supp}(\tilde{\mu}^*)$, then we appeal to Theorem 2.1(c)). Since $U^{\tilde{\mu}^*}$ is bounded on compact subsets of the z-plane, we see from (3.5) that the right-hand side of (3.5) is bounded on compact subsets of $\bar{\mathbb{C}}\backslash\{c\}$. This means that the corresponding $U^{\mu^*}(z)$ is bounded on the compact subsets of the z-plane. Furthermore, by Theorem 2.1(d) we have, for example,

$$U^{\tilde{\mu}^*}(\zeta) + Q(\zeta) \le \tilde{F}_1 \qquad \text{on } \operatorname{supp}(\tilde{\mu}_1^*),$$

which means that

$$U^{\mu^*}(z) \le \tilde{F}_1 - U^{\tilde{\mu}^*}(c) =: F_1 \qquad \text{on } \operatorname{supp}(\mu_1^*).$$

We have thus shown that there is a unique $\mu^* \in \mathcal{M}_\tau$ for which $V_\tau = I(\mu^*)$, and μ^* has compact support unless $\tau = \frac{1}{2}$ and one of the sets E_1 or E_2 is unbounded. Moreover, its potential U^{μ^*} satisfies (3.2) and (3.3) for suitable constants $F_1 = F_1(\tau)$, $F_2 = F_2(\tau)$. The assertion of (3.4) and the analogues of Theorems 2.3 and 2.4 also follow immediately from the properties of the transformed problem. ∎

Remark. If, say, E_1 is unbounded and $\tau > \frac{1}{2}$, then it is easy to see that $V_\tau = -\infty$ in (3.1).

With $F_1 = F_1(\tau)$, $F_2 = F_2(\tau)$ denoting the constants in (3.2) and (3.3), we define (cf. (3.4))

$$F(\tau) := F_1(\tau) + F_2(\tau).$$

We also define F for $\tau = 0$ (when E_2 is bounded) and for $\tau = 1$ (when E_1 is bounded), which are classical cases. For $\tau = 1$, $\mathcal{M}_\tau$ is the set of probability measures on E_1 and the corresponding equilibrium potential is given by

$$U^{\mu^*}(z) = \log \frac{1}{\operatorname{cap}(E_1)} - g_{D_\infty(E_1)}(z; \infty),$$

where $g_{D_\infty(E_1)}(z; \infty)$ is the Green function with pole at infinity for the unbounded component of the complement $\bar{\mathbb{C}}\backslash E_1$. In this case we set

$$F_1(1) := \log \frac{1}{\operatorname{cap}(E_1)}, \tag{3.8}$$

$$F_2(1) := -\log \frac{1}{\operatorname{cap}(E_1)} + \text{“}\inf_{z \in E_2}\text{”}\, g_{D_\infty(E_1)}(z; \infty).$$

$= 0$, $\mathcal{M}_\tau$ becomes the set of negative probability measures on E_2. In this case,

$$U^{\mu^*}(z) = -\log \frac{}{\operatorname{cap}(E_2)} + g_{D_\infty(E_2)}(z; \infty),$$

and we set

(3.11) $$F_1(0) := -\log \frac{1}{\operatorname{cap}(E_2)} + \text{“}\inf_{z \in E_1}\text{”}\, g_{D_\infty(E_2)}(z; \infty),$$

(3.12) $$F_2(0) := \log \frac{1}{\operatorname{cap}(E_2)}.$$

With the above definitions, the function $F(\tau)$ is defined on $[0, 1]$ when E_1 and E_2 are both bounded, on $[0, \frac{1}{2}]$ or $[\frac{1}{2}, 1]$ when E_1 or E_2 is unbounded.

An important property of the function F is established in the following theorem.

Theorem 3.2. *The function $F(\tau)$ of (3.6) is concave, continuous, and nonnegative on its closed interval of definition. Moreover, if $\tau_1 < \tau_2$ are such that the corresponding equilibrium measures have the same supports, then $F(\tau)$ is linear on $[\tau_1, \tau_2]$.*

Proof. Let τ_1, τ_2 belong to the domain of F and let

$$\tau = \alpha\tau_1 + (1 - \alpha)\tau_2 \qquad 0 < \alpha < 1.$$

With $\mu^*(\tau_1)$, $\mu^*(\tau_2)$ denoting the corresponding equilibrium measures, we define a signed measure μ by

(3.13) $$\mu := \alpha\mu^*(\tau_1) + (1 - \alpha)\mu^*(\tau_2).$$

Clearly, $\mu \in \mathcal{M}_\tau$ and so, by (3.4),

(3.14) $$F(\tau) \geq \text{“}\inf_{E_1}\text{”}\, U^\mu - \text{“}\sup_{E_2}\text{”}\, U^\mu$$

Applying (3.2) and (3.3) we obtain

(3.15) $$\begin{aligned}\text{“}\inf_{E_1}\text{”}\, U^\mu &= \text{“}\inf_{E_1}\text{”}[\alpha U^{\mu^*(\tau_1)} + (1 - \alpha)U^{\mu^*(\tau_2)}] \\ &\geq \alpha F_1(\tau_1) + (1 - \alpha)F_1(\tau_2),\end{aligned}$$

and similarly

(3.16) $$\text{“}\sup_{E_2}\text{”}\, U^\mu \leq -\alpha F_2(\tau_1) - (1 - \alpha)F_2(\tau_2).$$

Thus $F(\tau) \geq \alpha F(\tau_1) + (1 - \alpha)F_2(\tau_2)$, which proves that F is concave.

One consequence of concavity is the fact that $F(\tau)$ cannot be less than the smaller of its values at the endpoints of its interval of definition. Clearly,

(3.17) $$F(0) = \text{“}\inf_{z \in E_1}\text{”}\, g_{D_\infty(E_2)}(z; \infty), \qquad F(1) = \text{“}\inf_{z \in E_2}\text{”}\, g_{D_\infty(E_1)}(z; \infty)$$

are nonnegative (provided E_2 (resp. E_1) is bounded). In case one of the sets E_1, E_2 is unbounded, we note further that for $\tau = \frac{1}{2}$, which corresponds to the classical condenser problem, we have $F(\frac{1}{2}) > 0$ (see Section 4). Hence $F \geq 0$.

Now suppose that $\mu^*(\tau_1)$, $\mu^*(\tau_2)$ have the same support. Then with $\mu = \mu_1 - \mu_2$ as in (3.13) we have equality in (3.15) and

$$U^\mu(z) = \alpha F_1(\tau_1) + (1-\alpha)F_1(\tau_2) \qquad \text{q.e. on } \operatorname{supp}(\mu_1).$$

Likewise, equality holds in (3.16) and

$$U^\mu(z) = -\alpha F_2(\tau_1) - (1-\alpha)F_2(\tau_2) \qquad \text{q.e. on } \operatorname{supp}(\mu_2).$$

Thus by Theorem 2.3, we get $\mu = \mu^*(\tau)$ and equality holds in (3.14); that is,

$$F(\tau) = \alpha F(\tau_1) + (1-\alpha)F(\tau_2).$$

Hence F is linear on $[\tau_1, \tau_2]$.

The concavity of F implies, of course, that F is continuous, except, perhaps, at the endpoints of its interval of definition. So it remains to prove that continuity also holds at the endpoints. For this purpose we apply two simple lemmas.

Lemma 3.3. *If $\tau_n \to \tau$, where τ_n belongs to the interval of definition of F, then*

$$\limsup_{n\to\infty} V_{\tau_n} \leq V_\tau.$$

Proof. If $\tau \notin \{0, 1\}$, then the measure

$$\mu := \frac{\tau_n}{\tau}\mu_1^*(\tau) - \frac{1-\tau_n}{1-\tau}\mu_2^*(\tau) \tag{3.19}$$

belongs to $\mathcal{M}_{\tau_n}$. Thus, by the extremal property of $\mu^*(\tau_n)$, we get

$$V_{\tau_n} = I(\mu^*(\tau_n)) \leq I(\mu) = I(\mu^*(\tau)) + o(1) \qquad \text{as} \quad n \to \infty,$$

which yields (3.18).

If $\tau = 1$ (so that E_1 is bounded), we replace (3.19) by

$$\mu := \tau_n\mu_1^*(1) - (1-\tau_n)\sigma,$$

where σ is, say, the equilibrium measure for the set $E_2 \cap \{z: |z| \leq R\}$ with R large and fixed. Then we again deduce (3.18). In a similar fashion, we can also deal with the case $\tau = 0$. ■

Lemma 3.4. *If $\tau_n \to \tau$, where τ_n belongs to the interval of definition of F, then, in the weak-star topology,*

$$\mu^*(\tau_n) \to \mu^*(\tau) \qquad \textit{as} \quad n \to \infty.$$

Proof. Assume first that both E_1 and E_2 are compact. Let μ be any weak-star limit of the sequence $\{\mu^*(\tau_n)\}$. Then $\mu = \mu_1 - \mu_2 \in \mathscr{M}_\tau$. On taking a subsequence, if necessary, we may suppose that $\mu^*(\tau_n) \to \mu$ as $n \to \infty$. Then we have

(3.22) $$\lim_{n\to\infty} U^{\mu_2^*(\tau_n)} = U^{\mu_2} \qquad \text{on } E_1$$

and, by the lower envelope theorem (see Theorem 3.8 of [L]),

$$\liminf_{n\to\infty} U^{\mu_1^*(\tau_n)} = U^{\mu_1} \qquad \text{q.e. on } E_1.$$

$$\liminf_{n\to\infty} U^{\mu^*(\tau_n)} = U^{\mu} \qquad \text{q.e. on } E_1,$$

and, similarly,

(3.25) $$\liminf_{n\to\infty}(-U^{\mu^*(\tau_n)}) = -U^{\mu} \qquad \text{q.e. on } E_2$$

From Theorem 3.1, (3.24), and (3.25) we obtain that

(3.26) $$U^{\mu} \geq \liminf_{n\to\infty} F_1(\tau_n) \qquad \text{q.e. on } E_1,$$

(3.27) $$U^{\mu} \leq -\liminf_{n\to\infty} F_2(\tau_n) \qquad \text{q.e. on } E_2$$

Next assume that $\tau \neq 0$. Then $\operatorname{supp}(\mu_1)$ is nonempty. Given $\zeta \in \operatorname{supp}(\mu_1)$, it follows from the weak-star convergence that for any neighborhood Δ_ζ of ζ we have

$$\liminf_{n\to\infty} \mu_1^*(\tau_n)(\Delta_\zeta) \geq \mu_1(\Delta_\zeta) > 0.$$

Thus, given a sequence $\mathscr{N}$ of positive integers such that

$$\lim_{n\in\mathscr{N}} F_1(\tau_n) = \liminf_{n\to\infty} F_1(\tau_n),$$

a subsequence $\mathscr{N}' \subset \mathscr{N}$ and points $\{\zeta_n\}_{n\in\mathscr{N}'}$ can be selected such that $\zeta_n \to \zeta$ and $U^{\mu^*(\tau_n)}(\zeta_n) = F_1(\tau_n)$ for $n \in \mathscr{N}'$. By the (generalized) principle of descent (see Theorem 1.3 of [L]) we have

$$U^{\mu_1}(\zeta) \leq \liminf_{n\in\mathscr{N}'} U^{\mu_1^*(\tau_n)}(\zeta_n).$$

Thus, on recalling (3.22), it follows that

$$U^{\mu}(\zeta) \leq \liminf_{n\in\mathscr{N}'} U^{\mu^*(\tau_n)}(\zeta_n) = \liminf_{n\to\infty} F_1(\tau_n), \qquad \zeta \in \operatorname{supp}(\mu_1).$$

Similarly, if $\tau \neq 1$, it can be shown that

(3.29) $$U^{\mu}(\zeta) \geq -\liminf_{n\to\infty} F_2(\tau_n), \qquad \zeta \in \operatorname{supp}(\mu_2).$$

Since μ has finite energy, we deduce from (3.26), (3.28), (3.27), (3.29), and Theorem 2.3 that $\mu = \mu^*(\tau)$, and we also have

$$F_i(\tau) = \liminf_{n\to\infty} F_i(\tau_n), \qquad = 1, 2.$$

If $\tau = 1$, then (3.26) and (3.28) imply that $\mu = \mu^*(1)$ (see [T]) and that

$$\liminf_{n\to\infty} F_1(\tau_n) = F_1(1), \qquad \liminf_{n\to\infty} F_2(\tau_n) = F_2(1).$$

The case $\tau = 0$ is similar.

Finally, if E_1 or E_2 is unbounded, then on applying the transformation $z - c = (\zeta - c)^{-1}$ as in the proof of Theorem 3.1 and arguing as above we again obtain $\mu = \mu^*(\tau)$ as well as (3.30) and (3.31). ■

We can now complete the proof of Theorem 3.2 by showing that F is continuous at the endpoints of its interval of definition. For this purpose we use the representation

$$V_\tau = \tau F_1(\tau) + (1 - \tau)F_2(\tau),$$

which follows from (2.9), (3.8), and (3.12). Suppose first that $\tau_n \to 1$. By Lemma 3.3 we have

$$\begin{aligned} F_1(1) = V_1 &\geq \limsup_{n\to\infty}[\tau_n F_1(\tau_n) + (1 - \tau_n)F_2(\tau_n)] \\ &= \limsup_{n\to\infty}[(2\tau_n - 1)F_1(\tau_n) + (1 - \tau_n)F(\tau_n)] \\ &= \limsup_{n\to\infty}(2\tau_n - 1)F_1(\tau_n), \end{aligned}$$

where, in the last equality, we used the fact that F is bounded since it is nonnegative and concave. It follows from (3.33) that the sequence $\{F_1(\tau_n)\}$ is bounded from above. By (3.31), it is also bounded from below. Thus

$$F_1(1) \geq \limsup_{n\to\infty}(2\tau_n - 1)F_1(\tau_n) = \limsup_{n\to\infty} F_1(\tau_n).$$

Together with (3.31) this gives

$$\lim_{n\to\infty} F_1(\tau_n) = F_1(1),$$

and so

$$\begin{aligned} \liminf_{n\to\infty} F(\tau_n) &= \liminf_{n\to\infty}[F_1(\tau_n) + F_2(\tau_n)] \\ &= F_1(1) + \liminf_{n\to\infty} F_2(\tau_n) \\ &\quad F_1(1) + F_2(1) = F(1), \end{aligned}$$

where we have again used (3.31). Because F is concave and bounded on $[\frac{1}{2}, 1]$, it follows from (3.34) that

$$\lim_{n\to\infty} F(\tau_n) = F(1).$$

The case when $\tau_n \to 0$ is handled similarly.

So suppose now that $\tau_n \to \frac{1}{2}$. From Lemma 3.3 we have

$$\frac{1}{2}F(\tfrac{1}{2}) = V_{1/2} \geq \limsup_{n\to\infty}[\tfrac{1}{2}F(\tau_n) + (\tau_n - \tfrac{1}{2})(F_1(\tau_n) - F_2(\tau_n))]. \tag{3.35}$$

By (3.30), both sequences $F_1(\tau_n)$ and $F_2(\tau_n)$ are bounded from below. Since $F(\tau_n) = F_1(\tau_n) + F_2(\tau_n)$ is bounded, the same must be true for $F_1(\tau_n)$ and $F_2(\tau_n)$. Hence (3.35) can be rewritten as

$$F(\tfrac{1}{2}) \geq \limsup_{n\to\infty} F(\tau_n),$$

and from the concavity of F we deduce that

$$F(\tfrac{1}{2}) = \lim_{n\to\infty} F(\tau_n). \qquad \blacksquare$$

Remark. Instead of the energy problem (3.1) we could consider a more general one; namely,

$$V_{\tau,Q} := \inf\left\{I(\mu) + 2\int Q\, d\mu\colon \mu \in \mathcal{M}_\tau\right\},$$

where Q is some "admissible" weight function. Then, on applying the results of Section 2 in their full generality, it is possible to obtain extensions of Theorems 3.1 and 3.2. However, we do not pursue the details here.

4. Representation of the Equilibrium Potential

For the energy problem of Section 3, we obtain a simple representation for the equilibrium measure $\mu^* = \mu^*(\tau) = \mu_1^* - \mu_2^*$ and its potential U^{μ^*}. Observe that, for $\tau = 1$, we have $\mu^* = \mu_1^*$; in this case E_1 is assumed to be compact and the corresponding potential is given by (3.7). If $\tau = 0$, then $\mu^* = -\mu_2^*$; in this case E_2 is assumed to be compact and the corresponding potential is given by (3.10).

For $\tau = \frac{1}{2}$, we have the classical condenser case (see [B2]) except that each μ_i has total mass $\frac{1}{2}$ instead of 1 as in [B2]. Thus if we denote by $\sigma = \sigma_1 - \sigma_2(\|\sigma_i\| = 1$, $\operatorname{supp}(\sigma_i) \subset E_i)$ the equilibrium measure for the condenser (E_1, E_2), and denote by $u := U^\sigma$ the corresponding potential, we have

$$\mu^*(\tfrac{1}{2}) = \tfrac{1}{2}\sigma \quad \text{and} \quad U^{\mu^*(1/2)} = \tfrac{1}{2}u. \tag{4.1}$$

Let us now recall some properties of u (see [B2]).

First, nonnegative constants a_1, a_2 exist such that

$$u(z) = a_1 \quad \text{q.e. on } E_1, \qquad u(z) = -a_2 \quad \text{q.e. on } E_2$$

$$-a_2 \le u(z) \le a_1 \qquad \text{for all} \quad z \in \bar{\mathbf{C}},$$

and if both E_1, E_2 are compact, then $u(\infty) = 0$. Furthermore,

$$\frac{1}{\operatorname{cap}(E_1, E_2)} := a_1 + a_2 > 0.$$

Note that (4.1)–(4.3) imply that

$$F(\tfrac{1}{2}) = \frac{1}{2\operatorname{cap}(E_1, E_2)} > 0.$$

This strict positivity together with the concavity of the nonnegative function F imply that $F > 0$ on its interval of definition except, perhaps, at $\tau = 0$ or $\tau = 1$.

We now turn to representation for $U^{\mu^*(\tau)}$ when $\tau \notin \{\frac{1}{2}, 0, 1\}$. Let

$$E_i^* := \operatorname{supp}(\mu_i^*), \qquad \mu^* = \mu^*(\tau) = \mu_1^* - \mu_2^*$$

Then, by Theorem 3.1, $\operatorname{supp}(\mu^*) = E_1^* \cup E_2^*$ is compact, and both E_1^* and E_2^* have positive capacity (since $\|\mu_i^*\| > 0$, $i = 1, 2$). Thus we may consider the equilibrium potential u for the condenser (E_1^*, E_2^*). Also, since $E_1^* \cup E_2^*$ is compact we may consider its equilibrium potential; this is given by

$$U^\nu(z) = \log \frac{1}{\operatorname{cap}(E_1^* \cup E_2^*)} - g_{D_\infty(E_1^* \cup E_2^*)}(z; \infty),$$

where ν (≥ 0) is the equilibrium measure $(\|\nu\| = 1)$ on $E_1^* \cup E_2^*$. With the above notation, we now prove

Theorem 4.1. *Let E_1, E_2 be as in Theorem 3.1 and let $\tau \notin \{\frac{1}{2}, 0, 1\}$ belong to the interval of definition of F. Let $\mu^* = \mu^*(\tau) = \mu_1^* - \mu_2^*$ be the equilibrium distribution for the energy problem (3.1) with $\operatorname{supp}(\mu_i^*) = E_i^*$, $i = 1, 2$. Furthermore, let ν, φ denote, respectively, the equilibrium measure and equilibrium (conductor) potential for $E_1^* \cup E_2^*$, and let σ, u be the equilibrium measure and equilibrium potential for the condenser (E_1^*, E_2^*). Then*

$$\mu^* = (2\tau - 1)\nu + \gamma F(\tau)\sigma,$$

$$U^{\mu^*} = (2\tau - 1)\varphi + \gamma F(\tau) u \qquad \textit{everywhere in } \mathbf{C},$$

$$\gamma := \operatorname{cap}(E_1^*, E_2^*).$$

We wish to emphasize that all the symbols E_1^*, E_2^*, μ^*, ν, σ, u, φ, and γ depend on τ.

Proof. We establish that (4.8) holds q.e., which implies (4.7) and, consequently, that (4.8) holds everywhere (see Theorem 1.12 of [L]). Since $\operatorname{supp}(\mu^*)$ is compact, we have

$$\begin{aligned} U^{\mu^*}(z) &= \int \log \frac{1}{|z-t|}\, d\mu^*(t) \\ &= (2\tau \quad 1) \log|z| + \int \log \left| \frac{z}{z-t} \right| \\ &\quad (2\tau \quad 1) \log|z| + h_1(z), \end{aligned}$$

where h_1 is harmonic outside $\operatorname{supp}(\mu^*)$ and $h_1(\infty) = 0$. Similarly, we can write

$$\varphi(z) = U^{\nu}(z) = \quad \log|z| + h_2(z),$$

where h_2 is harmonic outside $\operatorname{supp}(\mu^*)$ and $h_2(\infty) = 0$. Furthermore, we have that u is harmonic outside $\operatorname{supp}(\mu^*)$ and $u(\infty) = 0$. Thus the function

$$H(z) \quad U^{\mu^*}(z) + (2\tau - 1)\varphi(z) \quad \gamma F(\tau) u(z) \tag{4.10}$$

is harmonic outside $\operatorname{supp}(\mu^*)$, including ∞, and $H(\infty) = 0$.

Next, since U^{μ^*} is lower semicontinuous on E_1^*, it follows from (3.2), (3.2′) that U^{μ^*}, considered as a function on E_1^*, is continuous at q.e. $z \in E_1^*$. Since $U^{\mu_2^*}$ is continuous in a neighborhood of E_1^*, we conclude (see Theorem III.2 of [T]) that U^{μ^*} is continuous (as a function on $\mathbf{C}$) q.e. on E_1^*. Similarly, U^{μ^*} is continuous q.e. on E_2^*. The same argument applies to u (recall (4.2), (4.2′)) and to φ (see Theorems III.12 and III.13 of [T]). Thus H is continuous q.e. on $\operatorname{supp}(\mu^*)$.

Now, since $\gamma = 1/(a_1 + a_2)$ and $F(\tau) = F_1(\tau) + F_2(\tau)$ we observe that

$$\begin{aligned} &F_1(\tau) + (2\tau \quad 1) \log \frac{1}{\operatorname{cap}(E_1^* \cup E_2^*)} \quad \gamma F(\tau) a_1 \\ &\qquad F_2(\tau) + (2\tau \quad 1) \log \frac{1}{\operatorname{cap}(E_1^* \cup E_2^*)} + \gamma F(\tau) a_2 =: M. \end{aligned}$$

Thus (see (4.10)), $H(z) = M$ for q.e. $z \in \operatorname{supp}(\mu^*)$. Consequently, the continuity property of H established above implies that for any component D of the complement of $\operatorname{supp}(\mu^*)$ we have

$$\lim_{\substack{\zeta \to z \\ \zeta \in D}} H(\zeta) \quad M \qquad \text{for q.e.} \quad z \in \partial D.$$

Since H is bounded in $\mathbf{C}$ we deduce from the generalized maximum principle (see Theorem III.28 of [T]) that H is constant q.e. in $\mathbf{C}$. Since $H(\infty) = 0$, it follows that (4.8) holds q.e. ∎

Corollary 4.2. *With the above notation, let ν_1, ν_2 denote the restrictions of ν on*

E_1^*, E_2^*, respectively. Then

$$F(\tau) = \frac{1}{\gamma}\,[(1 - \tau)\|\nu_1\| + \tau\|\nu_2\|], \tag{4.11}$$

where γ is given by (4.9).

Proof. Write (4.7) as a pair of equations

$$\mu_1^* = (2\tau - 1)\nu_1 + \gamma F(\tau)\sigma_1, \tag{4.12}$$

$$\mu_2^* = -(2\tau - 1)\nu_2 + \gamma F(\tau)\sigma_2 \tag{4.13}$$

If $\tau \geq \frac{1}{2}$, we obtain from (4.12) that

$$\tau = \|\mu_1^*\| = (2\tau - 1)\|\nu_1\| + \gamma F(\tau),$$

and since $\|\nu_1\| + \|\nu_2\| = 1$, we get (4.11). If $\tau \leq \frac{1}{2}$, we use (4.13) and again obtain (4.11). ∎

There are many important situations when $\operatorname{supp}(\mu^*(\tau))$ can be described explicitly. In these cases, the representations (4.8), (4.11) provide an efficient method to determine $F(\tau)$ and $U^{\mu^*(\tau)}$ numerically (see Section 8).

5. Two Examples

In this section we present two simple examples in which $F(\tau)$ can be explicitly determined. In the first example $F(\tau)$ is linear, while in the second $F(\tau)$ is a hat function.

Example 5.1. Let $E_1 = \{z : |z| = 1\}$, $E_2 = \{z : |z| = R > 1\}$. Since this configuration is invariant under rotations about 0, it follows, by the uniqueness of the equilibrium measure μ^*, that for any $0 \leq \tau \leq 1$ we have

$$d\mu_1^* = \tau\,\frac{1}{2\pi}\,d\theta \quad \text{on } E_1, \qquad d\mu_2^* = (1 - \tau)\,\frac{1}{2\pi}\,d\theta \quad \text{on } E_2$$

$$\frac{1}{2\pi}\int_0^{2\pi} \log|z - re^{i\theta}|\,d\theta = \begin{cases} \log|z|, & |z| > r, \\ \log r, & |z| \leq r, \end{cases} \tag{5.1}$$

we obtain

$$U^{\mu^*}(z) = \frac{\tau}{2\pi}\int_0^{2\pi} \log\frac{1}{|z - e^{i\theta}|}\,d\theta + \frac{1 - \tau}{2\pi}\int_0^{2\pi} \log\frac{1}{|z - Re^{i\theta}|}\,d\theta$$

$$= \begin{cases} (1 - \tau)\log R, & |z| = 1, \\ (1 - 2\tau)\log R, & |z| = R. \end{cases}$$

Thus $F_1(\tau) = (1 - \tau)\log R$, $F_2(\tau) = (2\tau - 1)\log R$, so that $F(\tau) = \tau\log R$.

Another way to find F is to use formula (4.11). Assume that $0 < \tau < 1$. Then $\operatorname{supp}(\mu^*) = E_1 \cup E_2$, and we have

$$\gamma = \operatorname{cap}(E_1, E_2) = \frac{1}{\log R}$$

Since v is supported on the outer boundary of $E_1 \cup E_2$, we have $\|v_2\| = 1$, $\|v_1\| = 0$. Therefore, by (4.11), $F(\tau) = \tau \log R$. By the continuity of F this formula is valid for $\tau = 0$, $\tau = 1$ as well.

Example 5.2. Let $E_1 = C_r \cup C_R$ and $E_2 = C_1$ where, generically, $C_\rho := \{z : |z| = \rho\}$, and we assume that $r < 1 < R$. Again, our configuration is invariant under rotations. Hence, given $0 \le \tau \le 1$, there is a constant α with $0 \le \alpha \le 1$ such that

$$d\mu_1^* = \begin{cases} \alpha\tau \dfrac{1}{2\pi}\, d\theta & \text{on} \quad |z| = r, \\[2ex] (1-\alpha)\tau \dfrac{1}{2\pi}\, d\theta & \text{on} \quad |z| = R. \end{cases}$$

$$d\mu_2^* = (1-\tau)\frac{1}{2\pi}\, d\theta \qquad \text{on} \quad |z| = 1$$

Applying (5.1) we obtain that

$$U^{\mu^*}(z) = \tau\left(\alpha \log \frac{1}{r} + (1-\alpha)\log\frac{1}{R}\right), \qquad |z| = r,$$

$$U^{\mu^*}(z) = (2\tau - 1)\log\frac{1}{R}, \qquad |z| = R,$$

$$U^{\mu^*}(z) = \tau(1-\alpha)\log\frac{1}{R}, \qquad |z| = 1 \tag{5.4}$$

To determine $F(\tau)$ we first assume that $0 < \tau < 1$ and we consider three cases.

Case 1. Assume $0 < \alpha < 1$. Then $\operatorname{supp}(\mu^*) = E_1 \cup E_2$ and $U^{\mu^*} = F_1$ on E_1, $U^{\mu^*} = -F_2$ on E_2. By (5.2), (5.3) we have

$$\tau\left(\alpha \log\frac{1}{r} + (1-\alpha)\log\frac{1}{R}\right) = (2\tau - 1)\log\frac{1}{R} = F_1$$

$$\alpha = \frac{-\tau \ \log R}{\log(R/r)}$$

Since $\alpha < 1$, this is possible only if

$$\tau > \frac{\log R}{\log R + \log(R/r)} =: \tau_0 \tag{5.6}$$

For such τ we have, by (5.2), (5.4), and (5.5), that

$$F(\tau) = \frac{\log(1/r)\log R}{\log(R/r)}(1-\tau). \tag{5.7}$$

Case 2. Assume $\alpha = 0$. Then μ_1^* is supported on C_R. Thus $U^{\mu^*}(z) \geq F_1$ on C_r, and $U^{\mu^*}(z) = F_1$ on C_R. From (5.2), (5.3) (with $\alpha = 0$) we obtain

$$\tau \log \frac{1}{R} \geq (2\tau - 1)\log \frac{1}{R},$$

which is impossible if $\tau < 1$

Case 3. Assume $\alpha = 1$. Then μ_1^* is supported on C_r. In this case $F_2 = 0$ (by (5.4)), and $F_1 = \tau\log(1/r)$ by (5.2)) and so $F(\tau) = \tau\log(1/r)$. Since $U^{\mu^*} \geq F_1$ on C_R we get

$$(2\tau - 1)\log \frac{1}{R} \geq \tau \log \frac{1}{r},$$

which implies that $\tau \leq \tau_0$, where τ_0 is given in (5.6).

Combining all cases and using the continuity of F we obtain that

$$F(\tau) = \begin{cases} \dfrac{\log(1/r)\log R}{\log(R/r)}(1-\tau) & \text{if} \quad \tau_0 < \tau \leq 1 \\[2ex] \tau \log\left(\dfrac{1}{r}\right) & \text{if} \quad 0 \leq \tau \leq \tau_0. \end{cases}$$

Notice that F is a hat function that attains its maximum on $[0, 1]$ at $\tau = \tau_0$. Since τ_0 corresponds to Case 3, $\mu^*(\tau_0)$ has no mass on $|z| = R$.

6. Asymptotics for Zolotarev Numbers

Our goal in this section is to describe the limiting behavior of ray sequences of the Zolotarev numbers

$$Z_{m,n} = Z_{m,n}(E_1, E_2) := \inf_{r \in \mathbf{R}_{m,n}} \left(\frac{\sup_{z \in E_1} |r(z)|}{\inf_{z \in E_2} |r(z)|} \right).$$

Theorem 6.1. *Let E_1, E_2 be disjoint compact subsets of* $\mathbf{C}$ *each having positive logarithmic capacity and let $\{(m_k, n_k)\}_{k=1}^{\infty}$ be a sequence of ordered pairs of nonnegative integers that satisfy*

$$\lim_{k\to\infty} \frac{m_k}{n_k} = \lambda, \qquad \lim_{k\to\infty}(m_k + n_k) = \infty,$$

where $0 < \lambda < \infty$. *Then the Zolotarev numbers* (6.1) *satisfy*

$$\lim_{k \to \infty} Z_{m_k, n_k}^{1/(m_k + n_k)} = e^{-F(\tau)},$$

where

$$\tau \quad \tau(\lambda) := \frac{\lambda}{\lambda + 1},$$

and F *is defined in* (3.6). *Furthermore, if* E_1 (E_2) *is closed but unbounded and* E_2 (E_1) *is compact, then* (6.3) *holds provided* $m_k \leq n_k$ $(m_k \geq n_k)$ *for all* k.

Before proceeding with the proof we first observe that if $m + n > 0$, then

$$Z_{m,n}^{1/(m+n)} \geq \exp\left\{ \quad F\left(\frac{m}{m+n}\right)\right\},$$

provided $m \leq n$ $(m \geq n)$ in case E_1 (E_2) is unbounded. Indeed, suppose $r \in \mathbf{R}_{m,n}$ has a numerator degree m and denominator degree n and let $\mu = \mu_1 - \mu_2$ be the discrete signed measure that has mass $1/(m + n)$ at each (finite) zero of r and has mass $-1/(m + n)$ at each (finite) pole of r. Then

$$\|\mu_1\| \quad \frac{m}{m+n} =: \tau_{m,n}, \qquad \|\mu_2\| = 1 \quad \tau_{m,n},$$

and hence by inequality (3.4) of Theorem 3. we have

$$\underset{z \in E_1}{\text{“inf”}}\, U^{\mu}(z) - \underset{z \in E_2}{\text{“sup”}}\, U^{\mu}(z) \leq F(\tau_{m,n}).$$

However,

$$U^{\mu}(z) \quad \frac{1}{m+n} \log \frac{1}{|r(z)|},$$

and so (6.6) yields

$$\left[\frac{\sup_{z \in E_1} |r(z)|}{\inf_{z \in E_2} |r(z)|}\right]^{1/(m+n)} \geq \quad F(\tau_{m,n}).$$

Since (6.7) holds for all r of the described form, it obviously holds for all $r \in \mathbf{R}_{m,n}$, which yields (6.5).

To prove Theorem 6.1 it is convenient to introduce the class $\hat{\mathbf{R}}_{\tau,k}$, $0 \leq \tau \leq 1$, $k = 1, 2, \ldots,$ of (multiple-valued) functions of the form

$$r(z) = \prod_{i=1}^{k} \frac{(z - z_i)^{\tau}}{(z - w_i)^{1-\tau}}$$

and to define the corresponding Zolotarev number by

$$\hat{Z}_{\tau,k} = \hat{Z}_{\tau,k}(E_1, E_2) := \inf_{r \in \hat{\mathbf{R}}_{\tau,k}} \left(\frac{\sup_{z \in E_1} |r(z)|}{\inf_{z \in E_2} |r(z)|} \right).$$

Observe that if $r_1 \in \hat{\mathbf{R}}_{\tau,k}$ and $r_2 \in \hat{\mathbf{R}}_{\tau,l}$, then $r_1 r_2 \in \hat{\mathbf{R}}_{\tau,k+l}$. Consequently, $\hat{Z}_{\tau,k+l} \le \hat{Z}_{\tau,k} \cdot \hat{Z}_{\tau,l}$ and therefore $\lim_{k\to\infty}(\hat{Z}_{\tau,k})^{1/k}$ exists. Moreover, the same argument used to prove (6.7) yields

$$(\hat{Z}_{\tau,k})^{1/k} \ge e^{-F(\tau)}. \tag{6.10}$$

We now show that equality holds in the limit.

Lemma 6.2. *Let E_1, E_2 be as in Theorem* 6.1 *and assume that $\tau \notin \{0, 1\}$ is in the domain of F. Then*

$$\lim_{k\to\infty} (\hat{Z}_{\tau,k})^{1/k} = e^{-F(\tau)}.$$

Proof. We assume that E_1, E_2 are compact. (If one of these sets is unbounded we can apply the reasoning used in the proof of Theorem 3.1.) We follow essentially the same argument used in [MS, p. 117].

With $0 < \tau < 1$, let U^{μ^*} be the potential for the equilibrium measure $\mu^* = \mu^*(\tau)$ of Theorem 3.1. Then

$$U^{\mu^*} \ge F_1 \quad \text{q.e. on } E_1, \qquad U^{\mu^*} \le -F_2 \quad \text{q.e. on } E_2 \tag{6.12}$$

Given $\varepsilon > 0$, we introduce the exceptional sets

$$E_1(\varepsilon) := \{z \in E_1 : U^{\mu^*}(z) \le F_1 - \varepsilon\},$$
$$E_2(\varepsilon) := \{z \in E_2 : U^{\mu^*}(z) \ge -F_2 + \varepsilon\}.$$

Since U^{μ^*} is lower (upper) semicontinuous on E_1 (on E_2) it follows that $E_1(\varepsilon)$, $E_2(\varepsilon)$ are compact. Furthermore, by (6.12), these sets have capacity zero. Thus by Evan's theorem (see Theorem III.27 of [T]), positive measures $\sigma_{1\varepsilon}$ on $E_1(\varepsilon)$ and $\sigma_{2\varepsilon}$ on $E_2(\varepsilon)$ exist such that for $\sigma_\varepsilon := \sigma_{1\varepsilon} - \sigma_{2\varepsilon}$ we have

$$\begin{cases} +\infty, & z \in E_1(\varepsilon), \\ & z \in E_2(\varepsilon). \end{cases} \tag{6.13}$$

τ and $\|\sigma_{2\varepsilon}\| = 1$ τ. Then, for any

$$\mu_{\alpha,\varepsilon} := (1-\alpha)\mu^* + \alpha\sigma_\varepsilon \tag{6.14}$$

belongs to $\mathcal{M}_\tau$. The corresponding potential $U^{\mu_{\alpha,\varepsilon}}$ is finite outside $E_1(\varepsilon) \cup E_2(\varepsilon)$, and since this union is compact, the potential is finite q.e. on $E_1 \cup E_2$ (recall that $\operatorname{cap}(E_i) > 0$, $i = 1, 2$).

Since we have assumed that E_1, E_2 are bounded, we have for some constants $c_1(\varepsilon) > 0$, $c_2(\varepsilon) > 0$ that

$$U^{\sigma_\varepsilon}(z) \ge \quad c_1(\varepsilon) \quad \text{on } E_1, \qquad U^{\sigma_\varepsilon}(z) \le c_2(\varepsilon) \quad \text{on } E_2$$

Thus from the definitions of $E_i(\varepsilon)$ and $\mu_{\alpha,\varepsilon}$ we obtain (recall (6.13))

$$(6.15)\qquad \begin{aligned} U^{\mu_{\alpha,\varepsilon}} &\geq (1-\alpha)(F_1-\varepsilon)-\alpha c_1(\varepsilon) \qquad \text{on } E_1, \\ U^{\mu_{\alpha,\varepsilon}} &\leq (1-\alpha)(-F_2+\varepsilon)+\alpha c_2(\varepsilon) \qquad \text{on } E_2 \end{aligned}$$

Next, it is easy to see that a triangular scheme of points $\{z_{k,1},\ldots,z_{k,m_k}\}_{k=1}^{\infty}$ on E_1 can be constructed such that the corresponding normalized counting measures defined by

$$\nu_k^{(1)}(B) := \frac{1}{m_k}\operatorname{card}\{i: z_{k,i}\in B\}, \qquad B\subset \mathbf{C},$$

converge weak-star to the measure

$$\frac{1}{\tau}\left[(1-\alpha)\mu_1^* + \alpha\sigma_{1\varepsilon}\right].$$

Similarly, a triangular scheme $\{w_{k,1},\ldots,w_{k,n_k}\}_{k=1}^{\infty}$ on E_2 can be constructed such that the corresponding counting measures $\nu_k^{(2)}$ converge weak-star to the measure

$$\frac{1}{1-\tau}\left[(1-\alpha)\mu_2^* + \alpha\sigma_{2\varepsilon}\right].$$

Thus, on setting

$$(6.16)\qquad \nu_k := \tau\nu_k^{(1)} - (1-\tau)\nu_k^{(2)}$$

we have $\nu_k \xrightarrow{*} \mu_{\alpha,\varepsilon}$ as $k\to\infty$.

Next we observe that the function

$$r(z) := \frac{\prod_{i=1}^{m_k}(z-z_{k,i})^{\tau n_k}}{\prod_{i=1}^{n_k}(z-w_{k,i})^{(1-\tau)m_k}}$$

belongs to the class $\hat{\mathbf{R}}_{\tau,m_k n_k}$ and, from the definitions of ν_k, $\nu_k^{(1)}$, and $\nu_k^{(2)}$, we have

$$(6.17)\qquad \frac{1}{m_k n_k}\log|r(z)| = -U^{\nu_k}(z).$$

Since $\lim_{j\to\infty}(\hat{Z}_{\tau,j})^{1/j}$ exists, we obtain from the definition of $\hat{Z}_{\tau,j}$ that

$$(6.18)\qquad \lim_{j\to\infty}\frac{1}{j}\log\hat{Z}_{\tau,j} \leq \limsup_{k\to\infty}\left[\sup_{E_1}(-U^{\nu_k}) + \sup_{E_2} U^{\nu_k}\right].$$

Let a_k, b_k satisfy

$$\sup_{E_1}(-U^{\nu_k}) = -U^{\nu_k}(a_k), \qquad \sup_{E_2} U^{\nu_k} = U^{\nu_k}(b_k),$$

and assume (passing if necessary to a subsequence) that $a_k \to a\in E_1$ and

$b_k \to b \in E_2$. By the principle of descent (note that $-U^{\nu_k}$ and U^{ν_k} are subharmonic on E_1 and E_2, respectively), we get

$$\limsup_{k\to\infty}(-U^{\nu_k}(a_k)) \le -U^{\mu_{\alpha,\varepsilon}}(a) \le -[(1-\alpha)(F_1-\varepsilon)-\alpha c_1(\varepsilon)],$$

where the last inequality follows from (6.15). Similarly,

$$\limsup_{k\to\infty} U^{\nu_k}(b_k) \le U^{\mu_{\alpha,\varepsilon}}(b) \le (1-\alpha)(-F_2+\varepsilon)+\alpha c_2(\varepsilon).$$

Using the estimates (6.19) and (6.20) in (6.18) we obtain

$$\lim_{j\to\infty}\frac{1}{j}\log \hat{Z}_{\tau,j} \le -(1-\alpha)F(\tau)+2(1-\alpha)\varepsilon+\alpha(c_1(\varepsilon)+c_2(\varepsilon)).$$

Thus, on first letting $\alpha \to 0$ and then $\varepsilon \to 0$, we obtain that

$$\lim_{j\to\infty}\frac{1}{j}\log \hat{Z}_{\tau,j} \le -F(\tau),$$

which together with (6.10) completes the proof. ■

Now we turn to the

Proof of Theorem 6.1. We give the proof for the case when E_1 and E_2 are compact and leave the unbounded cases for the reader. Let (m_k, n_k) be a sequence that satisfies (6.2) for some λ $(0<\lambda<\infty)$. Notice that from (6.5) we have

$$Z_{m_k,n_k}^{1/(m_k+n_k)} \ge \exp\{-F(\tau_k)\}, \qquad \tau_k := \frac{m_k}{m_k+n_k}$$

and, from (6.2), $\tau_k \to \tau = \lambda/(\lambda+1)$ as $k\to\infty$. Thus, on appealing to the continuity of F we get that

$$\liminf_{k\to\infty} Z_{m_k,n_k}^{1/(m_k+n_k)} \ge e^{-F(\tau)}.$$

To obtain the needed upper bound, assume first that $\lambda = p/q$ is rational. Then, given a sequence (m_k, n_k) that satisfies (6.2), nonnegative integers α_k, β_k can be found such that

$$\frac{m_k-\alpha_k}{n_k-\beta_k}=\lambda \tag{6.22}$$

and

$$\alpha_k+\beta_k = o(m_k+n_k). \tag{6.23}$$

Indeed, if $m_k/n_k \le \lambda$, set

$$\alpha_k = m_k - p\left[\frac{m_k}{p}\right], \qquad \beta_k = n_k - q\left[\frac{m_k}{p}\right],$$

and, if $m_k/n_k > \lambda$, set

$$\alpha_k = m_k - p\left[\frac{n_k}{q}\right], \qquad \beta_k = n_k - q\left[\frac{n_k}{q}\right],$$

where $[x]$ denotes the greatest integer not larger than x.

Now let $a, b \notin E_1 \cup E_2$. Since $E_1 \cup E_2$ is bounded, we have for some constants $A > 0$, $B > 0$ that

$$A \le |z - a| \quad \text{and} \quad B \ge |z - b|, \qquad z \in E_1 \cup E_2. \tag{6.24}$$

On multiplying an arbitrary $r \in \mathbf{R}_{m_k - \alpha_k, n_k - \beta_k}$ by $(z-a)^{\alpha_k}/(z-b)^{\beta_k}$ we obtain from (6.23) and (6.24) that

$$\limsup_{k\to\infty} Z_{m_k,n_k}^{1/(m_k+n_k)} \le \limsup_{k\to\infty} (Z_{m_k-\alpha_k, n_k-\beta_k})^{1/(m_k-\alpha_k+n_k-\beta_k)}.$$

On setting $\tau = p/(p+q)$ in Lemma 6.2 and recalling (6.22) we deduce the desired upper bound

$$\limsup_{k\to\infty} Z_{m_k,n_k}^{1/(m_k+n_k)} \le \lim_{j\to\infty} (\hat{Z}_{\tau,j})^{1/j} = e^{-F(\tau)}. \tag{6.25}$$

Now let λ be irrational. We split the sequence (m_k, n_k) into one for which $m_k/n_k < \lambda$ and another for which $m_k/n_k \ge \lambda$. We consider only the former sequence since the latter is similar.

Choose integers p, q such that $p/q > \lambda$. Then $m_k/n_k < p/q$ and hence

$$\beta_k := n_k - q\left[\frac{m_k}{p}\right] \ge 0.$$

Also we have

$$\alpha_k := m_k - p\left[\frac{m_k}{p}\right] \ge 0.$$

With this notation we obtain (as in the first part of the proof) that

$$Z_{m_k,n_k} \le C^{\alpha_k+\beta_k} Z_{p[m_k/p],\, p[m_k/p]} \tag{6.26}$$

for some constant $C > 1$. Note that

$$\lim_{k\to\infty} \frac{\alpha_k+\beta_k}{m_k+n_k} = 1 - \left(\quad p \qquad \right)$$

$$\frac{(p+q)[m_k/p]}{m_k+n_k} = \left(\qquad \right).$$

$$\limsup_{k\to\infty} Z_{m_k,n_k}^{1/(m_k+n_k)} \le C^{1-((p+q)/p)(\lambda/(\lambda+1))} \exp\left(-\left(\frac{p+q}{p}\right)\left(\frac{\lambda}{\lambda+1}\right) F\left(\frac{p}{p+q}\right)\right).$$

On setting in this inequality $p = p_i$, $q = q_i$, where p_i/q_i decreases to λ, and using the continuity of F we again obtain (6.25). Together with (6.21), this completes the proof.

Remark. It follows from (1.6) that the conclusion of Theorem 6.1 holds for the polynomials if and only if

$$d := \min_{z \in E_2} g_{D_\infty(E_1)}(z; \infty) = \text{“}\inf\text{”}_{z \in E_2} g_{D_\infty(E_1)}(z; \infty) = F(1).$$

This relation holds, in particular, if E_2 is regular. Trivial examples show that we may have $d < F(1)$, if E_2 is irregular. We conjecture that, in such a case, given any value ρ of $g_{D_\infty(E_1)}$ such that $d \le \rho \le F(1)$ we can construct a sequence (m_k, n_k) with $m_k/n_k \to \infty$ for which the limit in (6.3) is equal to $e^{-\rho}$. Several examples support this conjecture. Similar remarks apply to the case $\lambda = 0$.

7. Numerical Methods

Throughout this section we assume (for simplicity only) that E_1, E_2 are bounded. We start with a simple lemma (see also Proposition 3.3 of [LR]).

Lemma 7.1. *Given λ, $0 \le \lambda \le \infty$, let $\{m_k\}$, $\{n_k\}$, $\{p_k\}$, $\{q_k\}$ be sequences of positive integers that satisfy*

$$m_k + n_k \to \infty \quad \textit{and} \quad \frac{m_k p_k}{n_k q_k} \to \lambda \qquad \textit{as} \quad k \to \infty.$$

Let $\{z_{j,m_k}\}_{j=1}^{m_k}$, $\{w_{j,n_k}\}_{j=1}^{n_k}$, $k = 1, 2, \ldots$, be triangular schemes of (not necessarily distinct) points in E_1, E_2, respectively, and assume that the corresponding normalized counting measures $\nu_k^{(1)}$, $\nu_k^{(2)}$ satisfy

$$\tau\nu_k^{(1)} - (1 - \tau)\nu_k^{(2)} \xrightarrow{*} \mu^*(\tau) \qquad \textit{as} \quad k \to \infty,$$

where $\tau = \lambda/(\lambda + 1)$ and $\mu^*(\tau)$ is the equilibrium measure of Theorem 3.1. Set

$$r_k(z) := \frac{\prod_{j=1}^{m_k}(z - z_{j,m_k})^{p_k}}{\prod_{j=1}^{n_k}(z - w_{j,n_k})^{q_k}} \in \mathbf{R}_{m_k p_k, n_k q_k}.$$

$$\text{(7.4)} \qquad \limsup_{k \to \infty} \left(\frac{\sup_{E_1}|r_k|}{\inf_{E_2}|r_k|} \right)^{1/(m_k p_k + n_k q_k)} \le \exp\left\{ \left(\inf_{E_1} U^{\mu^*} - \sup_{E_2} U^{\mu^*} \right) \right\}.$$

Moreover, if E_1, E_2 are regular, then

$$\text{(7.5)} \qquad \lim_{k \to \infty} \left(\frac{\sup_{E_1}|r_k|}{\inf_{E_2}|r_k|} \right)^{1/(m_k p_k + n_k q_k)} \quad \exp(-F(\tau)).$$

Proof. On setting

$$\nu_k := \frac{m_k p_k}{m_k p_k + n_k q_k} \nu_k^{(1)} - \frac{n_k q_k}{m_k p_k + n_k q_k} \nu_k^{(2)}$$

we have by (7.1) and (7.2) that $\nu_k \xrightarrow{*} \mu^*$ as $k \to \infty$. Applying the principle of descent, we obtain (in exactly the same manner as in the proof of Lemma 6.2) that

$$\limsup_{k \to \infty} \left\{ \sup_{E_1}(-U^{\nu_k}) + \sup_{E_2}(U^{\nu_k}) \right\} \leq -\inf_{E_1} U^{\mu^*} + \sup_{E_2} U^{\mu^*}.$$

Since $\log|r_k|^{-1} = (m_k p_k + n_k q_k) U^{\nu_k}$, we get (7.4).

If E_1, E_2 are regular, we have by Theorem 3.1 (recall also (2.10), (2.11), (2.5), (2.7)) that

$$\inf_{E_1} U^{\mu^*} = \text{“}\inf\text{”}_{E_1} U^{\mu^*} = F_1, \qquad \sup_{E_2} U^{\mu^*} = \text{“}\sup\text{”}_{E_2} U^{\mu^*} = -F_2$$

Thus the right-hand side of (7.4) becomes $\exp(-F(\tau))$. Combining this fact with (6.5) gives (7.5). ■

Our goal in this section is to describe three methods that produce triangular schemes of points that satisfy (7.2). These methods are generalizations of classical methods due to Fekete, Leja, Fejér, Walsh, and Bagby.

a. *Fekete Points*

Given $0 \leq \tau \leq 1$, set

$$d_{k,\tau} := \sup_{\substack{z_i \in E_1 \\ w_i \in E_2}} \left\{ \prod_{\substack{i \neq j \\ 1 \leq i,j \leq k}} \frac{|z_i - z_j|^{\tau^2} |w_i - w_j|^{(1-\tau)^2}}{|z_i - w_j|^{2\tau(1-\tau)}} \right\}^{1/k(k-1)};$$

and let $\{z_{i,k}\}_{i=1}^k$, $\{w_{i,k}\}_{i=1}^k$ be points for which the supremum is attained. Let $\nu_k^{(1)}$, $\nu_k^{(2)}$ denote the corresponding normalized counting measures for these points. A standard argument shows that

$$d_{k,\tau} \geq \exp(-V_\tau), \qquad k = 2, 3, \ldots,$$

and consequently the measures $\nu_k^{(1)}$, $\nu_k^{(2)}$ satisfy (7.2) (see [B2] and [ST] for the case $\tau = \frac{1}{2}$).

Now, choose any sequences $\{p_k\}$, $\{q_k\}$ of positive integers such that $p_k/q_k \to \lambda$ and define r_k by (7.3) with $m_k = n_k = k$. Then (7.1) holds and Lemma 7.1 yields that these r_k satisfy (7.5), provided E_1, E_2 are regular. Alternatively, one can choose sequences $\{m_k\}$, $\{n_k\}$ such that $m_k/n_k \to \lambda$ and define r_k by (7.3) with $p_k = q_k = 1$. Note that in the latter construction the points z_{i,m_k}, w_{i,n_k} are derived from the solution to two extremal problems (7.6) (for $k = m_k$ and $k = n_k$).

Remark. By a standard argument it can be shown that $d_{k,\tau}$ decreases as $k \to \infty$ and that

$$\lim_{k \to \infty} d_{k,\tau} = \exp(-V_\tau).$$

Since $V_{1/2} = \frac{1}{2}F(\frac{1}{2})$, we obtain (see [ST]) that, for the case $\tau = \frac{1}{2}$, relation (7.5) holds for arbitrary E_1, E_2. In general, it is not clear whether the regularity assumption can be dropped. We also mention that the quantity $d_{k,\tau}$ was independently introduced in [LR].

b. Leja–Bagby Points

More convenient (from the numerical point of view) triangular schemes were introduced by Leja (for $\tau = 1$) and by Bagby (for $\tau = \frac{1}{2}$). These can be generalized as follows. For simplicity, assume that $\lambda = p/q$ is rational. Then

$$\tau = \frac{p}{p+q} \qquad -\tau = \frac{q}{p+q}$$

Pick any $z_1 \in E_1$, $w_1 \in E_2$ and successively define $z_n \in E_1$, $w_n \in E_2$ to maximize (over all $z \in E_1$, $w \in E_2$) the expression

$$\frac{\prod_{j=1}^{n-1}(|z - z_j|^{p^2}/|z - w_j|^{pq})}{\prod_{j=1}^{n-1}(|w - z_j|^{pq}/|w - w_j|^{q^2})}$$

Denote this maximum by A_n^{n-1}. Let $\nu_n^{(1)}$, $\nu_n^{(2)}$ denote the normalized counting measures for $\{z_j\}_{j=1}^n$, $\{w_j\}_{j=1}^n$, respectively, and let

$$\sigma_n := \tau\nu_n^{(1)} - (1 - \tau)\nu_n^{(2)}.$$

If we show that $\sigma_n \to \mu^*$, then we deduce, by Lemma 7.1, that the functions

$$r_n(z) := \frac{\prod_{j=1}^n (z - z_j)^p}{\prod_{j=1}^n (z - w_j)^q} \in \mathbf{R}_{np,nq}$$

satisfy

$$\lim_{n \to \infty} \left(\frac{\max_{E_1}|r_n|}{\min_{E_2}|r_n|} \right)^{1/n(p+q)} = \exp(-F(\tau)),$$

provided E_1, E_2 are regular.

By the definition of A_n, σ_n we have (recall also (7.8))

$$\begin{aligned}(p + q)^{-2} \log\left(\frac{1}{A_{n+1}}\right) &= \tau \inf_{E_1} U^{\sigma_n} - (1 - \tau) \sup_{E_2} U^{\sigma_n} \\ &\le \int U^{\sigma_n}\, d\mu^* = \int U^{\mu^*}\, d\sigma_n \le C,\end{aligned}$$

for some constant C (cf. Theorem 2.1(c)). Thus

$$A_{k+1}^{k/(p+q)^2} \geq \exp(-kC), \qquad k = 1, 2, \ldots.$$

Multiplying these inequalities together for $k = 1, 2, \ldots, n-1$, we get (recall again (7.8))

$$\prod_{\substack{i \neq j \\ 1 \leq i, j \leq n}} \frac{|z_i - z_j|^{\tau^2} |w_i - w_j|^{(1-\tau)^2}}{|z_i - w_j|^{2\tau(1-\tau)}} \geq e^{-n(n-1)C},$$

which implies that, for some constant C_1,

$$\iint_{z \neq t} \log \frac{1}{|z-t|} \, d\sigma_n(t) \, d\sigma_n(z) \leq C_1. \tag{7.11}$$

Now let σ be any weak-star limit measure of the σ_n's. We shall show that σ fulfills the hypotheses of Theorem 2.3. First, inequality (7.11) implies (see, for example, the reasoning in [ST] for the case of Fekete points) that σ has finite energy.

Next, write $\sigma_n^{(1)} := \tau \nu_n^{(1)}$, $\sigma_n^{(2)} := (1-\tau)\nu_n^{(2)}$ so that $\sigma_n = \sigma_n^{(1)} - \sigma_n^{(2)}$. Then for some subsequence of integers, which we continue to denote by n, we have

$$\sigma_n^{(1)} \xrightarrow{*} \sigma^{(1)} \qquad \sigma_n^{(2)} \xrightarrow{*} \sigma^{(2)} \qquad \sigma = \sigma^{(1)} - \sigma^{(2)}.$$

Thus $\sigma \in \mathcal{M}_\tau$. By the lower envelope theorem (see [L]), a set $K \subset \mathbf{C}$, with $\operatorname{cap}(K) = 0$, exists such that

$$U^{\sigma^{(2)}}(w) = \liminf_{n \to \infty} U^{\sigma_n^{(2)}}(w), \qquad w \in \mathbf{C} \setminus K. \tag{7.12}$$

Pick any $\alpha, \beta \in \operatorname{supp}(\sigma^{(2)}) \setminus K$. Since $\alpha \in \operatorname{supp}(\sigma^{(2)})$, a subsequence of the points $\{w_n\}$ that converges to α (recall that $\sigma_j^{(2)}$ is supported in the points $w_1, \ldots, w_j$) exists. For convenience in notation, assume $w_n \to \alpha$. Now by the defining property of w_n we have

$$U^{\sigma_n}(w_n) = \sup_{E_2} U^{\sigma_n} \geq U^{\sigma_n}(\beta). \tag{7.13}$$

This implies

$$U^{\sigma_n^{(2)}}(w_n) \leq U^{\sigma_n^{(1)}}(w_n) - U^{\sigma_n^{(1)}}(\beta) + U^{\sigma_n^{(2)}}(\beta). \tag{7.14}$$

Clearly,

$$\lim_{n \to \infty} U^{\sigma_n^{(1)}}(w_n) = U^{\sigma^{(1)}}(\alpha) \quad \text{and} \quad \lim_{n \to \infty} U^{\sigma_n^{(1)}}(\beta) = U^{\sigma^{(1)}}(\beta),$$

and so from (7.14) we get

$$\liminf_{n \to \infty} U^{\sigma_n^{(2)}}(w_n) \leq U^{\sigma^{(1)}}(\alpha) - U^{\sigma^{(1)}}(\beta) + \liminf_{n \to \infty} U^{\sigma_n^{(2)}}(\beta).$$

Thus, from (7.12) and the generalized principle of descent (see [L]) it follows that

$$U^{\sigma^{(2)}}(\alpha) \leq U^{\sigma^{(1)}}(\alpha) - U^{\sigma^{(1)}}(\beta) + U^{\sigma^{(2)}}(\beta),$$

i.e.,

$$U^{\sigma}(\beta) \le U^{\sigma}(\alpha). \tag{7.15}$$

Interchanging α and β, we see that equality holds in (7.15). Thus $U^{\sigma} = \text{const.}$ q.e. on $\operatorname{supp}(\sigma^{(2)})$ and, in a similar manner we deduce that $U^{\sigma} = \text{const.}$ q.e. on $\operatorname{supp}(\sigma^{(1)})$.

Finally, if in the preceding argument we replace β by a point $w \in E_2 \backslash \operatorname{supp}(\sigma^{(2)})$, then we get (cf. (7.15))

$$U^{\sigma}(w) \le U^{\sigma}(\alpha), \qquad \alpha \in \operatorname{supp}(\sigma^{(2)}) \backslash K.$$

Since a corresponding inequality holds for $z \in E_1 \backslash \operatorname{supp}(\sigma^{(1)})$, we have shown that the conditions of Theorem 2.3 are satisfied and so $\sigma = \mu^*$. From the arbitrariness of σ we conclude that $\sigma_n \xrightarrow{*} \mu^*$.

c. *Fejér–Walsh Points*

In some important situations the potential U^{μ^*} can be found either explicitly or numerically (see Section 8). In such a case the following method, which generalizes those due to Fejér (for $\tau = 1$) and Walsh (for $\tau = \frac{1}{2}$), is more efficient than (a) or (b).

Assuming that E_1^*, E_2^* (the supports of μ_1^*, μ_2^*) are "nice enough" (say consist of rectifiable Jordan arcs or curves), we select m_k μ_1^*-equally spaced points $\{z_{j,m_k}\}_{j=1}^{m_k}$ on E_1^*, and select n_k μ_2^*-equally spaced points $\{w_{j,n_k}\}_{j=1}^{n_k}$ on E_2^*. Then the corresponding counting measures will obviously satisfy (7.2). Thus, on choosing $p_k = q_k = 1$ and $m_k/n_k \to \lambda$ we obtain that the rational functions of (7.3) will satisfy (7.5).

The points $z_{j,m}$, $w_{j,n}$ (we drop the index k in the following) can be found as follows. Let $\tilde{U}^{\mu^*}$ denote the (multiple-valued) conjugate of U^{μ^*} in $D_\infty(E_1^* \cup E_2^*)$. Then the function

$$\zeta = \Phi(z) := \exp(U^{\mu^*}(z) + i\tilde{U}^{\mu^*}(z))$$

maps the boundary of $D_\infty(E_1^* \cup E_2^*)$ onto the boundary of the annulus

$$e^{-F_2} < |\zeta| < e^{F_1}.$$

Assume, for example, that E_1^* is a simple Jordan *curve*, while E_2^* is a simple Jordan *arc*. Then, as z moves along E_1^*, the μ_1^*-measure of the portion of E_1^* traversed is given by the change of the argument of Φ divided by $2\pi/\tau$. Similarly, the μ_2^*-measure of the portion of E_2^* traversed is given by the change of the argument of Φ divided by $\pi/(1-\tau)$. Thus, on setting

$$z_{j,m} := \Phi^{-1}(\zeta_{j,m}^{(1)}), \qquad \zeta_{j,m}^{(1)} = \exp\left(F_1 + \frac{2\pi i}{m} j\right), \qquad j = 0, \ldots, m-1,$$

$$w_{j,n} := \Phi^{-1}(\zeta_{j,n}^{(2)}), \qquad \zeta_{j,n}^{(2)} = \exp\left(-F_2 + \frac{\pi i}{n}(j + \tfrac{1}{2})\right), \qquad j = 0, \ldots, n-1$$

we obtain the desired (Fejér–Walsh) points on E_1^*, E_2^*.

To find Φ, we appeal to Theorem 4.1, according to which

$$U^{\mu^*} = (2\tau - 1)U^{\nu} + \gamma F(\tau) U^{\sigma} = -(2\tau - 1)g + \gamma F(\tau) U^{\sigma} + c$$

where ν is the equilibrium measure for $E_1^* \cup E_2^*$, $g := g_{D_\infty(E_1^* \cup E_2^*)}$, and c is a real constant. Thus

$$\Phi = e^c G^{2\tau - 1} f^{\gamma F(\tau)}, \tag{7.18}$$

where $G := \exp(-g - i\tilde{g})$ is a complex Green function for $D_\infty(E_1^* \cup E_2^*)$ with zero at ∞, while

$$f := \exp(U^\sigma + i\tilde{U}^\sigma)$$

maps this domain onto the annulus $e^{-a_2} < |\zeta| < e^{a_1}$ (cf. (4.2)). (Of course, in order to find the argument of Φ we may drop the factor e^c in (7.18).)

If $\tau = \frac{1}{2}$, then $\gamma F(\tau) = \frac{1}{2}$ (by (4.11)), so that (7.18) reduces to $\Phi = e^c f^{1/2}$. In this case it is simpler to define $z_{j,m}$, $w_{j,n}$ (for $m = n$) by (7.17) with Φ replaced by $\Phi_1 := f$ and with F_1, F_2 replaced by a_1, a_2, respectively. This construction appears (for the case of two Jordan *curves*) in [S2]. We remark that in this special case, the points $z_{j,n}$, $w_{j,n}$ not only produce the relation (7.5), but the corresponding rational functions $r_n(z)$ are nearly optimal for a given n. Indeed, it was proved by T. Ganelius [Ga] that if E_1, E_2 have bounded boundary rotation, then with r_n as above the following holds:

$$e^{-2nF(1/2)} \le \frac{\sup_{E_1}|r_n|}{\inf_{E_2}|r_n|} \le Ce^{-2nF(1/2)}, \tag{7.19}$$

where C is independent of n.

It seems plausible that, for any $0 < \tau < 1$, the corresponding Fejér–Walsh points z_{jm}, w_{jn} $(m/n = \lambda)$ produce rational functions that satisfy

$$e^{-(m+n)F(\tau)} \le \frac{\sup_{E_1}|r_{mn}|}{\inf_{E_2}|r_{mn}|} \le Ce^{-(m+n)F(\tau)}.$$

This is correct, at least, if E_1, E_2 are bounded by a finite number of piecewise analytic Jordan curves. (For polygonal lines, this follows from a recent result due to Levin and Lubinsky [LL, Theorem 9.1]. For the more general case, some straightforward modifications in the proof in [LL] have to be made.)

Finally, we mention that our computed examples show that, for Fejér–Walsh points, the left-hand side of (7.5) (with $p_k = q_k = 1$) tends to $\exp(-F(\tau))$ in a monotonically increasing fashion. As yet, we have no explanation for this nice phenomenon.

8. The Case of Two Real Intervals

In this section we examine more closely the case when E_1, E_2 are intervals on the real axis. Let

$$E_1 = [a_1, b_1], \qquad E_2 = [a_2, b_2], \qquad b_1 < a_2$$

We allow that $a_1 = -\infty$ (if $\tau \le \frac{1}{2}$) or $b_2 = \infty$ (if $\tau \ge \frac{1}{2}$). If E_1, E_2 are of the same length, then we obviously have

$$F(1-\tau) = F(\tau), \qquad 0 \le \tau \le 1$$

and since F is concave, we deduce that

$$\max_{\tau} F(\tau) = F(\tfrac{1}{2}).$$

(It can be shown that in this case $F(\tau) = F(\frac{1}{2})$ for $|\tau - \frac{1}{2}| \le \varepsilon$, for some $\varepsilon > 0$).

In the general case Leja–Bagby points (for a given τ) can be computed to determine $F(\tau)$ numerically. Then, by repeating this procedure for various values of τ, the maximum of F can be found numerically. In this section we describe a much simpler procedure, which can also be applied to other important configurations (e.g., two circles).

First we describe the support of the equilibrium distribution.

Theorem 8.1. *Let $\mu^* = \mu^*(\tau) = \mu_1^*(\tau) - \mu_2^*(\tau)$ be the equilibrium distribution for the energy problem of Section 3 and let $E_i^* = E_i^*(\tau) := \operatorname{supp}(\mu_i^*)$, $i = 1, 2$.*

(i) *If $\tau = \frac{1}{2}$, then $E_i^* = E_i$, $i = 1, 2$.*
(ii) *If $\frac{1}{2} < \tau < 1$, then $E_1^* = E_1$ and $E_2^* = [a_2, s_\tau]$ for some s_τ, $a_2 < s_\tau \le b_2$. Also,*

$$U^{\mu^*}(x) < -F_2 \qquad \textit{for} \quad x > s_\tau.$$

Moreover, s_τ is continuous on $[\frac{1}{2}, 1)$ and $\lim_{\tau \to 1^-} s_\tau = a_2$.
(iii) *If $0 < \tau < \frac{1}{2}$, then $E_2^* = E_2$ and $E_1^* = [\tilde{s}_\tau, b_1]$ for some $\tilde{s}_\tau$, $a_1 \le \tilde{s}_\tau < b_1$. Also,*

$$U^{\mu^*}(x) > F_1 \qquad \textit{for} \quad x < \tilde{s}_\tau.$$

Moreover, $\tilde{s}_\tau$ is continuous on $(0, \frac{1}{2}]$ and $\lim_{\tau \to 0^+} \tilde{s}_\tau = b_1$.

Proof. Case (i) is classical. Also, by symmetry, it suffices to prove (ii). So, let $\frac{1}{2} < \tau < 1$ and let $U^{\mu^*} =: U$ be the corresponding equilibrium potential. Then $U(\infty) = -\infty$ and E_1 is compact. Since E_1, E_2 are regular, U is continuous in $\mathbf{C}$ and satisfies

$$U = F_1 \quad \text{on } E_1^*, \qquad U \ge F_1 \quad \text{on } E_1,$$
$$U = -F_2 \quad \text{on } E_2^*, \qquad U \le -F_2 \quad \text{on } E_2$$

Also (see Section 4), $F_1 > -F_2$.

Consider the level curve $\Gamma := \{z \colon U(z) = -F_2\}$. Since $U > -F_2$ on E_1, there is a bounded component G_1 of $\mathbf{C}\backslash\Gamma$ that contains E_1, and there is an unbounded component, G_2. Since U is subharmonic outside E_1 and $U \not\equiv$ const., there cannot be other components. Thus, $\mathbf{C}\backslash\Gamma = G_1 \cup G_2$, and the maximum principle yields

$$U > -F_2 \quad \text{in } G_1, \qquad U < -F_2 \quad \text{in } G_2. \tag{8.3}$$

Now U is harmonic in $\Delta := G_1 \backslash (E_1^* \cup E_2^*)$ and is equal either to F_1 or $-F_2$ on $\partial\Delta$. Thus $U < F_1$ in Δ and we get, by (8.1), that $E_1 = E_1^*$.

Next we show that:

(a) If $U(x_0) < -F_2$ for some $x_0 \in E_2$, then $U(x) < -F_2$ for $x > x_0$.
(b) If $U \equiv -F_2$ on an interval $I \subseteq E_2$, then $I \subseteq E_2^*$.

From (8.2) and the fact that E_2^* has positive capacity, it is easy to see that (a) and (b) imply that E_2^* is of the form $[a_2, s_\tau]$ and that $U(x) < -F_2$ for $x > s_\tau$.

Proof of (a). Since $x_0 \in G_2$, a polygonal line $L \subset G_2$ joining x_0 and ∞ can be found. Since $\operatorname{supp}(\mu^*) \subset \mathbf{R}$, we have

$$U(\bar{z}) = U(z), \qquad z \in \mathbf{C}. \tag{8.4}$$

Therefore we may assume (using reflection about $\mathbf{R}$, if necessary, and then applying a continuity argument) that $L \backslash \{x_0\}$ lies in the (open) upper half-plane. Let D be the domain bounded by L and by its reflection about $\mathbf{R}$, and containing the ray $\{x: x > x_0\}$. Then $U < -F_2$ on ∂D and U is subharmonic in $D \cup \{\infty\}$. Thus $U < -F_2$ in D.

Proof of (b). Assuming the contrary, an interval $I_1 \subset I$ can be found such that $U \equiv -F_2$ there and U is harmonic at each point in I_1. Since $U_x \equiv 0$ in I_1 and U is not a constant in $\mathbf{C}$, an $x_1 \in I_1$ where $U_y \neq 0$ can be found. However, this contradicts the symmetry property (8.4).

Finally, let $\tau_n \to \tau < 1$ and $s_{\tau_n} \to s'$. Since $\mu^*(\tau_n) \xrightarrow{*} \mu^*(\tau)$ (Lemma 3.4), we have $s(\tau) \leq s'$. Next, $U^{\mu^*(\tau_n)} = -F_2(\tau_n)$ on $[a_2, s_{\tau_n}]$. Thus,

$$U^{\mu^*(\tau)}(x) = \liminf_{\tau_n \to \tau} U^{\mu^*(\tau_n)}(x) = \text{const.} \qquad \text{for} \quad a_2 \leq x \leq s'.$$

However, $U^{\mu^*(\tau)} = -F_2(\tau)$ on $[a_2, s_\tau]$ and $U^{\mu^*(\tau)} < -F_2(\tau)$ if $x > s_\tau$. Therefore $s' = s_\tau$. For $\tau = 1$, we have similarly (recall (3.9)) that $s' = a_2$. ∎

Next we recall the representation (see (4.11))

$$F(\tau) = \frac{1}{\gamma}((1-\tau)\|\nu_1\| + \tau\|\nu_2\|),$$

where $\nu = \nu_1 + \nu_2$ is the equilibrium measure on $E_1^* \cup E_2^*$ and $\gamma = \operatorname{cap}(E_1^*, E_2^*)$. Assuming $\frac{1}{2} < \tau < 1$ we can write (by Theorem 8.1(ii))

$$E_1^* = [a_1, b_1], \qquad E_2^* = [a_2, s_\tau]$$

and therefore

$$\gamma = \operatorname{cap}([a_1, b_1], [a_2, s_\tau]).$$

We now show that if the length of E_1 is less than that of E_2, then the maximum of F on $[0, 1]$ is attained only on $(\frac{1}{2}, 1]$. Indeed, by (4.4),

$$F(\tfrac{1}{2}) = \frac{1}{2\operatorname{cap}([a_1, b_1], [a_2, b_2])}$$

On the other hand, it follows from Theorem 8.1 that the function $\tau \to s_\tau$ maps $(\frac{1}{2}, 1)$ onto (a_2, b_2). Thus if we define

$$b := a_2 + b_1 \quad a_1,$$

then $a_2 < b < b_2$ and therefore $b = s_{\tau_0}$ for some $\frac{1}{2} < \tau_0 < 1$. For this τ_0, the intervals E_1^*, E_2^* have the same length (see (8.6), (8.9)), so that $\|\nu_1\| = \|\nu_2\| = \frac{1}{2}$ and we get from (8.5) that

$$F(\tau_0) \quad \frac{1}{2\,\mathrm{cap}([a_1, b_1], [a_2, s_{\tau_0}])}$$

Since $[a_2, s_{\tau_0}] \subset [a_2, b_2]$, we obtain (see (8.8)) that $F(\tau_0) > F(\frac{1}{2})$. Finally, because F is concave, it must increase on the interval $[0, \frac{1}{2}]$. (Curiously, our computed examples show that the actual maximum of F is very close to the value $F(\tau_0)$.)

In what follows we assume (clearly without loss of generality) that the length of E_1 is less than that of E_2, and consider only $\tau \in (\frac{1}{2}, 1]$. Let $x \in (a_2, b_2)$. We know that $x = s_\tau$ for some $\frac{1}{2} < \tau < 1$. The following theorem enables us to find this τ explicitly. Therefore the corresponding $F(\tau)$ can be calculated from (8.5).

Theorem 8.2. *Let $x \in (a_2, b_2]$ and let $\nu = \nu_1 + \nu_2$ and $\sigma = \sigma_1 - \sigma_2$ be the equilibrium measures for the set $[a_1, b_1] \cup [a_2, x]$ and for the condenser $([a_1, b_1], [a_2, x])$, respectively. Then:*

(i) *For $x \in (a_2, b_2)$, a unique $\tau_x > \frac{1}{2}$ exists for which $x \quad s_{\tau_x}$, and τ_x is the solution of the equation*

$$\frac{2\tau \quad 1}{(1-\tau)\|\nu_1\| + \tau\|\nu_2\|} \quad \lim_{t \to x+0} \left(\frac{(d/dt)U^\sigma(t)}{(d/dt)U^\nu(t)} \right).$$

(ii) *It holds that*

$$\hat{\tau} := \lim_{x \to b_2^-} \tau_x > \tfrac{1}{2} \qquad \text{if} \quad b_2 < \infty,$$

so that $F(\tau)$ is linear on $[\frac{1}{2}, \hat{\tau}]$.

Proof. (i) Let $\tau > \frac{1}{2}$ be such that $x = s_\tau$ By (4.13), we have the following representation for the corresponding $\mu^*(\tau)$:

$$\mu_2^* \quad -(2\tau - 1)\nu_2 + \gamma F(\tau)\sigma_2 \tag{8.12}$$

Because E_1^*, E_2^* are segments, it is well known that ν_2 and σ_2 have the forms $\alpha(t)[(x-t)(t-a_2)]^{-1/2}\,dt$ and $\beta(t)[(x-t)(t-a_2)]^{-1/2}\,dt$, respectively, where α, β are positive and continuous on $[a_2, x]$. Since $\mu_2^* \geq 0$, we have in particular (see (8.12))

$$\frac{2\tau \quad 1}{\gamma F(\tau)} \leq \frac{\beta(x)}{\alpha(x)} = \lim_{t \to x-0} \frac{d\sigma_2(t)}{d\nu_2(t)}$$

On the other hand, $U^{\mu^*}(t) < -F_2$ for $t > x$ while $U^{\mu^*}(x) = -F_2$. Applying (4.8) of Theorem 4.1, we thus obtain, for $t > x$,

$$0 > U^{\mu^*}(t) - U^{\mu^*}(x) = (2\tau - 1)[U^{\nu}(t) - U^{\nu}(x)] + \gamma F(\tau)[U^{\sigma}(t) - U^{\sigma}(x)].$$

For t outside $\operatorname{supp}(\nu)$ we have

$$U^{\nu}(t) < \log \frac{1}{\operatorname{cap}([a_1, b_1] \cup [a_2, x])} = U^{\nu}(x),$$

and so we deduce that

$$\frac{2\tau - 1}{\gamma F(\tau)} > -\frac{U^{\sigma}(t) - U^{\sigma}(x)}{U^{\nu}(t) - U^{\nu}(x)}, \qquad t > x. \tag{8.14}$$

Now,

$$U^{\sigma}(t) - U^{\sigma}(x) = \int \log \left| \frac{x - y}{t - y} \right| d\sigma(y),$$

and a similar representation holds for $U^{\nu}(t) - U^{\nu}(x)$. As $t \to x + 0$, the integrand approaches 0 uniformly for $a_1 \le y \le x - \varepsilon$. In view of the special form of σ_2, ν_2, we easily deduce that

$$\lim_{t \to x+0} \left\{ \frac{U^{\sigma}(t) - U^{\sigma}(x)}{U^{\nu}(t) - U^{\nu}(x)} \right\} = \lim_{t \to x+0} \left\{ + \frac{U^{\sigma_2}(t) - U^{\sigma_2}(x)}{U^{\nu_2}(t) - U^{\nu_2}(x)} \right\} \tag{8.15}$$

$$\lim_{t \to x-0} \frac{d\sigma_2(t)}{d\nu_2(t)}$$

The conclusion of Theorem 8.2(i) now follows from (8.13), (8.14), and (8.15).

(ii) We have shown that the correspondence $x \to \tau_x$ is one-to-one on (a_2, b_2). Since the inverse function $\tau \to s_\tau$ is continuous on $[\frac{1}{2}, 1)$, we deduce that the function $x \to \tau_x$ is strictly decreasing. Thus, $\hat{\tau} := \lim_{x \to b_2^-} \tau_x$ must satisfy $s_{\hat{\tau}} = b_2$.

If we had $\hat{\tau} = \frac{1}{2}$, we would obtain (see (8.14)) that $U^{\sigma}(t) - U^{\sigma}(b_2) < 0$ for $t > b_2$. However, this contradicts (4.2′), and so $\hat{\tau} > \frac{1}{2}$. Finally, since $\operatorname{supp} \mu^*(\hat{\tau}) = E_1 \cup E_2 = \operatorname{supp} \mu^*(\frac{1}{2})$, we obtain (by Theorem 3.2) that F is linear on $[\frac{1}{2}, \hat{\tau}]$. ■

Next we show how to find the limit in (8.11). By linear transformation, we transfer $[a_1, b_1]$, $[a_2, x]$ into $[-1, \alpha]$, $[\beta, 1]$, $\alpha < \beta$, where

$$\alpha = \frac{2(b_1 - a_1)}{x - a_1} - 1, \qquad \beta = \frac{2(a_2 - a_1)}{x - a_1} - 1$$

Thus, our task is to find the limit

$$\lim_{t \to 1+} \left\{ \frac{(d/dt) \log |f(t)|}{(d/dt) g(t)} \right\}, \tag{8.17}$$

g is the Green function for $\mathbf{C} \backslash ([-1, \alpha] \cup [\beta, 1])$, and $\zeta = f(z)$ is a conformal mapping of this domain onto an annulus, which can be assumed to have a form

$$e^{-1/\gamma} < |\zeta| < 1,$$

where $|\zeta| = 1$ corresponds to $[-1, \alpha]$ in the z-plane. These functions are well known, and were described by N. Akhiezer. A convenient reference is a recent paper by B. Fischer [F].

First we introduce some auxiliary quantities. Define k, $0 < k < 1$, by

(8.19) $$k^2 := \frac{2(\beta - \alpha)}{(1-\alpha)(1+\beta)},$$

and let K, K' denote the complete elliptic integrals (of the first kind) for moduli k and $k' := (1 - k^2)^{1/2}$, respectively. Next calculate

$$\gamma = \frac{K'}{\pi K}$$

(this is the same γ that appears in (8.18) and (8.5)). Next define $0 < \rho < K$ by

$$\operatorname{sn}(\rho; k) = \left(\frac{1-\alpha}{2}\right)^{1/2}$$

that is,

(8.21′) $$\rho = \int_0^{((1-\alpha)/2)^{1/2}} \frac{dt}{(1-t^2)^{1/2}(1-k^2t^2)^{1/2}}.$$

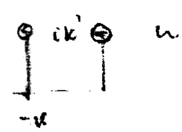
Now the conformal mapping $\zeta = f(z)$ of $\mathbf{C}\backslash([-1, \alpha] \cup [\beta, 1])$ onto the annulus (8.18) is given by the pair of equations

$$z = \varphi(u) = \frac{\operatorname{sn}^2 u \operatorname{cn}^2 \rho + \operatorname{cn}^2 u \operatorname{sn}^2 \rho}{\operatorname{sn}^2 u - \operatorname{sn}^2 \rho}$$

$$= \alpha + \frac{1 - \alpha^2}{2 \operatorname{sn}^2 u + \alpha - 1},$$

$$u = \frac{K'}{\pi} \log \zeta \qquad (\log 1 = 0).$$

It is important to note that φ maps $[-1, \alpha]$ and $[\beta, 1]$ in the z-plane onto the vertical segments



(8.24) $$[0, iK'] \quad \text{and} \quad [-K, -K + iK'],$$

respectively. Also,

(8.25) $$z \to 1 + 0 \quad \Leftrightarrow \quad u \to -K + 0 \quad \Leftrightarrow \quad \zeta \to e^{-\pi K/K'} + 0.$$

Thus, we obtain

$$\frac{d}{dt}U^{\sigma}(t)=\frac{d}{dt}\log|f(t)|=\left(\frac{\pi}{K'}+o(1)\right)\frac{1}{\varphi'(u)}\qquad\text{as}\quad t\to 1+0.$$

Next, the Green function for $\mathbf{C}\backslash([-1,\alpha]\cup[\beta,1])$ is given by

$$g(z)=\log\left|\frac{H(u-\rho)}{H(u+\rho)}\right|,\qquad z=\varphi(u), \tag{8.27}$$

where

$$H(u)=2q^{1/4}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}\sin\left(\frac{2n+1}{2K}\pi u\right),$$

$$q:=e^{-\pi K'/K}.$$

Now (8.27), (8.28), (8.25) yield

$$\frac{d}{dt}U^{\nu}(t)=-\frac{d}{dt}g(t)=\left[2\frac{H'(-K+\rho)}{H(-K+\rho)}+o(1)\right]\frac{1}{\varphi'(u)}\qquad\text{as}\quad t\to 1+0.$$

Combining this with (8.26) we can rewrite (8.11) in the form (see also (8.28))

$$\frac{2\tau-1}{(1-\tau)\|v_1\|+\tau\|v_2\|}$$
$$=\frac{\pi}{2K'}\left(\sum_{n=0}^{\infty}q^{n(n+1)}\cos\frac{2n+1}{2K}\pi\rho\right)\left(\sum_{n=0}^{\infty}q^{n(n+1)}\frac{2n+1}{2K}\pi\sin\frac{2n+1}{2K}\pi\rho\right).$$

Finally, we obtain from (8.27) that

$$\|v_1\|=1-\frac{\rho}{K},\qquad \|v_2\|=\frac{\rho}{K}.$$

Now formula (8.30) enables us to calculate τ, and then to find the corresponding $F(\tau)$, by (8.5).

Thus, on scanning the values of x in (a_2,b_2) and determining the corresponding τ, $F(\tau)$, we can numerically find the maximal value of F. As we mentioned above, this value is very close to $F(\tau_0)$, where τ_0 corresponds to $x=a_2+b_1-a_1$ (see (8.9)). Thus we suggest choosing this particular x as a starting point.

Now choose some rational $\lambda=p/q$, such that the corresponding $\tau=p/(p+q)$ is close to the value that maximizes F. Let x denote the corresponding s_τ and let α, β be defined by (8.16). Since our problem is invariant under linear transformation, it suffices to find the Fejér–Walsh points for $[-1,\alpha]\cup[\beta,1]$. To apply the method of Section 7, we need to know the complex Green function for $D_\infty([-1,\alpha]\cup[\beta,1])$. This is given by

$$G(z)=\frac{H(u+\rho)}{H(u-\rho)},\qquad z=\varphi(u),$$

where H, φ, and ρ were defined above. As z moves along $[-1,\alpha]$, from -1 to α, u moves along the vertical line, from 0 to iK'. Thus, the argument of $G(z)$ (we choose its branch that takes values in $(-\pi,0)$) is growing from $-\pi$ to $-\pi\rho/K$. (This explains, by the way, the relation $\|v_1\|=1-\rho/K$.) The argument of f (the conformal mapping given by (8.22), (8.23)) is growing from 0 to π. Thus, define $t_{jm}^{(1)}\in(0,K')$, $j=0,\ldots,m-1$, as the solution of the equation

(8.33)
$$(2\tau-1)\left(1+\frac{1}{\pi}\arg\frac{H(it+\rho)}{H(it-\rho)}\right)+\left[(1-\tau)\left(1-\frac{\rho}{K}\right)+\tau\frac{\rho}{K}\right]\frac{t}{K'}$$
$$=\left(\frac{1}{2m}+\frac{j}{m}\right)\tau,\qquad j=0,\ldots,m-1,$$

and set

(8.34)
$$z_{jm}:=\varphi(it_{jm}^{(1)}),\qquad j=0,\ldots,m-1$$

Similarly, define $t_{jn}^{(2)}\in(0,K')$, $j=0,\ldots,n-1$, as a solution of the equation

(8.35)
$$(2\tau-1)\frac{1}{\pi}\arg\frac{H(-K+it+\rho)}{H(-K+it-\rho)}+\left[(1-\tau)\left(1-\frac{\rho}{K}\right)+\tau\frac{\rho}{K}\right]\frac{t}{K'}$$
$$=\left(\frac{1}{2n}+\frac{j}{n}\right)(1-\tau),\qquad j=0,\ldots,n-1$$

and set

(8.36)
$$w_{jn}:=\varphi(-K+it_{jn}^{(2)}),\qquad j=0,\ \ .,n-1$$

Table 8.1. The results of the computations of Example 8.3.

$\lambda=m/n$	$m+n$	Fejér–Walsh	Leja–Bagby
	20	1.5683	1.4189
	60	1.6145	1.5590
	120	1.6260	1.5910
	∞	1.6376	1.6376
2	18	1.8440	1.6114
	60	1.8967	1.7668
	120	1.9083	1.8503
	∞	1.9203	1.9203
4	20	1.9411	1.4774
	60	1.9858	1.8149
	120	1.9969	1.9395
	∞	2.0090	2.0090

Note that arg in (8.33), (8.35) is chosen in $(-\pi, 0)$. As was explained in Section 7, the points z_{jm}, w_{jn} (with $m/n = \lambda$) are Fejér–Walsh points for $[-1, \alpha] \cup [\beta, 1]$. Using a linear transformation, we find the corresponding points on $[a_1, b_1] \cup [a_2, x]$.

Example 8.3. Let $E_1 = [-1, -\frac{5}{6}]$, $E_2 = [-\frac{2}{3}, 1]$. Applying our method we first find the maximum of F. This turns out to be equal to 2.0095 and it corresponds to $\lambda = 4.28$. Next, for $\lambda = 4, 2, 1$, we compute the quantities $\sup_{E_1}|r_{mn}|/\inf_{E_2}|r_{mn}|$ (for various m, n, $m/n = \lambda$) using the Fejér–Walsh method and also, using the Leja–Bagby method. The results of the computations are given in Table 8.1.

9. Applications

a. Approximation of a Signum-Type Function

Let the closed sets E_1, E_2 be as in Section 3. Given a function f on $E_1 \cup E_2$, set

$$\rho_{m,n}(f) = \rho_{m,n}(f; E_1, E_2) := \inf_{r \in \mathbf{R}_{m,n}} \|f - r\|_{E_1 \cup E_2},$$

where $\|\cdot\|$ denotes the sup-norm. We consider the characteristic function

(9.1)
$$\chi(z) := \begin{cases} 0, & z \in E_1, \\ 1, & z \in E_2. \end{cases}$$

For $m \leq n$, the correspondence $r \to r/(1 + r)$ is a bijection of $\mathbf{R}_{m,n}$ onto itself. With the help of this observation, it can be easily deduced (see [Go]), that, for $Z_{m,n} = Z_{m,n}(E_1, E_2)$,

(9.2)
$$\frac{Z_{m,n}^{1/2}}{1 + Z_{m,n}^{1/2}} \leq \rho_{m,n}(\chi) \leq \frac{Z_{m,n}^{1/2}}{1 - Z_{m,n}^{1/2}}, \qquad m \leq n.$$

Thus we obtain from Theorem 6.1 the following.

Theorem 9.1. *Let $0 < \lambda \leq 1$ and $\{(m_k, n_k)\}_{k=1}^{\infty}$ be a sequence of nonnegative integers that satisfy*

$$\frac{m_k}{n_k} \to \lambda, \qquad m_k + n_k \to \infty, \qquad m_k \leq n_k.$$

Then

(9.4)
$$\lim_{k \to \infty} [\rho_{m_k, n_k}(\chi)]^{1/(m_k + n_k)} = \exp(-\tfrac{1}{2}F(\tau)),$$

where $\tau = \lambda/(\lambda + 1)$.

If E_1 is regular, then (9.3) also holds for $\lambda = 0$ (see the Remark at end of Section 6). We further note that from the symmetry property (1.2) it follows that

(9.5)
$$\lim_{k \to \infty} [\rho_{m_k, n_k}(1 - \chi)]^{1/(m_k + n_k)} = \exp(-\tfrac{1}{2}F(1 - \tau))$$

for any ray sequence (m_k, n_k) satisfying (9.3).

At present, it is not known how (9.4), (9.5) can be generalized to treat functions of the form $\chi + c(1-\chi)$, where c is a constant different from 0 and 1 and $0 \le \lambda \le 1$. Moreover, for $\lambda > 1$, even the cases $c = 0$, $c = 1$ are not settled yet.

We further remark that if E_1, E_2 are intervals on the real axis and the length of E_1 is greater than that of E_2, then $F(\tau)$ attains its maximum (over $0 \le \tau \le 1$) at some $\tau_0 < \frac{1}{2}$. It then follows from (9.4) that the corresponding $\lambda_0 := \tau_0/(1-\tau_0)$ is the optimal one in the sense that the limit in (9.4) is the smallest possible, at least for all $\lambda \in [0, 1]$.

b. Application to the ADI Method

For a more comprehensive description of the alternating direction iteration (ADI) method, its applications, and its relation to the Zolotarev problem, see, for example, [W], [S1]–[S3], and [LR]. Here we give only a brief discussion.

Consider Sylvester's equation

$$AX - XB = C,$$

where A, B, and C are given real matrices of orders $m \times m$, $n \times n$, and $m \times n$, respectively, and the matrix X (of order $m \times n$) is to be determined. Assume that the spectra $\sigma(A)$, $\sigma(B)$ of A, B are disjoint. Then (9.6) has a unique solution for any C.

Assuming $t \notin \sigma(A)$, we may rewrite (9.6) as

$$X = (A - tI)^{-1}C + (A - tI)^{-1}X(B - tI).$$

Similarly, if $s \notin \sigma(B)$, we rewrite (9.6) as

$$X = -C(B - sI)^{-1} + (A - sI)X(B - sI)^{-1}.$$

These representations for X suggest the following two types of iterations:

$$X_k = (A - t_kI)^{-1}C + (A - t_kI)^{-1}X_{k-1}(B - t_kI), \qquad t_k \notin \sigma(A),$$

$$X_k = -C(B - s_kI)^{-1} + (A - s_kI)^{-1}X_{k-1}(B - s_kI)^{-1}, \qquad s_k \notin \sigma(B).$$

In the generalized ADI method, we begin with an initial approximation X_0 to X, perform j iterations of type (9.7′) and then perform l iterations of type (9.8′) (with $X_0 := X_j$). If X_{j+l} denotes the resulting matrix, then we easily obtain that the error $X_{j+l} - X$ is given by

$$X_{j+l} - X = r(A)(X_0 - X)r(B)^{-1},$$

where

$$r(z) := \prod_{k=1}^{l} (z - s_k) \cdot \prod_{k=1}^{j} (z - t_k)^{-1} \tag{9.10}$$

It then follows (under quite general assumptions on A, B) that

$$\|r(A)\| \le c_A \max_{z\in\sigma(A)} |r(z)|, \qquad \|r(B)^{-1}\| \le c_B \max_{z\in\sigma(B)} \left|\frac{1}{r(z)}\right|,$$

where $\|\cdot\|$ now stands for the matrix spectral norm and c_A, c_B are positive constants. Thus

$$\|X_{j+l} - X\| \le \kappa\left\{\frac{\max_{\sigma(A)}|r|}{\min_{\sigma(B)}|r|}\right\} \tag{9.11}$$

for some constant κ.

In most applications we only know that $\sigma(A) \subset E_1$, $\sigma(B) \subset E_2$, for some disjoint compact sets E_1, E_2. Then for the upper bound in (9.11) (with $\sigma(A)$, $\sigma(B)$ replaced by E_1, E_2) we can use the results of this paper to determine numerically asymptotically optimal ratios j/l as well as the corresponding parameters t_k, s_k in the generalized ADI algorithm.

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