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Vol. 9, No. 2, October 1973
Printed in Belgium

On the Zeros of the Error Function for Tchebycheff Approximation on a Disk

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Communicated by T. J. Rivlin

Received June 20, 1971

For the case of best Tchebycheff approximation to a real continuous function on an interval $[a, b]$ by a polynomial of degree n it is well-known that the error function has at least $n + 1$ zeros on $[a, b]$. No analogous result, however, is known for the case of Tchebycheff approximation to an analytic function on a disk of the complex plane. Indeed it is easy to construct a function f analytic on $\Delta\colon |z| \leqslant 1$ for which the constant of best uniform approximation to f on Δ is not in the range $f(\Delta)$. On the other hand, there are known examples [1, 4] of functions f for which the n-th error function has precisely $n + 1$ zeros in Δ.

The purpose of this paper is to exhibit a class of entire functions f, which includes the exponential function, with the property that for each n sufficiently large the polynomial of degree n of best Tchebycheff approximation to f on Δ interpolates to f in precisely $n + 1$ points in Δ (Theorem 5). We also study the distribution of these interpolation points (Theorem 4).

In [3] Motzkin and Walsh also investigate the zeros of the n-th error function, but their results pertain to Tchebycheff approximation on small disks whose radii depend on n.

Throughout this paper $f(z)$ denotes a function analytic on Δ, $p_n(z)$ is the polynomial of degree n of best Tchebycheff approximation to f on Δ, and

$$E_n(f) \equiv \|f - p_n\| \equiv \max_{|z|\leqslant 1} |f(z) - p_n(z)|.$$

We begin with a result on the degree of best approximation.

THEOREM 1. *Let* $f(z) = \sum_0^\infty a_k z^k$, *where* $a_{n+1}/a_n \to 0$ *as* $n \to \infty$. *Then*

$$E_n(f) = |a_{n+1}|[(1 + 4|a_{n+2}/a_{n+1}|^2)^{1/2} + O(|a_{n+3}/a_{n+1}|)] \tag{1}$$

as $n \to \infty$.

* The research of the author was supported, in part, by NSF Grant GF-19275 and by a University of South Florida Release-Time Grant.

Proof. Let $\lambda = e^{i\phi}$, where $\phi = -\arg(a_{n+2}/a_{n+1})$, and set

$$f_1(z) \equiv f(\lambda z), \qquad q(z) \equiv \sum_0^n a_k \lambda^k z^k + a_{n+2}\lambda^{n+2}z^n.$$

Then we have

$$\begin{aligned} E_n(f) = E_n(f_1) &\leqslant \| f_1 - q \| \\ &\leqslant | a_{n+1} | \left\| z + \frac{a_{n+2}}{a_{n+1}} \lambda z^2 - \frac{a_{n+2}}{a_{n+1}} \lambda \right\| + \sum_{n+3}^{\infty} | a_k |. \end{aligned} \tag{2}$$

It is trivial to verify that

$$\| z + \tau z^2 - \tau \| = (1 + 4\tau^2)^{1/2}, \qquad \tau \text{ real},$$

and hence from (2) we obtain

$$E_n(f) \leqslant | a_{n+1} | \left[(1 + 4 | a_{n+2}/a_{n+1} |^2)^{1/2} + | a_{n+3}/a_{n+1} | \sum_{n+3}^{\infty} |a_k/a_{n+3} | \right].$$

This last inequality implies (1).

For comparison we mention the lower estimate

$$| a_{n+1} |(1 + | a_{n+2}/a_{n+1} |^2)^{1/2} \leqslant E_n(f),$$

which follows from the inequalities

$$\sum_{n+1}^{\infty} | a_k |^2 \leqslant \frac{1}{2\pi} \int_{|z|=1} | f(z) - p_n(z)|^2 | dz | \leqslant E_n(f)^2.$$

The above methods give the bounds (compare [2, p. 80])

$$\begin{aligned} \left(1 + \frac{1}{(n+2)^2}\right)^{1/2} &\leqslant (n+1)!\, E_n(e^z) \\ &\leqslant \left(1 + \frac{4}{(n+2)^2}\right)^{1/2} + \frac{n+4}{(n+2)(n+3)^2}, \end{aligned}$$

$$\begin{aligned} \left(1 + \frac{1}{(2n+3)^2(2n+4)^2}\right)^{1/2} &\leqslant (2n+2)!\, E_{2n}(\cos z) \\ &\leqslant \left(1 + \frac{4}{(2n+3)^2(2n+4)^2}\right)^{1/2} \\ &\quad + \left(\frac{56}{55}\right) \frac{(2n+2)!}{(2n+6)!}, \end{aligned}$$

$$\begin{aligned} \left(1 + \frac{1}{(2n+4)^2(2n+5)^2}\right)^{1/2} &\leqslant (2n+3)!\, E_{2n+1}(\sin z) \\ &\leqslant \left(1 + \frac{4}{(2n+4)^2(2n+5)^2}\right)^{1/2} \\ &\quad + \left(\frac{72}{71}\right) \frac{(2n+3)!}{(2n+7)!}. \end{aligned}$$

There are certain analytic functions f for which the polynomials $p_n(z)$ of best Tchebycheff approximation on Δ turn out to be the sections of the Taylor development for f (see [5]). This is not the case for the functions of Theorem 1. Indeed we have the following.

COROLLARY. *Let f be as in Theorem 1. Then for each n sufficiently large the polynomial $s_n(z) \equiv \sum_0^n a_k z^k$ is not the polynomial of degree n of best Tchebycheff approximation to f on Δ.*

Proof. Suppose that for some increasing sequence of integers m we have $s_m(z) \equiv p_m(z)$. Since

$$| a_{m+1} | + | a_{m+2} | - \sum_{m+3}^{\infty} | a_k | \leqslant \| f - s_m \|,$$

there follows from (1)

$$| a_{m+1} | + | a_{m+2} | \leqslant | a_{m+1} | [(1 + 4 | a_{m+2}/a_{m+1} |^2)^{1/2} + A | a_{m+3}/a_{m+1} |]$$

and consequently

$$1 \leqslant 4 | a_{m+2}/a_{m+1} | + A | a_{m+3}/a_{m+2} |,$$

where A is a constant independent of m. But the right-hand member of the last inequality approaches zero as $m \to \infty$, which gives the desired contradiction.

A result on the zeros of the error function will follow from

THEOREM 2. *If f is defined as in Theorem 1, then*

$$\lim_{n\to\infty}((f(z) - p_n(z))/a_{n+1}z^{n+1}) = 1, \tag{3}$$

uniformly for z on each compact set in $| z | > 1$.

Proof. It is easy to show that

$$\lim_{n\to\infty} ((f(z) - s_n(z))/a_{n+1}z^{n+1}) = 1, \tag{4}$$

uniformly for z on each compact set in the plane. Since

$$\frac{1}{2\pi} \int_{|z|=1} | s_n(z) - p_n(z)|^2 | dz | = \frac{1}{2\pi} \int_{|z|=1} |f(z) - p_n(z)|^2 | dz | - \sum_{n+1}^{\infty} | a_k |^2$$
$$\leqslant E_n(f)^2 - | a_{n+1} |^2 - | a_{n+2} |^2,$$

it follows from (1) that

$$\frac{1}{2\pi}\int_{|z|=1}\left|\frac{s_n(z)-p_n(z)}{a_{n+1}z^{n+1}}\right|^2 |dz| \leqslant A_1\left|\frac{a_{n+3}}{a_{n+1}}\right| + A_2\left|\frac{a_{n+2}}{a_{n+1}}\right|^2 \to 0, \quad (5)$$

as $n \to \infty$. But $(s_n(z) - p_n(z))/a_{n+1}z^{n+1}$ is analytic for $|z| \geqslant 1$, even at ∞, and so the last inequality implies that

$$\lim_{n\to\infty} \frac{s_n(z)-p_n(z)}{a_{n+1}z^{n+1}} = 0, \quad (6)$$

uniformly for z on each closed set in $|z| > 1$. From (4) and (6) we deduce (3).

By applying the Argument Principle to (3) one can establish

THEOREM 3. *Let f be as in Theorem 1 and let $\epsilon > 0$. Then for each $n > n_\epsilon$ the error function $f(z) - p_n(z)$ has exactly $n + 1$ zeros in $|z| < 1 + \epsilon$.*

Concerning the distribution of these zeros of the error function we prove

THEOREM 4. *Let f be as in Theorem 1 and for $n > n_1$ let $\alpha_1^{(n)}, \alpha_2^{(n)},..., \alpha_{n+1}^{(n)}$ be the $n + 1$ zeros of $f(z) - p_n(z)$ in the disk $|z| < 2$. Then*

$$\lim_{n\to\infty}\left|\prod_{i=1}^{n+1}(z-\alpha_i^{(n)})\right|^{1/n} = |z|, \quad (7)$$

uniformly for z on each compact set in $|z| > 1$.

The limit (7) implies that for an arbitrary function F analytic on Δ the sequence of polynomials of respective degrees n found by interpolation to F in the points $\alpha_i^{(n)}$ converges maximally [6, Sect. 7.2] to F on Δ.

Proof. Write $f(z) - p_n(z) = \omega_n(z)\, g_n(z)$, where $\omega_n(z) \equiv \prod_{i=1}^{n+1}(z - \alpha_i^{(n)})$, and set $M_n \equiv \max_{|z|\leqslant 1}|\omega_n(z)|$. We shall show that

$$\limsup_{n\to\infty} M_n^{1/n} \leqslant 1, \quad (8)$$

which is equivalent to (7), see [6, Sect. 7.4].

Let $1 < \rho < R$. Since $|\omega_n(z)| \leqslant (R + 2)^{n+1}$ for $|z| = R$, it follows from Theorem 2 that

$$\frac{R}{R+2} \leqslant \liminf_{n\to\infty}\min_{|z|=R}\left|\frac{g_n(z)}{a_{n+1}}\right|^{1/n}.$$

By the "minimum modulus principle" we have for n sufficiently large

$$\min_{|z|=R}|g_n(z)| \leqslant \min_{|z|=\rho}|g_n(z)|,$$

and so

$$\limsup_{n\to\infty} \max_{|z|=\rho} \left| \frac{\omega_n(z)}{z^{n+1}} \right|^{1/n} \liminf_{n\to\infty} \min_{|z|=R} \left| \frac{g_n(z)}{a_{n+1}} \right|^{1/n}$$

$$\leqslant \limsup_{n\to\infty} \max_{|z|=\rho} \left| \frac{\omega_n(z)\, g_n(z)}{a_{n+1} z^{n+1}} \right|^{1/n} = 1.$$

Hence

$$\limsup_{n\to\infty} M_n^{1/n} \leqslant \limsup_{n\to\infty} \max_{|z|=\rho} | \omega_n(z)|^{1/n} \leqslant \rho(R+2)/R.$$

Letting $R \to \infty$ and $\rho \to 1$ in the last inequality we deduce (8). This proves Theorem 4.

We now show that under certain conditions the n-th error function has exactly $n+1$ zeros in Δ.

THEOREM 5. *Let* $f(z) = \sum_0^\infty a_k z^k$, *where* $n^{1/2} a_{n+1}/a_n \to 0$ *as* $n \to \infty$. *Then for each n sufficiently large* $f(z) - p_n(z)$ *has exactly* $n+1$ *zeros in* $|z| < 1$ *and no zeros on* $|z| = 1$.

Proof. It suffices to show that the limit (3) holds uniformly for $|z| = 1$. By the Cauchy–Schwarz inequality and (5) we have for $|z| = 1$

$$\left| \frac{s_n(z) - p_n(z)}{a_{n+1} z^{n+1}} \right|^2 \leqslant \frac{(n+1)}{2\pi} \int_{|z|=1} \left| \frac{s_n(z) - p_n(z)}{a_{n+1} z^{n+1}} \right|^2 | dz |$$

$$\leqslant (n+1) \left[A_1 \left| \frac{a_{n+3}}{a_{n+1}} \right| + A_2 \left| \frac{a_{n+2}}{a_{n+1}} \right|^2 \right].$$

The hypothesis on the coefficients a_k implies that the right-hand member of the last inequality approaches zero as $n \to \infty$. Hence (6) and therefore (3) hold uniformly for $|z| = 1$. This proves Theorem 5.

Using the estimates obtained in this paper one can deduce from Rouché's theorem that *for* $n \geqslant 6$ *the polynomial of degree n of best Tchebycheff approximation to* e^z *on* Δ *interpolates to* e^z *in exactly* $n+1$ *points in* Δ.

Added in proof. In contrast, S. Ja. Al'per has shown [Mathematical Analysis and its Applications (Russian), pp. 3–6. Izdat. Rostov. Univ., Rostov-na-Donu, 1969] that for each n there exists a function $f(z)$ (depending on n) which is analytic in $|z| < 1$ continuous on $|z| \leqslant 1$, and such that $f(z)-p_k(z) \neq 0$, z in Δ, $k \leqslant n$-1.

ACKNOWLEDGMENT

The author wishes to thank Professor J. L. Walsh for many valuable discussions.

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Printed by the St Catherine Press Ltd., Tempelhof 37, Bruges, Belgium.