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The Zeros of Faber Polynomials for an m-Cusped Hypocycloid

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The Faber polynomials for a region of the complex plane are of interest as a basis for polynomial approximation of analytic functions. In this paper we determine the location, density, and asymptotic behavior of the zeros of Faber polynomials associated with the closed region bounded by the m-cusped hypocycloid with parametric equation

$$z = \exp(i\theta) + \frac{1}{(m-1)} \exp(-(m-1)i\theta), \qquad 0 \le \theta < 2\pi, m = 2, 3, 4, \ldots .$$

For $m = 2$, the Faber polynomials are simply the classical Chebyshev polynomials for the segment $[-2, 2]$; thus our results can be viewed as a study of the algebraic and asymptotic properties of generalized Chebyshev polynomials.

1. INTRODUCTION

Let E be any compact set (not a single point) in the complex plane $\mathbf{C}$ whose complement with respect to the extended complex plane $\overline{\mathbf{C}}$ is simply connected. The Riemann mapping theorem asserts that there exists a conformal mapping $w = \Phi(z)$ of $\overline{\mathbf{C}} \setminus E$ onto the exterior of a circle

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$|w| = \rho_E$ in the w-plane. For a unique choice of ρ_E, we insist that

$$\Phi(\infty) = \infty, \qquad \Phi'(\infty) = 1$$

so that, in a neighborhood of infinity,

$$\Phi(z) = z + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots . \tag{1.1}$$

With this normalization, the constant ρ_E is the logarithmic capacity or transfinite diameter of the set E.

The polynomial part of $\Phi(z)^n$, denoted by $F_n(z) = z^n + \cdots$, is called the *Faber polynomial* of degree n generated by the set E. (For a survey of the theory of Faber polynomials see [C2].)

Let

$$\Psi(w) = w + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots \tag{1.2}$$

be the inverse function of $w = \Phi(z)$. Then $\Psi(w)$ maps the domain $|w| > \rho_E$ conformally onto $\overline{\mathbf{C}} \setminus E$ and Faber [F] proved that

$$\frac{\Psi'(w)}{\Psi(w) - z} = \sum_{n=0}^{\infty} \frac{F_n(z)}{w^{n+1}}, \qquad |w| > \rho_E,\ z \in E. \tag{1.3}$$

For the unit disk the Faber polynomial of degree n is z^n and the corresponding Faber series for an analytic function is its Taylor series about the origin.

If $E = [-2, 2]$, then

$$\Phi(z) = \frac{z + \sqrt{z^2 - 4}}{2}$$

with inverse

$$\Psi(w) = w + w^{-1}.$$

For $n \geq 1$, the polynomial part of $\Phi(z)^n$ is the same as the polynomial part of

$$\Phi(z)^n + \Phi(z)^{-n} = w^n + w^{-n}$$

which reduces to $2\cos n\theta$, when $w = e^{i\theta}$. Thus the Faber polynomials are the same as the classical monic Chebyshev polynomials $T_n(x) = 2\cos n(\cos^{-1}(x/2))$ for the segment $[-2, 2]$.

The following properties for $T_n(z)$ are well known:

(i) The $T_n(z)$'s satisfy the 3-term recurrence relation

$$T_{n+2}(z) = zT_{n+1}(z) - T_n(z), \qquad T_0(z) = 2,\, T_1(z) = z.$$

(ii) All zeros of $T_n(z)$ are located on $(-2,2)$ for every $n \geq 1$.

(iii) The zeros of $\{T_n(z)\}_{n=1}^{\infty}$ are dense on $[-2,2]$ and have limiting distribution

$$d\mu(t) = \frac{1}{\pi}\frac{1}{\sqrt{4-t^2}}\,dt, \qquad t \in [-2,2],$$

which is the equilibrium distribution for $E = [-2,2]$.

(iv) For $n \geq 1$, the polynomial $T_n(z)$ satisfies the second-order differential equation

$$(4 - z^2)T_n'' - zT_n' + n^2 T_n = 0.$$

We remark that the segment $E = [-2,2]$ can be considered as a two-cusped hypocycloid with parametric equation

$$z = \exp(i\theta) + \exp(-i\theta), \qquad 0 \leq \theta < 2\pi.$$

As we shall see, the Faber polynomials for the closed region H_m bounded by the m-cusped hypocycloid with parametric equation

$$z = \exp(i\theta) + \frac{1}{(m-1)}\exp(-(m-1)i\theta),$$
$$0 \leq \theta < 2\pi,\ m = 2,3,4,\ldots \qquad (1.4)$$

enjoy certain properties that are similar to Chebyshev polynomials. For example, these polynomials also satisfy a three-term m-th order recurrence formula and their zeros lie on the m line segments joining the cusps to the origin (See Figs. 1–6). In particular, the m cusp points attract zeros of the Faber polynomials, and for $m \geq 3$, the zeros of $F_n(z)$ stay away from the analytic portions of the boundary.

The paper is organized as follows: Section 2 describes some basic algebraic properties of $F_n(z)$. In Sections 3 and 4 we determine the location, density, and distribution of zeros of $F_n(z)$. To illustrate our results we have plotted in Figs. 1–6 the zeros of certain Faber the polynomials associated with m-cusped hypocycloids ($m = 3,4,5,6$).

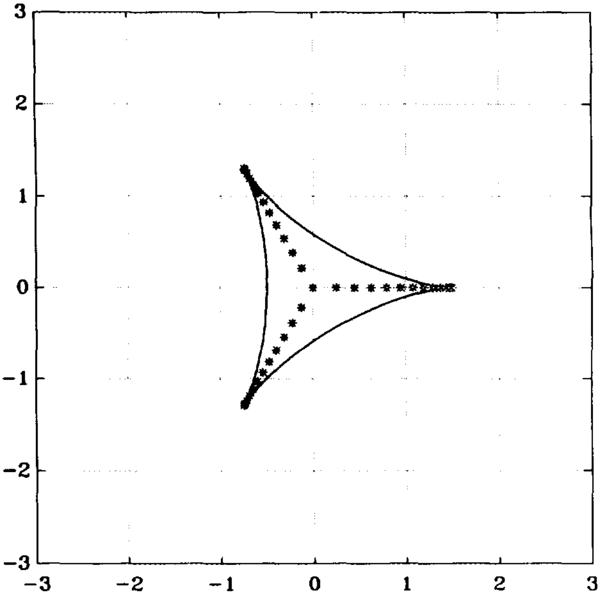

FIG. 1. Zeros of $F_n(z)$ of 3-cusped hypocycloid when $n = 38$.

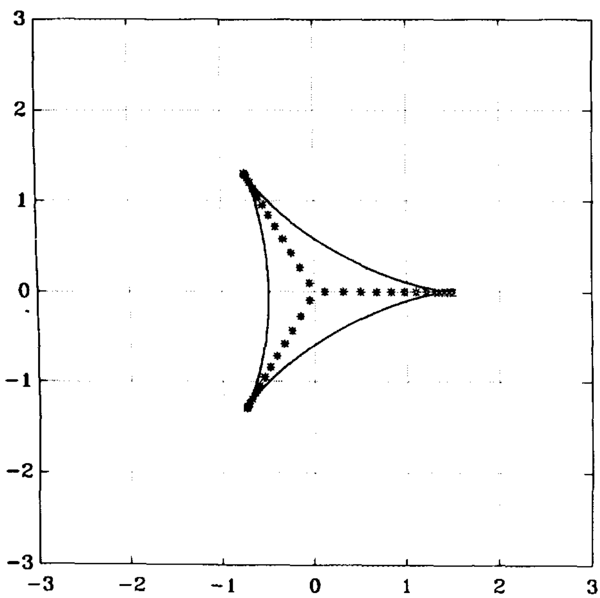

FIG. 2. Zeros of $F_n(z)$ of 3-cusped hypocycloid when $n = 39$.

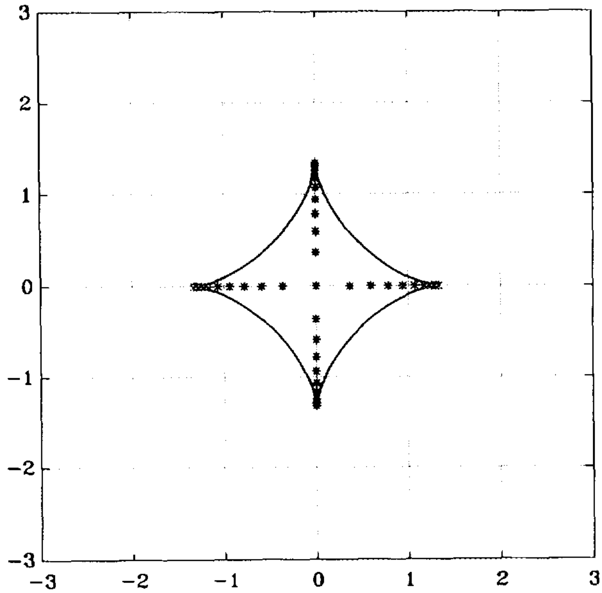

FIG. 3. Zeros of $F_n(z)$ of 4-cusped hypocycloid when $n = 39$.

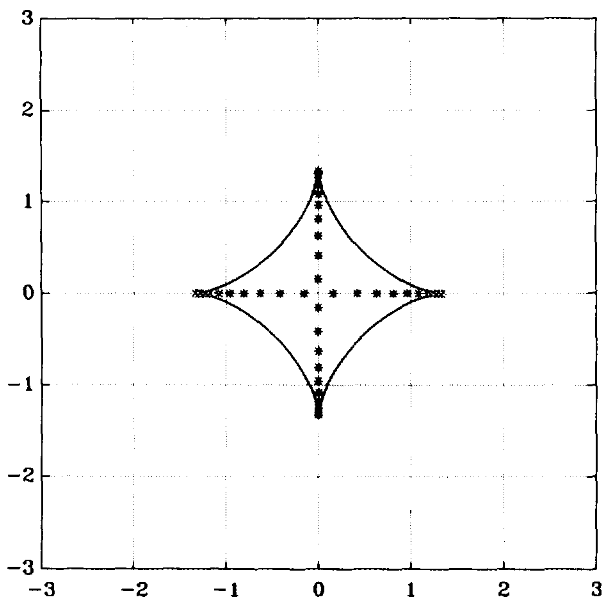

FIG. 4. Zeros of $F_n(z)$ of 4-cusped hypocycloid when $n = 40$.

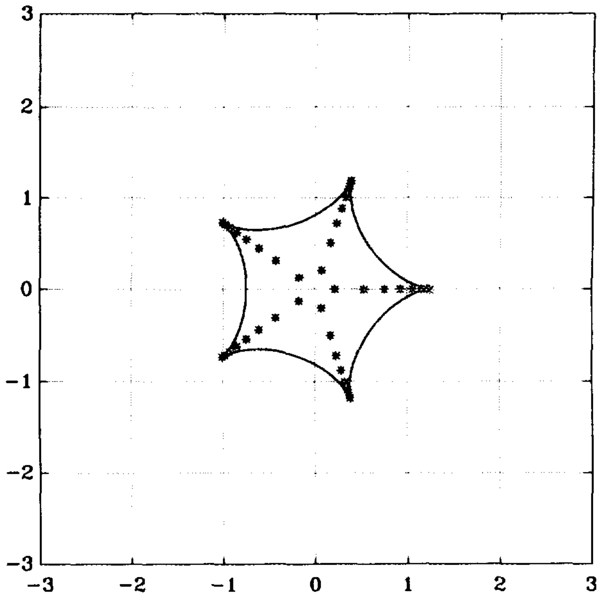

FIG. 5. Zeros of $F_n(z)$ of 5-cusped hypocycloid when $n = 40$.

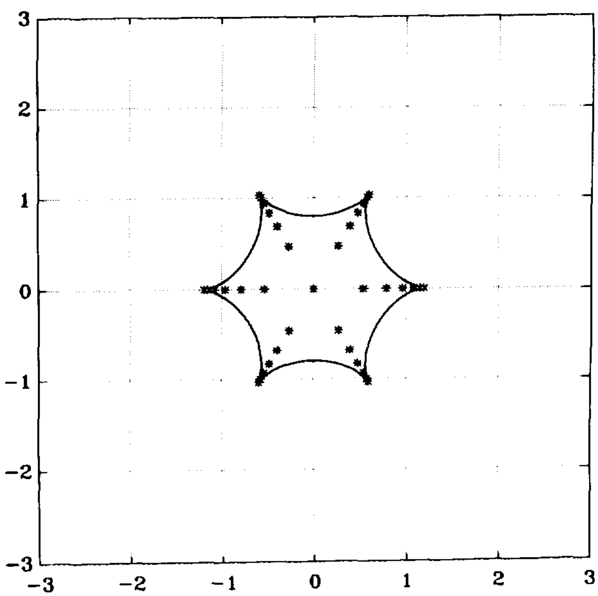

FIG. 6. Zeros of $F_n(z)$ of 6-cusped hypocycloid when $n = 40$.

We remark that in recent years there has been a growing interest in studying the Faber polynomials for specific regions. Ellacott [E] computed the coefficients of some Faber polynomials for the semi-disk $|z| \leq 1$, $\operatorname{Re} z \geq 0$ and for the square $|\operatorname{Re} z| \leq 1$, $|\operatorname{Im} z| \leq 1$. Coleman and Smith [CS] as well as Gatermann, Hoffman, and Opfer [GHO] have studied the coefficients of the Faber polynomials on circular sectors. For starlike domains, Papamichael, Soares, and Stylianopoulos [PSS] describe a simple process for computing approximations to the Faber polynomials.

The asympototic distribution of the zeros of Faber polynomials for general sets E was recently investigated by Kuijlaars and Saff [KS]. In particular, they showed that if E is bounded by a piecewise analytic curve that has a singularity other than an outward cusp, then there is a subsequence of Faber polynomials whose zeros have limiting distribution equal to equilibrium distribution of E (in particular, evey point of the boundary of E attracts zeros of Faber polynomials).

2. Algebraic Properties of $F_n(z)$

Here and throughout the remainder of this paper, $F_n(z) = F_n(z; H_m)$ denotes the Faber polynomial of degree n associated with the closed region H_m bounded by the m-cusped hypocycloid given in (1.4).

In this section we derive some basic algebraic properties of these Faber polynomials and show that $F_n(z)$ satisfies an m-th order linear differential equation.

Proposition 2.1. *Let $\omega = \exp(2\pi i/m)$ denote the primitive m-th root of unity. Then we have*

$$F_n(z\omega^k) = \omega^{kn} F_n(z), \qquad k = 0, 1, \ldots, m-1, n = 0, 1, \ldots .$$

Proof. It is easy to verify that for $m = 2, 3, \ldots,$

$$z = \Psi(w) = w + \frac{1}{(m-1)w^{m-1}}$$

maps $|w| > 1$ conformally onto $\overline{\mathbf{C}} \setminus H_m$. It follows from (1.3) that the $F_n(z)$'s satisfy the equation

$$g(w, z) := \frac{\Psi'(w)}{\Psi(w) - z} = \frac{w^m - 1}{w^{m+1} - zw^m + w/(m-1)} = \sum_{n=0}^{\infty} \frac{F_n(z)}{w^{n+1}}. \tag{2.1}$$

It is easy to see that

$$\omega^k g(w, z\omega^k) = g(\omega^{-k}w, z), \qquad k = 1, 2, \ldots, m-1,$$

and using this identity in (2.1), the result follows immediately. ∎

PROPOSITION 2.2. *For $n = 1, 2, \ldots,$ we have*

$$F_n(0) \neq 0 \Leftrightarrow n = 0 \mod(m). \tag{2.2}$$

Furthermore,

$$F_{mk}(0) = \frac{(-1)^k m}{(m-1)^k}, \qquad k = 1, 2, \ldots. \tag{2.3}$$

Proof. Let $z = 0$ in (2.1). Then

$$\frac{w^m - 1}{w^{m+1} + w/(m-1)} = \sum_{n=0}^{\infty} \frac{F_n(0)}{w^{n+1}}, \qquad |w| > 1.$$

Expanding the left-hand side as a series in w^{-n} and comparing the coefficients of w^{-n}, we get

$$F_0 = 1, \qquad F_n(0) = 0, n \neq 0 \mod(m),$$

$$F_{mk}(0) = \frac{(-1)^k m}{(m-1)^k} \neq 0, \qquad k = 1, 2, \ldots. \quad ∎$$

PROPOSITION 2.3. *The polynomials $F_n(z)$ satisfy the recurrence relation*

$$F_{n+m}(z) = zF_{n+m-1}(z) - \frac{1}{m-1}F_n(z) \tag{2.4}$$

with the initial conditions

$$F_0(z) = m, \qquad F_1(z) = z, \qquad F_2(z) = z^2, \ldots, F_{m-1}(z) = z^{m-1}.$$

Furthermore, for each $n \geq 0$,

$$F_n(z) = \Phi_1(z)^n + \Phi_2(z)^n + \cdots + \Phi_m(z)^n, \tag{2.5}$$

where $\Phi_1(z), \Phi_2(z), \ldots, \Phi_m(z)$ are the roots of the equation

$$w^m - zw^{m-1} + \frac{1}{m-1} = 0. \tag{2.6}$$

Note that it is convenient here to take $F_0(z) \equiv m$ instead of $F_0(z) \equiv 1$. We remark that (2.6) has m distinct roots except when z is a vertex of H_m; that is, $z = \omega^k m/(m-1)$, $k = 0, \dots, m-1$. We also note that the representation (2.5) extends the classical representation of Chebyshev polynomials for $[-2, 2]$ mentioned in the Introduction.

Proof. Since $z = \Phi(z) + 1/(m-1)\Phi(z)^{m-1}$, we have

$$z\Phi(z)^{m+n-1} = \Phi(z)^{n+m} + \frac{\Phi(z)^n}{m-1}$$

and by taking the polynomial parts of the left-hand and right-hand sides and using Proposition 2.2, it is easy to see that the $F_n(z)$'s satisfy the relation (2.4).

The characteristic equation of (2.4) is

$$w^m - zw^{m-1} + \frac{1}{m-1} = 0.$$

Clearly it has m solutions, say $\Phi_1(z), \Phi_2(z), \dots, \Phi_m(z)$, and the functions

$$\Phi_1(z)^n + \Phi_2(z)^n + \cdots + \Phi_m(z)^n$$

satisfy the relation (2.4) with the same initial conditions. ∎

PROPOSITION 2.4. *Let*

$$v_n := F_n\left(\frac{m}{m-1}\right), \qquad n = 0, 1, 2, \dots .$$

Then

$$v_n \geq 1, \qquad \text{for } n = 0, 1, 2, \dots .$$

In order to prove Proposition 2.4, we need the following lemma:

LEMMA 2.1. [C1]. *Let*

$$\Psi(w) = w + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots, \qquad |w| > 1,$$

$$F_n(\Psi(w)) = w^n + \sum_{k=1}^{\infty} \alpha_{n,k} w^{-k}, \qquad |w| > 1,$$

where $\alpha_{n,k}$ are the so-called Grunsky coefficients. Then

$$\alpha_{n,k+1} = \alpha_{n+1,k} - \sum_{j=1}^{k-1} b_j \alpha_{n,k-j} + \sum_{j=1}^{n-1} b_j \alpha_{n-j,k} - b_{k+n}, \qquad (2.7)$$

with initial values $\alpha_{1,k} = b_k$, $\alpha_{j,1} = jb_j$, for $k \geq 1$, $j \geq 1$.

Furthermore, if $b_k \geq 0$ for all $k = 0, 1, \ldots,$ then $\alpha_{n,k} \geq 0$ for all n, k.

Proof of Proposition 2.4. Using (2.4), we see that v_n satisfies the recurrence relation

$$v_{n+m} = \frac{m}{m-1} v_{n+m-1} - \frac{1}{m-1} v_n \qquad (2.8)$$

with initial values

$$v_0 = m, \qquad v_1 = \frac{m}{m-1}, \qquad v_2 = \left(\frac{m}{m-1}\right)^2, \ldots, v_{m-1} = \left(\frac{m}{m-1}\right)^{m-1}.$$

Define

$$r_0 := m - 1, \qquad r_n := \sum_{k=1}^{\infty} \alpha_{n,k}, \quad n = 1, 2, \ldots,$$

where, as in Lemma 2.1, $\alpha_{n,k}$ are the Grunsky coefficients associated with $\Psi(w)$, with initial values

$$\alpha_{1,k} = b_k, \, k = 1, 2, \ldots, \qquad \alpha_{n,1} = nb_n, \, n = 1, 2, \ldots .$$

We observe that the coefficients of mapping function

$$z = \Psi(w) = w + \frac{1}{(m-1)w^{m-1}}$$

are

$$b_0 = b_1 = b_2 = \cdots = b_{m-2} = 0, \qquad b_{m-1} = \frac{1}{m-1}, \, b_n = 0, \, n \geq m.$$

Clearly,

$$r_1 = \sum_{k=1}^{\infty} \alpha_{1,k} = \sum_{k=1}^{\infty} b_k = \frac{1}{m-1}.$$

For all $n > 1$ the series $\sum_{k=1}^{\infty} \alpha_{n,k}$ converges absolutely by Theorem 4.4 in

[C2]. Taking the sum of both sides of (2.7) for $k = 1, 2, \ldots$ yields

$$r_n - nb_n = r_{n+1} - \left(\sum_{j=1}^{n-1} b_j r_n + \sum_{j=n}^{\infty} b_j r_n \right) + \sum_{j=1}^{n-1} b_j r_{n-j} - \left(\sum_{j=n}^{\infty} b_j - b_n \right)$$

or

$$r_{n+1} - r_n = \sum_{j=1}^{n-1} b_j (r_n - r_{n-j}) + \sum_{j=n}^{\infty} b_j (r_n + 1) - (n+1) b_n. \quad (2.9)$$

Replacing n $(n \geq 1)$ by $n + m - 1$ in (2.9) gives

$$r_{n+m} = \frac{m}{m-1} r_{n+m-1} - \frac{1}{m-1} r_n, \qquad n = 1, 2, \ldots . \quad (2.10)$$

For $n = 0$, (2.10) is valid since we defined $r_0 = m - 1$. We verify that

$$r_k = \frac{m^k - (m-1)^k}{(m-1)^k}, \qquad \text{for } 1 \leq k \leq m - 1 \quad (2.11)$$

by induction on k.

For $k = 1$, $r_1 = 1/(m-1) = (m - (m-1))/(m-1)$.

For $k = 2$, $r_2 = (2m-1)/(m-1)^2 = (m^2 - (m-1)^2)/(m-1)^2$, from (2.9).

Assume that (2.11) holds for $1 \leq k \leq l \leq m - 2$. We show that it holds for $k = l + 1$. Substituting n for l in (2.9) and using our assumption yields

$$\begin{aligned} r_{l+1} &= r_l + b_{m-1}(r_l + 1) \\ &= \frac{m^l - (m-1)^l}{(m-1)^l} + \frac{1}{m-1} \left(\frac{m^l - (m-1)^l}{(m-1)^l} + 1 \right) \\ &= \frac{m^{l+1} - (m-1)^{l+1}}{(m-1)^{l+1}}. \end{aligned}$$

We now observe that

$$r_0 + 1 = m, \qquad r_k + 1 = \left(\frac{m}{m-1} \right)^k, \text{ for } 1 \leq k \leq m - 1,$$

$$r_{n+m} + 1 = \frac{m}{m-1}(r_{n+m-1} + 1) - \frac{1}{m-1}(r_n + 1), \qquad n = 0, 1 \ldots .$$

Comparing this with (2.8) gives

$$v_n = 1 + r_n, \qquad \text{for } n = 0, 1, 2, \dots .$$

Since $b_k \geq 0$ for all $k = 0, 1, \dots$, it follows from Lemma 2.1 that for all n, k

$$\alpha_{n,k} \geq 0.$$

Hence $r_n \geq 0$, and thereby $v_n \geq 1$ for $n = 0, 1, \dots$. ∎

Remark. For $m = 2$, 3, and 4, the values of $F_n(z)$ for each n at the vertex $z = m/(m-1)$ can be easily determined from (2.8). When $m = 2$, then $F_n(2) = 2$, for all $n = 0, 1, 2, \dots$. When $m = 3$, we get $F_n(3/2) = 2 + (-1/2)^n$, for all $n = 0, 1, 2, \dots$, and for $m = 4$, we obtain

$$F_n\left(\frac{4}{3}\right) = 2 + \frac{1}{3^{n/2}}(\lambda^n + \bar{\lambda}^n),$$

where $\lambda := (-1 + \sqrt{2}i)/\sqrt{3}$.

PROPOSITION 2.5. *The Faber polynomial $F_n(z)$ of degree $n \geq 1$ for H_m satisfies the following m-th order linear differential equation,*

$$\left[\frac{1}{m-1}D^m + \frac{1}{m^m}(n - zD)\prod_{k=0}^{m-2}(n + mk + (m-1)zD)\right]F_n = 0, \tag{2.12}$$

where $D := d/dz$.

Proof. Setting $y = e^{\pi i/m}(m-1)^{1/m}w$ and $x = -e^{\pi i/m}(m-1)^{1/m}z$, Eq. (2.6) becomes

$$y^m + xy^{m-1} - 1 = 0. \tag{2.13}$$

By using the Mellin transform

$$Y(\zeta) := \int_0^\infty y(x)^n x^{\zeta - 1}\,dx,$$

it is shown in [H, p. 85] that for $|x|$ sufficiently small, $y(x)^n$ satisfies the equation

$$\left(-\frac{d}{dx}\right)^m y(x)^n = p\left(-x\frac{d}{dx} - m\right)y(x)^n, \tag{2.14}$$

where

$$p(\zeta) := \left(\frac{n+m}{m} + \frac{\zeta}{m}\right)\prod_{k=0}^{m-2}\left(\frac{n-(m-1)\zeta}{m} - (m-1) + k\right)$$

is a polynomial of degree m. From the relations between y and w, as well as x and z, Eq. (2.12) is an immediate consequence of (2.5) and (2.14). ∎

3. Location of Zeros of $F_n(z)$

In this section we prove that the zeros of $F_n(z) = F_n(z; H_m)$ lie on m rays emanating from the origin.

THEOREM 3.1. *For each $n \geq 1$, all zeros of $F_n(z)$ are located on the set*

$$S_m = \left\{x\omega^k; 0 \leq x < \frac{m}{m-1}, k = 0,1,2,\dots,m-1, m \geq 2\right\}, \quad (3.1)$$

where $\omega := e^{2\pi i/m}$.

Our proof of Theorem 3.1 was inspired by a method due to Uchimura [U]. Some careful modifications of this method and some results from Section 2 are needed to prove the theorem.

Proof. For $z = 0$, we know from Proposition 2.2 that for $n = 1, 2, \dots,$

$$F_n(0) \neq 0 \Leftrightarrow n = 0 \mod(m).$$

For $z \neq 0$, we consider $G_n(z) := F_n(z)/z^n$. It follows from (2.4) that

$$G_0(z) = m, \qquad G_1(z) = 1, \qquad G_2(z) = 1, \dots, G_{m-1}(z) = 1,$$

$$G_{n+m}(z) = G_{n+m-1}(z) - \frac{1}{(m-1)z^m} G_n(z).$$

We now define $U_n(x)$ by the recurrence relation

$$U_{n+m}(x) = U_{n+m-1}(x) - \left(x + \frac{(m-1)^{(m-1)}}{m^m}\right)U_n(x), \quad (3.2)$$

with the initial conditions

$$U_0(x) = m, \qquad U_1(x) = 1, \qquad U_2(x) = 1, \dots, U_{m-1}(x) = 1$$

and with

$$x = \frac{(m-1)^{(m-1)}}{m^m}\left(\frac{1}{z^m}\left(\frac{m}{m-1}\right)^m - 1\right). \tag{3.3}$$

Clearly,

$$x = 0 \Leftrightarrow z^m = \left(\frac{m}{m-1}\right)^m,$$
$$x \to \infty \Leftrightarrow z \to 0,$$
$$x \in \mathbf{R}^+ \Leftrightarrow z \in S_m \setminus \{0\}.$$

We show by induction that every zero of $U_n(x)$ is a positive real number.
We first prove that $U_n(0) > 0$ for any $n \geq 0$. Notice that

$$F_n(z) = z^n G_n(z) = z^n U_n(x). \tag{3.4}$$

By Propositions 2.1 and 2.4 we obtain for $n = 0, 1, \ldots,$

$$U_n(0) = \frac{F_n(\omega^k m/(m-1))}{(\omega^k m/(m-1))^n} = \frac{F_n(m/(m-1))}{(m/(m-1))^n} > 0. \tag{3.5}$$

Next we observe from (3.2) that the degree of $U_j(x)$ is 0 for $0 \leq j \leq m-1$, the degree of $U_{m+j}(x)$ is 1 for $0 \leq j \leq m-1$, and, in general, the degree of $U_{nm+j}(x)$ is n for $0 \leq j \leq m-1$. Also, the signs of the leading coefficients of $U_{nm+j}(x)$ $(0 \leq j \leq m-1)$ are $(-1)^n$. Furthermore, $U_{nm+j}(x)$ satisfies the recurrence relation

$$U_{nm+j}(x) = U_{nm+j-1}(x) - aU_{(n-1)m+j}(x),$$

where

$$a := \left(x + \frac{(m-1)^{(m-1)}}{m^m}\right). \tag{3.6}$$

Let $x_{nm+j,i}$ $(0 \leq j \leq m-1,\ 1 \leq i \leq n)$ be the zeros $U_{nm+j}(x)$. We now proceed to prove that every zero of $U_{nm+j}(x)$ is a positive real number by induction on n and j. We divide our induction into several steps.

Step 1. For $n = 1$ and $0 \leq j \leq m-1$, we have

$$U_{m+j}(x) = U_{m+j-1}(x) - aU_j(x),$$

and so $U_{m+j}(x) = 1 - a(m+j)$, and

$$x_{m+j,1} = \frac{1}{m+j} - \frac{(m-1)^{(m-1)}}{m^m}, \qquad 0 \le j \le m-1.$$

Thus it suffices to show that

$$\frac{1}{m+m-1} - \frac{(m-1)^{(m-1)}}{m^m} > 0,$$

i.e.,

$$\left(\frac{m}{m-1}\right)^m > \frac{2m-1}{m-1}. \tag{3.7}$$

Using an elementary Bernoulli inequality,

$$b^m > 1 + m(b-1), \qquad b > 1, m > 1,$$

for $b = m/(m-1)$ we get (3.7). Furthermore, we note that

$$0 < x_{m+m-1,1} < x_{m+m-2,1} < \cdots < x_{m,1}.$$

Step 2. For $n = 2$ and $0 \le j \le m-1$, we have

$$U_{2m+j}(x) = U_{2m+j-1}(x) - aU_{m+j}(x).$$

(i) For $j = 0$, we have

$$U_{2m}(x) = U_{2m-1}(x) - aU_m(x).$$

Recall that $U_{2m}(0) > 0$, and note that

$$U_{2m}(x_{2m-1,1}) = -aU_m(x_{2m-1,1}) < 0,$$
$$U_{2m}(x_{m,1}) = U_{2m-1}(x_{m,1}) < 0,$$

because $x_{2m-1,1} < x_{m,1}$ from Step 1. Since the sign of the leading coefficient of $U_{2m}(x)$ is positive, there exists a zero of U_{2m} between 0 and $x_{2m-1,1}$ and a zero between $x_{m,1}$ and ∞. More precisely we have

$$0 < x_{2m,1} < x_{2m,2},$$

where

$$x_{2m,1} < x_{2m-1,1}, \qquad x_{m,1} < x_{2m,2}.$$

(ii) For $j = 1$, we have

$$U_{2m+1}(x) = U_{2m}(x) - aU_{m+1}(x).$$

Note that $U_{2m+1}(0) > 0$ and

$$U_{2m+1}(x_{2m,1}) = -aU_{m+1}(x_{2m,1}) < 0,$$

because $x_{2m,1} < x_{m+1,1}$ $(x_{2m,1} < x_{2m-1,1} < x_{m+1,1}$ by (i) and Step 1). Furthermore,

$$U_{2m+1}(x_{2m,2}) = -aU_{m+1}(x_{2m,2}) > 0,$$

because $x_{2m,2} > x_{m+1,1}$ $(x_{2m,2} > x_{m,1} > x_{m+1,1}$ by (i) and Step 1). Also,

$$U_{2m+1}(x_{m+1,1}) = U_{2m}(x_{m+1,1}) < 0,$$

because $x_{2m,1} < x_{m+1,1} < x_{2m,2}$. Since the sign of the leading coefficient of U_{2m+1} is positive, we see that it has a zero between 0 and $x_{2m,1}$ and a zero between $x_{2m,1}$ and $x_{2m,2}$. Consequently, we have

$$0 < x_{2m+1,1} < x_{2m+1,2},$$

and

$$x_{2m+1,i} < x_{2m,i}, \text{ for } i = 1,2; \qquad x_{m+1,i-1} < x_{2m+1,i} \text{ for } i = 2.$$

(ii) Fix s $(1 \leq s \leq m-2)$ and assume that for all $1 \leq j \leq s$ we have

$$0 < x_{2m+j,1} < x_{2m+j,2},$$
$$0 < x_{2m+j,i} < x_{2m+j-1,i}, \qquad i = 1,2,$$
$$x_{m+j,i-1} < x_{2m+j,i}, \qquad i = 2.$$

We show that these properties hold for the zeros of $U_{2m+s+1}(x)$.

Now

$$U_{2m+s+1}(x) = U_{2m+s}(x) - aU_{m+s+1}(x)$$

and so

$$U_{2m+s+1}(x_{2m+s,i}) = -aU_{m+s+1}(x_{2m+s,i}) < 0, \qquad \text{for } i = 1,$$

because $x_{2m+s,1} < x_{m+s+1,1}$ $(x_{2m+s,1} < x_{2m+s-1,1} < x_{2m,1} < x_{m+s+1,1}$ by the assumption and Step 1). Also

$$U_{2m+s+1}(x_{2m+s,i}) = -aU_{m+s+1}(x_{2m+s,i}) > 0, \qquad \text{for } i = 2,$$

because $x_{2m+s,2} > x_{m+s,1} > x_{m+s+1,1}$ from our assumption. Since $x_{2m+s,1} < x_{m+s+1,1} < x_{2m+s,2}$, we have

$$U_{2m+s+1}(x_{m+s+1,i}) = U_{2m+s}(x_{m+s+1;i}) < 0, \qquad \text{for } i = 1.$$

Note that the sign of the leading coefficient of U_{2m+s+1} is positive and $U_{2m+s+1}(0) > 0$. Thus $U_{2m+s+1}(x)$ has a zero between 0 and $x_{2m+s,1}$, and a zero between $x_{2m+s,1}$ and $x_{2m+s,2}$ such that

$$0 < x_{2m+s+1,1} < x_{2m+s+1,2},$$

and

$$x_{2m+s+1,i} < x_{2m+s,i} \qquad \text{for } i = 1,2,$$
$$x_{m+s+1,i-1} < x_{2m+s+1,i}, \qquad \text{for } i = 2.$$

We have therefore proved that the zeros of $U_{2m+j}(x)$ $(j = 0,1,2,\ldots, m-1)$ have the properties

$$0 < x_{2m+j,1} < x_{2m+j,2},$$
$$0 < x_{2m+m-1,i} < x_{2m+m-2,i} < \cdots < x_{2m,i}, \qquad \text{for } i = 1,2,$$
$$x_{2m,1} < x_{2m-1,1}, \qquad x_{2m+j-1,i} < x_{m+j,i}, \qquad \text{for } i = 1,$$

and

$$x_{m+j,i-1} < x_{2m+j,i}, \qquad \text{for } i = 2.$$

Step 3. For fixed $n \geq 1$ and $j = 0,1,2,\ldots, m-1$, we assume that the n zeros of U_{nm+j} satisfy

$$0 < x_{nm+j,1} < x_{nm+j,2} < \cdots < x_{nm+j,n}, \tag{3.8}$$
$$0 < x_{nm+m-1,i} < x_{nm+m-2,i} < \cdots < x_{nm,i}, \qquad i = 1,2,\ldots,n, \tag{3.9}$$
$$x_{nm,i} < x_{(n-1)m+m-1,i}, \qquad n \geq 2,\ i = 1,2,\ldots,n-1, \tag{3.10}$$
$$x_{nm+j-1,i} < x_{(n-1)m+j,i}, \qquad i = 1,2,\ldots,n-1,\ 0 \leq j \leq m-1, \tag{3.11}$$

and for $2 \leq i \leq n$,

$$0 < x_{(n-1)m+j,i-1} < x_{nm+j,i}, \qquad 0 \leq j \leq m-1. \tag{3.12}$$

We show that (3.8)–(3.12) hold for $n+1$, and $0 \leq j \leq m-1$.

We know that

$$U_{(n+1)m+j}(x) = U_{(n+1)m+j-1}(x) - aU_{nm+j}(x).$$

(i) For $j = 0$, we have

$$U_{(n+1)m}(x) = U_{(n+1)m-1}(x) - aU_{nm}(x).$$

Recall that $U_{(n+1)m}(0) > 0$. By (3.9) we have $x_{nm+m-1,i} < x_{nm,i}$. Also

(3.10) and (3.12) give $x_{nm,i} < x_{nm+m-1,i+1}$ for $i = 1, 2, \dots, n-1$. Therefore

$$U_{(n+1)m}(x_{nm+m-1,i}) = -aU_{nm}(x_{nm+m-1,i}) < 0, \qquad \text{if } i \text{ is odd},$$
$$U_{(n+1)m}(x_{nm+m-1,i}) = -aU_{nm}(x_{nm+m-1,i}) > 0, \qquad \text{if } i \text{ is even}.$$

Also

$$U_{(n+1)m}(x_{nm,i}) = U_{nm+m-1}(x_{nm,i}) < 0, \qquad \text{if } i \text{ is odd},$$
$$U_{(n+1)m}(x_{nm,i}) = U_{nm+m-1}(x_{nm,i}) > 0, \qquad \text{if } i \text{ is even}.$$

Note that the signs of the leading coefficients of $U_{(n+1)m}(x)$ and $U_{nm+j}(x)$ are, respectively, $(-1)^{(n+1)}$ and $(-1)^n$. So there exists a zero of $U_{(n+1)m}(x)$ between 0 and $x_{nm+m-1,1}$, a zero between $x_{nm+m-1,i}$ and $x_{nm+m-1,i+1}$ ($i = 1, 2, \dots, n-1$) and a zero between $x_{nm+m-1,n}$ and ∞. Thus there exist $n+1$ zeros of $U_{(n+1)m}(x)$ with the properties

$$0 < x_{(n+1)m,1} < x_{(n+1)m,2} < \cdots < x_{(n+1)m,n+1},$$
$$x_{(n+1)m,i} < x_{nm+m-1,i}, \qquad x_{nm+m-1,i} < x_{nm,i}, \; 1 \le i \le n \tag{3.13}$$

and

$$x_{nm,i-1} < x_{(n+1)m,i}, \qquad \text{for } 2 \le i \le n+1. \tag{3.14}$$

(ii) For $j = 1$, we have

$$U_{(n+1)m+1}(x) = U_{(n+1)m}(x) - aU_{nm+1}(x).$$

Combining (3.13) and (3.9) yields $x_{(n+1)m,i} < x_{nm+m-1,i} < x_{nm+1,i}$, $i = 1, 2, \dots, n$. Also (3.9) and (3.14) give $x_{nm+1,n} < x_{nm,n} < x_{(n+1)m,n+1}$ and $x_{nm+1,i-1} < x_{nm,i-1} < x_{(n+1)m,i}$ for $i = 2, 3, \dots, n+1$. Thus

$$U_{(n+1)m+1}(x_{(n+1)m,i}) = -aU_{nm+1}(x_{(n+1)m,i}) < 0, \qquad \text{if } i \text{ is odd},$$
$$U_{(n+1)m+1}(x_{(n+1)m,i}) = -aU_{nm+1}(x_{(n+1)m,i}) > 0, \qquad \text{if } i \text{ is even}.$$

Also

$$U_{(n+1)m+1}(x_{nm+1,i}) = U_{(n+1)m}(x_{nm+1,i}) < 0, \qquad \text{if } i \text{ is odd},$$
$$U_{(n+1)m+1}(x_{nm+1,i}) = U_{(n+1)m}(x_{nm+1,i}) > 0, \qquad \text{if } i \text{ is even}.$$

Using the same argument as in (i), we see that there exist $n + 1$ zeros of $U_{(n+1)m+1}$ with the properties

$$0 < x_{(n+1)m+1,1} < x_{(n+1)m+1,2} < \cdots < x_{(n+1)m+1,n+1},$$
$$0 < x_{(n+1)m+1,i} < x_{(n+1)m,i}, \qquad \text{for } i = 1,2,\ldots,n+1,$$
$$x_{(n+1)m+1,i} < x_{nm+2,i} < x_{nm+1,i}, \qquad i = 1,2,\ldots,n \ (m \geq 3)$$

because $x_{(n+1)m+1,i} < x_{(n+1)m,i} < x_{nm+m-1,i} \leq x_{nm+2,i} < x_{nm+1,i}$ from (3.13) and (3.9). Also

$$x_{nm+1,i-1} < x_{(n+1)m+1,i}, \qquad \text{for } i = 2,3,\ldots,n+1.$$

(iii) Fix s $(1 \leq s \leq m-2)$, and assume that for all $0 \leq j \leq s$ the zeros of $U_{(n+1)m+j}$ have the properties.

$$0 < x_{(n+1)m+j,1} < x_{(n+1)m+j,2} < \cdots < x_{(n+1)m+j,n+1}, \tag{3.15}$$
$$0 < x_{(n+1)m+j,i} < x_{(n+1)m+j-1,i}, \qquad \text{for } 1 \leq i \leq n+1, \tag{3.16}$$
$$x_{(n+1)m+j,i} < x_{nm+j+1,i}, \qquad i = 1,2,\ldots,n, \tag{3.17}$$

and

$$x_{nm+j,i-1} < x_{(n+1)m+j,i}, \qquad \text{for } i = 2,3,\ldots,n+1. \tag{3.18}$$

We prove that (3.15)–(3.18) hold for $0 \leq j \leq s+1$.

Now

$$U_{(n+1)m+s+1}(x) = U_{(n+1)m+s}(x) - aU_{nm+s+1}(x).$$

Recall that $U_{(n+1)m+s+1}(0) > 0$. By (ii) of this step and (3.17) we have $x_{(n+1)m+s,i} < x_{nm+s+1,i}$, for $i = 1,2,\ldots,n$. In the case when $s = m-2$, we have $x_{(n+1)m+m-2,i} < x_{(n+1)m,i} < x_{nm+m-1,i}$ because of (3.16) and (3.13). Using (3.9) and (3.18) we get $x_{nm+s+1,i-1} < x_{nm+s,i-1} < x_{(n+1)m+s,i}$ for $i = 2,3,\ldots,n+1$. Therefore

$$U_{(n+1)m+s+1}(x_{(n+1)m+s,i}) = -aU_{nm+s+1}(x_{(n+1)m+s,i}) < 0, \qquad \text{if } i \text{ is odd},$$
$$U_{(n+1)m+s+1}(x_{(n+1)m+s,i}) = -aU_{nm+s+1}(x_{(n+1)m+s,i}) > 0, \qquad \text{if } i \text{ is even}.$$

Also

$$U_{(n+1)m+s+1}(x_{nm+s+1,i}) = U_{(n+1)m+s}(x_{nm+s+1,i}) < 0, \qquad \text{if } i \text{ is odd},$$
$$U_{(n+1)m+s+1}(x_{nm+s+1,i}) = U_{(n+1)m+s}(x_{nm+s+1,i}) > 0, \qquad \text{if } i \text{ is even}.$$

So there exists a zero of $U_{(n+1)m+s+1}(x)$ between 0 and $x_{(n+1)m+s,1}$, as well as a zero between $x_{(n+1)m+s,i}$ and $x_{(n+1)m+s,\,i+1}$ $(i = 1, 2, \ldots, n)$. Note that the sign of the leading coefficient of $U_{(n+1)m+s+1}(x)$ is $(-1)^{(n+1)}$. Thus we see that there exist $n+1$ zeros of $U_{(n+1)m+s+1}(x)$ such that

$$0 < x_{(n+1)m+s+1,1} < x_{(n+1)m+s+1,2} < \cdots < x_{(n+1)m+s+1,n+1},$$

$$0 < x_{(n+1)m+s+1,i} < x_{(n+1)m+s,i}, \qquad \text{for } 1 \le i \le n+1,$$

$$x_{(n+1)m+s,i} < x_{nm+s+1,i}, \qquad i = 1, 2, \ldots, n,$$

and for $2 \le i \le n+1$,

$$x_{nm+s+1,i-1} < x_{(n+1)m+s+1,i}.$$

Combining (i), (ii), and (iii), we have shown that (3.8)–(3.12) holds with n replaced by $n+1$ and $0 \le j \le m-1$. This completes the induction which proves that all the zeros of $U_n(x)$, $n \ge 1$, are real and positive.

Finally we recall that for $z \ne 0$, $F_n(z) = z^n U_n(x)$. Therefore the zeros of $F_n(z)$ are determined by the zeros of $U_n(x)$. If the parameter x starts from zero and tends to plus infinity in the real domain, then the variable z describes the set S_m. For $z = 0$, we know that $F_n(0) \ne 0 \Leftrightarrow n = 0 \bmod(m)$. Therefore we have proved that the zeros of the Faber polynomials $F_n(z)$ of the m-cusped hypocycloid are located on the set S_m for $n = 1, 2, \ldots$. ∎

4. Density and Distribution of Zeros of $F_n(z)$

In Section 3 we showed that zeros of $F_n(z) = F_n(z; H_m)$ are located on the set S_m for every n. What can be said about the limiting behavior of the zeros of $F_n(z)$ as $n \to \infty$? The answer to this question is provided by the following theorem.

Theorem 4.1. (i) *The zeros of $\{F_n(z)\}_1^\infty$ are dense in the set S_m.*

(ii) *Let $z_{n,k}$, $k = 1, 2, 3, \ldots, n$, be the zeros of $F_n(z)$. Then for $z \in \overline{\mathbf{C}} \setminus S_m$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{z - z_{n,k}} = S(z),$$

where $S(z)$ is the analytic continuation to $\overline{\mathbf{C}} \setminus S_m$ of the power series

$$S(z) = \sum_{k=0}^{\infty} \frac{(mk)!}{k!((m-1)k)!(m-1)^k} z^{-mk-1}, \qquad |z| > m/(m-1). \tag{4.1}$$

(iii) *Let* ν_n *be the normalized counting measure in the zeros of* $F_n(z)$*; that is,*

$$\nu_n := \frac{1}{n} \sum_{k=1}^{n} \delta_{z_{n,k}},$$

where δ_z *denotes the Dirac delta measure with unit mass at* z. *If* ν^* *is any weak-star limit measure of* $\{\nu_n\}$, *then the balayage (cf.* [*L*]) *of* ν^* *to the boundary of* H_m *is the equilibrium measure* μ_{H_m} *for* H_m.

Consequently, if f *is harmonic in the interior of* H_m *and continuous on its closure, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(z_{n,k}) = \int f \, d\mu_{H_m}.$$

Proof of (i). From the relation between $F_n(z)$ and $U_n(x)$, given in the proof of Theorem 3.1, we see that proving (i) is equivalent to proving that the zeros of $U_n(x)$ are dense in $(0,\infty)$. The proof of the latter follows from the method of proof of Theorem 4.2 in [U], so we omit the details. ∎

Proof of (ii). Let Let $z_{n,k}$, $k = 1,2,3,\ldots,n$, be the zeros of $F_n(z)$, and set

$$f_n(z) := [F_n(z)]^{1/n} = z \prod_{k=1}^{n} \left(1 - \frac{z_{n,k}}{z}\right)^{1/n},$$

where the root has its principal value 1 at $z = \infty$. Notice that each function $f_n(z)$ is single-valued and analytic in $\mathbf{C} \setminus S_m$ (all its branch points are in S_m). Moreover, $\{f_n\}$ forms a normal family in $\mathbf{C} \setminus S_m$ and

$$\lim_{n\to\infty} f_n(z) = \Phi(z) \tag{4.2}$$

for z outside H_m (cf. [M, p. 108]). Hence $\{f_n\}$ converges to the analytic continuation of $\Phi(z)$ to $\mathbf{C} \setminus S_m$. (We again use $\Phi(z)$ to denote this continuation.)

Taking the logarithmic derivative in (4.2), we get

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{z - z_{n,k}} = \frac{\Phi'(z)}{\Phi(z)} \tag{4.3}$$

for $z \in \overline{\mathbf{C}} \setminus S_m$. Thus it remains to show that $S(z) := \Phi'(z)/\Phi(z)$ has the expansion (4.1). Since $S(z)$ is analytic at ∞ and $S(\infty) = 0$, the expansion of

$S(z)$ about ∞ has the form

$$S(z) = \frac{\Phi'(z)}{\Phi(z)} = \sum_{k=0}^{\infty} \frac{M_k}{z^{k+1}}. \tag{4.4}$$

The coefficients M_k of (4.4) can be evaluated from the formula

$$M_k = \frac{1}{2\pi} \int_{|w|=r>1} \Psi^k(w) \frac{dw}{w}, \qquad k = 0, 1, \ldots. \tag{4.5}$$

Substituting $\Psi(w) = w + 1/[(m-1)w^{m-1}]$ into (4.5) and using the Cauchy formula, we get

$$M_k = \begin{cases} k!/\left[(k(m-1)/m)!(k/m)!(m-1)^{k/m}\right], & \text{if } k = 0 \bmod(m), \\ 0, & \text{otherwise}, \end{cases}$$

which yields (4.1). ∎

Proof of (iii). It follows from (4.2) that for the sup norm on H_m

$$\limsup_{n\to\infty} \|F_n\|_{H_m}^{1/n} \le \rho_{H_m} = \text{capacity}(H_m).$$

Hence assertion (iii) is an immediate consequence of Theorem 2.3 of [MS]. ∎

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