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MARKOV–BERNSTEIN AND NIKOLSKII INEQUALITIES, AND CHRISTOFFEL FUNCTIONS FOR EXPONENTIAL WEIGHTS ON (–1,1)*

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Abstract. Exponential weights $w := e^{-Q}$ are considered, where $Q : (-1, 1) \to \mathbf{R}$ is even, convex, and sufficiently smooth. For example, the results may be applied to

$$w(x) := (1-x^2)^{\alpha}, \qquad \alpha > 0,$$
$$w(x) := \exp(-(1-x^2)^{-\alpha}), \qquad \alpha > 0, \quad \text{or}$$
$$w(x) := \exp(-\exp_k(1-x^2)^{-\alpha}), \quad \alpha > 0, \quad k \geq 1,$$

where $\exp_k = \exp(\exp(\cdots))$ denotes the kth iterated exponential.

Weighted Markov and Bernstein inequalities such as

$$\|P'w\|_{L_\infty[-1,1]} \leq CQ'(a_{2n})\|Pw\|_{L_\infty[-1,1]},$$

and

$$|P'w|(x) \leq \frac{Cn}{\sqrt{1-|x|/a_n}}\|Pw\|_{L_\infty[-1,1]}, \qquad |x| < a_n,$$

are established for polynomials P of degree at most n. Here a_n is the nth Mhaskar–Rahmanov–Saff number for Q. For the special weights listed above, a more explicit form is given to $Q'(a_{2n})$. Estimates are deduced for Christoffel functions such as

$$\sup_{x\in[-1,1]} \lambda_n^{-1}(w^2, x)w(x) \leq CQ'(a_{2n}),$$

and also Nikolskii inequalities.

Key words. Markov–Bernstein inequalities, Nikolskii inequalities, non-Szegő weights, Christoffel functions, orthogonal polynomials

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1. Introduction and statement of results.

Throughout, $\mathcal{P}_n$ denotes the class of real polynomials of degree at most n, and $\|\cdot\|_{L_p(S)}$ denotes the L_p norm over any measurable $S \subset \mathbf{R}$ $(0 < p \leq \infty)$. Furthermore, $C, C_1, C_2, \ldots,$ denote positive constants independent of n, $P \in \mathcal{P}_n$, and $x \in \mathbf{R}$, which are not necessarily the same in different occurrences.

The classical inequalities of Markov and Bernstein are, respectively,

$$\|P'\|_{L_\infty[-1,1]} \leq n^2\|P\|_{L_\infty[-1,1]} \tag{1.1}$$

and

$$|P'(x)| \leq n(1-x^2)^{-1/2}\|P\|_{L_\infty[-1,1]} \tag{1.2}$$

for $P \in \mathcal{P}_n$ and $|x| < 1$. The interest in these inequalities lies in their application to rates of approximation by polynomials, to discretisation procedures, to approximation processes, and so on.

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Naturally, such important inequalities have been generalized to treat a host of situations, such as in L_p spaces, and with weights inserted. We cannot hope here to review the history and the contributions of the many authors. A fairly typical example of those in the literature is [1, Thm. 8.4.7, p. 107]:

$$\|P'(x)\sqrt{1-x^2}w(x)\|_{L_p[-1,1]} \le Cn\|Pw\|_{L_p[-1,1]}, \tag{1.3}$$

$P \in \mathcal{P}_n$, $n \ge 1$. Here $0 < p \le \infty$, and $w(x)$ is either a Jacobi weight $(1-x)^\alpha(1+x)^\beta$, where α, $\beta > -1$, or something similar. See [15, Thm. 19, p. 164], [1], [2], [3] for further discussion and references.

The aim of this paper is to treat weights $w(x)$ that may decay more rapidly at ± 1 than Jacobi weights; for example,

$$w(x) := W_{0,\alpha}(x) := \exp(-(1-x^2)^{-\alpha}), \qquad \alpha > 0,$$

or

$$w(x) := W_{k,\alpha}(x) := \exp(-\exp_k[(1-x^2)^{-\alpha}]), \quad \alpha > 0, \quad k \ge 1$$

where

$$\exp_k := \exp(\exp(\exp\cdots)) \quad (k \text{ times})$$

denotes the kth iterated exponential. However, our results apply equally well to the classical ultraspherical weight

$$w(x) := (1-x^2)^\alpha, \qquad \alpha > 0.$$

For $\alpha = \frac{1}{2}$, the weight $W_{0,\alpha}$ of (1.4) is similar to a Pollaczek weight, and its orthogonal polynomials were considered in [15, pp. 82–83]. Asymptotics, and spacing of zeros of the orthogonal polynomials for $W_{k,\alpha}$, have been considered in [2], [12].

To the best of our knowledge, the Markov–Bernstein inequalities in this paper are new for the weights $W_{k,\alpha}$ for all the range of parameters. To those with an interest in orthogonal polynomials, it is noteworthy that Szegő's condition

$$\int_{-1}^{1} \frac{\log w(x)}{\sqrt{1-x^2}}dx > -\infty \tag{1.6}$$

is violated by $W_{0,\alpha}$ of (1.4) if $\alpha \ge \frac{1}{2}$ and by all the weights $W_{k,\alpha}$ of (1.5). Since the Markov–Bernstein inequalities have various applications to orthogonal polynomials associated with non-Szegő weights, such as estimates for Christoffel functions, they are of particular interest.

In fact, the results of this paper bear a close resemblance to results for exponential weights on the real line [4], [5], [6], [9], [10], [17], and more specifically to Erdös weights $W^2 = e^{-2Q}$ on $\mathbf{R}$: These have the property that Q grows faster than any polynomial at infinity.

In the analysis of those weights, and the ones treated in this paper, the Mhaskar–Rahmanov–Saff number plays an important role. Let us suppose that $w = e^{-Q}$, where $Q : (-1,1) \to \mathbf{R}$ is even, and differentiable in (0,1). Suppose, furthermore, that $tQ'(t)$ is positive and increasing in (0,1) with limits zero and infinity at zero and 1, respectively, and

$$\int_0^1 \frac{tQ'(t)}{\sqrt{1-t^2}}dt = \infty. \tag{1.7}$$

Then the uth Mhaskar–Rahmanov–Saff number, $a_u = a_u(Q)$, is defined to be the root of

$$u \quad \frac{2}{\pi}\int_0^1 \frac{a_u t Q'(a_u t)}{\sqrt{1-t^2}}\,dt, \qquad u > 0.$$

The importance of a_u lies in the identity (cf. [13], [14])

$$\|Pw\|_{L_\infty[-1,1]} = \|Pw\|_{L_\infty[-a_n,a_n]}, \quad P \in \mathcal{P}_n, \quad n \geq 1,$$

which we refer to as the *Mhaskar–Saff identity.* Of course $a_n \to 1$ as $n \to \infty$. As an illustration of its rate of approach, we note that for $w = W_{0,\alpha}$,

(1.10) $$1 - a_n \sim n^{-1/(\alpha+1/2)} \qquad n \to \infty,$$

and for $w = W_{k,\alpha}$,

(1.11) $$1 - a_n \sim (\log_k n)^{-1/\alpha} \qquad n \to \infty,$$

where $\log_k = \log(\log(\log\cdots))$ (k times) denotes the kth iterated logarithm. Furthermore, we are using $\sim$ in the sense of [15]: if $\{c_n\}_{n=1}^\infty$ and $\{d_n\}_{n=1}^\infty$ are real positive sequences, then

$$c_n \sim d_n, \qquad n \to \infty,$$

means that for n large enough,

$$C_1 \leq \frac{c_n}{d_n} \leq C_2$$

We are now ready to define our class of weights.

DEFINITION 1.1. Let $w := e^{-Q}$, where

(i) Q is even and continuously differentiable in (-1,1), while Q'' is continuous in (0,1);

(ii) $Q' \geq 0$ and $Q'' \geq 0$ in (0,1);

(iii) $\int_0^1 (tQ'(t))/(\sqrt{1-t^2})dt = \infty$;

(iv) Let

$$T(x) := 1 + \frac{xQ''(x)}{Q'(x)} \qquad x \in (0,1$$

We assume that

(1.13)
$$\begin{array}{ll} \text{(a)} & T \text{ is increasing in } (0,1); \\ \text{(b)} & T(0+) > 1; \\ \text{(c)} & T(x) = O(Q'(x)),\ x \to 1^-. \end{array}$$

Under the above conditions, we write $w \in \mathcal{W}$.

We remark that (1.13) is a rather weak regularity condition, while (iii) is required for the existence of a_n. Further, we note that most of our results really only require the above hypotheses to be satisfied for x near 1. However, for simplicity, we shall not pursue this point. In any event, $W_{k,\alpha} \in \mathcal{W}$, $k \geq 0$, $\alpha > 0$.

Following are our Markov–Bernstein inequalities for $P'w$.

THEOREM 1.2. *Let $w \in \mathcal{W}$.*
(i) *For $n \geq 1$ and $P \in \mathcal{P}_n$,*

$$\|P'w\|_{L_\infty[-1,1]} \leq CnT(a_{2n})^{1/2}\|Pw\|_{L_\infty[-1,1]}. \tag{1.14}$$

(ii) *For $n \geq 1$, $P \in \mathcal{P}_n$, and $|x| < a_n$,*

$$|P'w|(x) \leq \frac{Cn}{\sqrt{1-|x|/a_n}}\|Pw\|_{L_\infty[-1,1]}.$$

We remark that under mild additional conditions, which are satisfied for $W_{k,\alpha}$ for all $k \geq 0$, $\alpha > 0$,

$$nT(a_{2n})^{1/2} \sim Q'(a_{2n}), \qquad n \to \infty.$$

(see Lemma 3.2(ii) below), so one may reformulate (1.14) as

$$\|P'w\|_{L_\infty[-1,1]} \leq CQ'(a_{2n})\|Pw\|_{L_\infty[-1,1]} \tag{1.17}$$

One may combine (1.14) and (1.15) as follows:

COROLLARY 1.3. *Let $w \in \mathcal{W}$. For $n \geq 1$, $P \in \mathcal{P}_n$, and $x \in [-1,1]$,*

$$|P'w|(x) \leq \frac{Cn}{|1-|x|/a_n|^{1/2} + T(a_{2n})^{-1/2}}\|Pw\|_{L_\infty[-1,1]}. \tag{1.18}$$

The above is the analogue of the classical consequence of (1.1) and (1.2):

$$|P'(x)| \leq \frac{Cn}{|1-|x||^{1/2}+n^{-1}}\|P\|_{L_\infty[-1,1]}, \quad P \in \mathcal{P}_n, \quad x \in (-1,1).$$

As examples of Theorem 1.2, we present the following.

COROLLARY 1.4. (i) *Let $\alpha > 0$ and $W_{0,\alpha}$ be given by (1.4) Then for $n \geq 1$ and $P \in \mathcal{P}_n$,*

$$\|P'W_{0,\alpha}\|_{L_\infty[-1,1]} \leq Cn^{(2\alpha+2)/(2\alpha+1)}\|PW_{0,\alpha}\|_{L_\infty[-1,1]}$$

(ii) *Let $k \geq 1$, $\alpha > 0$, and $W_{k,\alpha}$ be given by (1.5). Then for $n \geq 1$ and $P \in \mathcal{P}_n$,*

$$\|P'W_{k,\alpha}\|_{L_\infty[-1,1]} \leq Cn\left[\prod_{j=1}^{k}\log_j n\right]^{1/2}(\log_k n)^{(\alpha+1)/2\alpha}\|PW_{k,\alpha}\|_{L_\infty[-1,1]}.$$

(iii) *Let $\alpha > 0$ and $w(x) := (1-x^2)^\alpha$. Then for $n \geq 1$ and $P \in \mathcal{P}_n$,*

$$\|P'w\|_{L_\infty[-1,1]} \leq Cn^2\|Pw\|_{L_\infty[-1,1]}. \tag{1.21}$$

We remark that under mild additional conditions involving Q''', which hold for $W_{k,\alpha}$, $k \geq 0$, $\alpha > 0$, we can show that (1.14) is sharp with respect to the dependence on n. The proof is lengthy, and involves analysis of L_∞ extremal polynomials for w. For the proof in a closely related situation, we refer the reader to [7, pp. 71–78]. Note that (1.21) is a classical inequality for ultraspherical weights [1].

Next, we turn to inequalities for $(Pw)'$. These are different from those for $P'w$, since for x close to a_n, $|(Pw)'|(x)$ admits a far better estimate than $|P'w|(x)$. A similar situation occurs for Erdös weights on $\mathbf{R}$ (cf. [6]).

THEOREM 1.5. *Let $w \in \mathcal{W}$. Then*
(a) *For $n \geq 1$, $P \in \mathcal{P}_n$ and $|x| < a_n$*

$$|(Pw)'(x)| \leq \frac{Cn}{(1-|x|/a_n)^{1/2} + T(a_{2n})^{-1/2}} \|Pw\|_{L_\infty[-1,1]};$$

(b) *For $n \geq 1$, $P \in \mathcal{P}_n$ and $|x| \leq a_n$*

$$|(Pw)'(x)| \leq CnT(a_{2n}) \left\{ \left(1 - \frac{|x|}{a_n}\right)^{1/2} + (nT(a_{2n}))^{-1/3} \right\} \|Pw\|_{L_\infty[-1,1]}$$

In particular,

$$|(Pw)'(a_n)| \leq C(nT(a_{2n}))^{2/3} \|Pw\|_{L_\infty[-1,1]}$$

Note that for suitable A, and $|x|/a_n \geq 1 - AT(a_{2n})^{-1}$, (1.23) provides a better estimate than (1.22), while for $|x|/a_n < 1 - AT(a_{2n})^{-1}$, (1.22) provides a better estimate.

Next, we turn to estimates of *Christoffel functions.* Recall that the nth Christoffel function for w^2 is

$$\text{(1.25)} \qquad \lambda_n(w^2, x) := \inf_{P \in \mathcal{P}_{n-1}} \frac{1}{P^2(x)} \int_{-1}^{1} (Pw)^2(t)\,dt, \qquad x \in (-1,1)$$

It turns out that a particularly simple way to find upper bounds for λ_n^{-1} involves the Markov–Bernstein inequality for w. This idea has been used elsewhere [7].

THEOREM 1.6. *Let $w \in \mathcal{W}$. Then for $n \geq 1$,*

$$\text{(1.26)} \qquad \sup_{x \in [-1,1]} \lambda_n^{-1}(w^2, x) w^2(x) \leq CnT(a_{2n})^{1/2}.$$

We note that this result is sharp under mild additional conditions on Q; see [9] for the proof in a similar situation. As a corollary of the above estimate, we obtain Nikolskii inequalities.

COROLLARY 1.7. *Let $w \in \mathcal{W}$ and $0 < p < q \leq \infty$. Then for $n \geq 1$ and $P \in \mathcal{P}_n$,*

$$\text{(1.27)} \qquad \|Pw\|_{L_q[-1,1]} \leq C(nT(a_{2n})^{1/2})^{1/p-1/q} \|Pw\|_{L_p[-1,1]}$$

This paper is organized as follows. In §2, we present the basic ingredients of the proof—contour integral estimates, and integral equations with logarithmic kernel. In §3, we prove some technical lemmas. In §4, we present estimates for the measure $\mu_n(x)$, and in §5, we estimate the majorisation function $U_n(x+iy)$. In §6, we prove the Markov–Bernstein inequalities, and in §7, we prove Theorems 1.6 and Corollary 1.7.

2. The two basic ingredients. The first ingredient is potential theory and an integral equation used in majorisation of weighted polynomials.

LEMMA 2.1. *Let $w := e^{-Q} \in \mathcal{W}$. For $n \geq 1$, let $a_n = a_n(Q)$ be the root of (1.8). For $x \in (-1,1) \backslash \{0\}$, let*

$$\text{(2.1)} \qquad \mu_n(x) := \frac{2}{\pi^2} \int_0^1 \frac{\sqrt{1-x^2}}{\sqrt{1-s^2}} \frac{a_n s Q'(a_n s) - a_n x Q'(a_n x)}{n(s^2 - x^2)} ds$$

Furthermore, let

$$\chi_n = \frac{2}{\pi} \int_0^1 \frac{Q(a_n t)}{\sqrt{1-t^2}}\, dt + n \log 2,$$

and for $z \in \mathbf{C}$ *such that* $|z| < 1/a_n$, *let*

$$U_n(z) := \int_{-1}^{1} (\log |z-t|) \mu_n(t)\, dt - \frac{Q(a_n|z|)}{n} + \frac{\chi_n}{n}$$

(a) *Then for almost every* $x \in (-1, 1)$,

$$0 < \mu_n(x) < \infty,$$

$$\int_{-1}^{1} \mu_n(x) dx = 1,$$

and

$$\int_{-1}^{1} \frac{\mu_n(x)}{1-x} dx = \frac{Q'(a_n)}{n}$$

(b) *Furthermore,*

$$U_n(x) = 0, \qquad x \in [-1, 1],$$

$$(xU_n'(x))' < 0 \quad U_n(x) < 0, \quad U_n'(x) < 0, \quad x \in \left(1 \; \frac{1}{a_n}\right)$$

(c) *For* $n \geq 1$, $P \in \mathcal{P}_n$ *and* $z \in \mathbf{C}$ *such that* $|z| < 1$

$$|P(z)w(|z|)| \leq \|Pw\|_{L_\infty[-a_n, a_n]} \exp\left(\quad \left(\frac{z}{\quad}\right)\right)$$

Furthermore,

$$\|Pw\|_{L_\infty[-1,1]} = \|Pw\|_{L_\infty[-a_n, a_n]},$$

and if P *is not identically zero,*

$$|Pw|(x) < \|Pw\|_{L_\infty[-a_n, a_n]}, \qquad |x| > a_n.$$

Proof. See [11, Lemmas 5.1, 5.2, pp. 28–34]. One sets $f(x) := Q(a_n x)$, $x \in [-1/a_n, 1/a_n]$ in [11].

(a) The constant B in [11, eqn. (5.4), p. 28] is zero in view of (1.8), and the function μ_n above is the function $g(f; t)$ or $L[f'](t)$ of [11]. Then (2.5) and (2.6) are, respectively, (5.6) and (5.23) in [11].

(b) The identity (2.7) follows from [11, eqn. (5.1), p. 28]. Furthermore, (5.25) and (5.26) in [11, p. 23] ensure that $U_n'(a_n) = U_n(a_n) = 0$ and the first relation in (2.8), which is (5.27) in [11, p. 32], then implies the other two.

(c) See [13, pp. 74–75], or see [11, p. 51]. □

Next, we turn to estimates for derivatives of weighted polynomials, derived via Cauchy's integral formula. This method has been used elsewhere [4], [6].

LEMMA 2.2. *Let $w \in \mathcal{W}$. Let $x \in (-1,1)$, $\epsilon \in (0, 1-|x|)$, and $P \in \mathcal{P}_n$ for some $n \geq 1$. Then*

$$|(Pw)'(x)| \leq \epsilon^{-1} e^{\tau} \|Pw\|_{L_\infty[-1,1]} \max_{|t-x|=\epsilon} \exp\left(nU_n\left(\frac{t}{a_n}\right)\right)$$

where

$$\tau := \begin{cases} Q'(3\epsilon)2\epsilon, & \text{if } |x| \leq 2\epsilon, \\ \left[\frac{Q'(|x|+\epsilon)}{|x|} + Q''(|x|+\epsilon)\right]2\epsilon^2 & \text{if } |x| > 2\epsilon. \end{cases}$$

Proof. Fix $x \in (-1,1)$ and define the entire function

$$\hat{w}(t) := \exp(-Q(x) - Q'(x)(t-x)), \qquad t \in \mathbf{C},$$

so that $\hat{w}^{(j)}(x) = w^{(j)}(x)$, $j = 0, 1$. Then

$$\begin{aligned} |(Pw)'(x)| = |(P\hat{w})'(x)| &= \left| \frac{1}{2\pi i} \int_{|t-x|=\epsilon} \frac{(P\hat{w})(t)}{(t-x)^2} dt \right| \\ &\leq \frac{1}{\epsilon} \max_{|t-x|=\epsilon} |P\hat{w}|(t) \\ &\leq \frac{1}{\epsilon} \max_{|t-x|=\epsilon} |\hat{w}(t)/w(|t|)| \cdot \max_{|t-x|=\epsilon} |P(t)w(|t|)| \\ &\leq \frac{1}{\epsilon} \max_{|t-x|=\epsilon} |\hat{w}(t)/w(|t|)| \cdot \|Pw\|_{L_\infty[-1,1]} \cdot \max_{|t-x|=\epsilon} \exp(nU_n(t/a_n)), \end{aligned} \tag{2.14}$$

by (2.9). It remains to estimate $|\hat{w}(t)/w(|t|)|$. Suppose first that $|x| \leq 2\epsilon$. Then for $|t-x| = \epsilon$,

$$\begin{aligned} |\hat{w}(t)/w(|t|)| &= \exp(-Q(x) - Q'(x)(\mathrm{Re}\ t - x) + Q(|t|)) \\ &\leq \exp(-Q(|x|) + Q'(2\epsilon)\epsilon + Q(|x|+\epsilon)), \end{aligned} \tag{2.15}$$

where we have used the monotonicity of Q and Q'. Finally,

$$Q(|x|+\epsilon) - Q(|x|) \leq Q'(|x|+\epsilon)\epsilon \leq Q'(3\epsilon)\epsilon,$$

and (2.13) follows for $|x| \leq 2\epsilon$.

Next, suppose $x > 2\epsilon$. Then

$$\begin{aligned} -Q(x) - Q'(x)(\mathrm{Re}\,t - x) + Q(|t|) &= Q(|t|) - Q(\mathrm{Re}\ t) \\ &\quad + Q(\mathrm{Re}\ t) - Q(x) - Q'(x)(\mathrm{Re}\,t - x) \\ &= Q'(\xi)(|t| - \mathrm{Re}\,t) + \tfrac{1}{2}Q''(\eta)(\mathrm{Re}\ t - x)^2, \end{aligned} \tag{2.16}$$

where ξ lies between $\mathrm{Re}\,t$ and $|t|$, and η lies between $\mathrm{Re}\,t$ and x. Here $\xi, \eta \in [x-\epsilon, x+\epsilon]$ Furthermore, by the elementary inequality $(a^2+b^2)^{1/2} \leq a + b^2/a$, for $a, b > 0$,

$$|t| - \mathrm{Re}\ t \leq \frac{(\mathrm{Im}\,t)^2}{\mathrm{Re}\,t} \leq \frac{\epsilon^2}{x-\epsilon} \leq \frac{2\epsilon^2}{x},$$

so

$$Q'(\xi)(|t| - \mathrm{Re}\, t) \le \frac{2Q'(x+\epsilon)\epsilon^2}{x} \tag{2.17}$$

Furthermore, $uQ''(u) = Q'(u)(T(u) - 1)$ is increasing in (0,1), so

$$\eta Q''(\eta) \le (x+\epsilon)Q''(x+\epsilon),$$

and hence

$$Q''(\eta) \le \frac{x+\epsilon}{x-\epsilon} Q''(x+\epsilon) \le 3Q''(x+\epsilon)$$

$$\tfrac{1}{2}Q''(\eta)(\mathrm{Re}\, t - x)^2 \le \tfrac{3}{2}\epsilon^2 Q''(x+\epsilon). \tag{2.18}$$

Combining (2.14) to (2.18) yields (2.13) for $x \ge 2\epsilon$, and the case $x \le -2\epsilon$ is similar. □

3. Technical lemmas. In this section, we present some elementary consequences of the hypothesis $w \in \mathcal{W}$.

LEMMA 3.1. *Let* $w = e^{-Q} \in \mathcal{W}$.

(i) *We have for* $0 < x \le Lx < 1$,

$$L^{T(x)} \le \frac{LxQ'(Lx)}{xQ'(x)} \le L^{T(Lx)}$$

(ii) $Q'(x)$ *and* $xQ''(x)$ *are increasing in* $(0,1)$

(iii)

$$\frac{xQ''(x)}{Q'(x)} \sim T(x) \quad \textit{in } (0,1].$$

(iv)

$$\frac{Q'(x)}{x} \le (T(0+) - 1)^{-1} Q''(x) \quad \textit{in } (0,1],$$

and

$$Q'(0) = 0.$$

(v)

$$\int_0^{1/2} \frac{Q'(x)}{x} dx < \infty.$$

Proof. (i) Now

$$\frac{LxQ'(Lx)}{xQ'(x)} = \exp\left(\int_x^{Lx} \frac{d}{dt}\log[tQ'(t)]dt\right) = \exp\left(\int_x^{Lx} \frac{T(t)}{t}dt\right)$$

Here the monotonicity of T ensures that

$$T(x)\log L \le \int_x^{Lx} \frac{T(t)}{t}dt \le T(Lx)\log L.$$

Then (3.1) follows.

(ii) Since

$$xQ''(x) = (T(x) - 1)Q'(x), \tag{3.6}$$

the monotonicity of T and Q' yields the monotonicity of $xQ''(x)$.

(iii) By (3.6), for $x \in (0, 1)$,

$$\left(1 - \frac{1}{T(x)}\right)\Bigg\{ \qquad\qquad\qquad\qquad \Bigg)$$

(iv) Firstly, (3.6), the monotonicity of T and the fact that $T(0+) > 1$ yield (3.3). Next, the evenness of Q and continuity of Q' force (3.4).

(v) If $0 < \delta < \frac{1}{2}$, inequality (3.3) yields

$$\int_\delta^{1/2} \frac{Q'(s)}{s} ds \le (T(0+) - 1)^{-1} [Q'(1/2) - Q'(\delta)]$$

Now let $\delta \to 0+$. □

LEMMA 3.2. *Let $w = e^{-Q} \in \mathcal{W}$ and $a_n = a_n(Q)$, $n \ge 1$*

(i) *For $j = 1, 2$ and n large enough,*

$$Q^{(j)}(a_n) = O(nT(a_n)^{j-1/2}).$$

(ii) *If also*

$$\frac{Q'(x)}{Q(x)} \sim T(x), \qquad x \text{ near } 1,$$

then for $j = 0, 1, 2$, and n large enough,

$$Q^{(j)}(a_n) \sim nT(a_n)^{j-1/2}$$

(iii) *We have*

$$Q'(a_n) = O(n^2); \tag{3.10}$$

$$T(a_n) = O(n^2). \tag{3.11}$$

(iv) *For $t \in (0, \infty)$,*

$$\frac{1}{tT(0+)} \ge \frac{a_t'}{a_t} \ge \frac{1}{tT(a_t)} \tag{3.12}$$

(v) *For $u \in (0, \infty)$ and $r \ge 1$,*

$$\frac{a_{ru}}{a_u} \ge 1 + \frac{\log r}{T(a_{ru})}. \tag{3.13}$$

Proof.

(i) $$\frac{n}{a_n Q'(a_n)} = \frac{2}{\pi} \int_0^1 \frac{a_n t Q'(a_n t)}{a_n Q'(a_n)} \frac{1}{\sqrt{1 - t^2}} dt$$

$$(3.14)\qquad \begin{aligned} &\geq \frac{2}{\pi}\int_0^1 t^{T(a_n)}\frac{1}{\sqrt{1-t^2}}dt \quad \text{(by (3.1))} \\ &\geq \frac{2}{\pi}\left(1-\frac{1}{T(a_n)}\right)^{T(a_n)}\int_{1-1/T(a_n)}^1 \frac{1}{\sqrt{1-t^2}}dt \\ &\geq C\left(1-\frac{1}{T(0+)}\right)^{T(0+)} T(a_n)^{-1/2} \end{aligned}$$

Then (3.7) follows for $j=1$, and (3.2) implies it for $j=2$.

(ii) From (3.14),

$$\begin{aligned} \frac{n}{a_nQ'(a_n)} &\leq \frac{2}{\pi}\int_0^{1-1/T(a_n)} \frac{Q'(a_nt)}{Q'(a_n)}\frac{1}{\sqrt{1-t^2}}dt + \frac{2}{\pi}\int_{1-1/T(a_n)}^1 \frac{1}{\sqrt{1-t^2}}dt \\ &\leq C_1T(a_n)^{1/2}\int_0^1 \frac{Q'(a_nt)}{Q'(a_n)}dt + C_1T(a_n)^{-1/2} \\ &= C_1T(a_n)^{1/2}\frac{Q(a_n)-Q(0)}{a_nQ'(a_n)} + C_1T(a_n)^{-1/2} \\ &\leq C_2T(a_n)^{1/2}\frac{Q(a_n)}{a_nQ'(a_n)} + C_1T(a_n)^{-1/2} \leq C_3T(a_n)^{-1/2} \end{aligned}$$

n large enough, by (3.8) and monotonicity of Q. Hence

$$a_nQ'(a_n) \sim nT(a_n)^{1/2}.$$

Then (3.2) yields (3.9) for $j=2$, and (3.8) yields it for $j=0$.

(iii) From (3.7) and (1.13),

$$Q'(a_n) = O(nT(a_n)^{1/2}) = O(nQ'(a_n)^{1/2}),$$

whence (3.10) follows. Then (1.13) yields (3.11).

(iv) Differentiating (1.8) with respect to u yields

$$\begin{aligned} 1 &= \frac{2}{\pi}\int_0^1 \frac{d}{du}(a_utQ'(a_ut))\frac{dt}{\sqrt{1-t^2}} \\ &= \frac{2}{\pi}\int_0^1 T(a_ut)Q'(a_ut)a_u't\frac{dt}{\sqrt{1-t^2}} \\ &\frac{a_u'}{a_u}\frac{2}{\pi}\int_0^1 T(a_ut)a_utQ'(a_ut)\frac{dt}{\sqrt{1-t^2}}. \end{aligned}$$

Since

$$T(0+) \leq T(a_ut) \leq T(a_u), \qquad t\in(0,1],$$

we obtain (3.12) from (1.8).

(v)

$$\frac{a_{ru}}{a_u} = \exp\left(\int_u^{ru} \frac{a_t'}{a_t} dt\right) \geq \exp\left(\int_u^{ru} \frac{dt}{tT(a_t)}\right)$$
$$\geq \exp\left(\frac{\log r}{T(a_{ru})}\right) \geq 1 + \frac{\log r}{T(a_{ru})} \qquad \square$$

LEMMA 3.3. *Let $w \in \mathcal{W}$.*

(i) *For $0 < \alpha < \Delta$, we have*

$$\frac{a_{\Delta n} Q'(a_{\Delta n})}{a_{\alpha n} Q'(a_{\alpha n})} \geq \frac{\Delta}{\alpha} > 1, \qquad n \geq 1. \tag{3.15}$$

(ii) *For $n \geq 1$ and $x \in [-1, 1]$,*

$$a_n x Q'(a_n x)(1 - |x|)^{1/2} \leq Cn. \tag{3.16}$$

Furthermore, for $n \geq 1$ and $s \in [-a_n, a_n]$

$$sQ'(s)\left(1 - \frac{|s|}{a_n}\right)^{1/2} \leq Cn. \tag{3.17}$$

Proof.

(i)

$$\frac{a_{\Delta n} Q'(a_{\Delta n})}{a_{\alpha n} Q'(a_{\alpha n})} = \exp\left(\int_{\alpha n}^{\Delta n} \frac{d}{dt}[\log(a_t Q'(a_t))] dt\right)$$
$$= \exp\left(\int_{\alpha n}^{\Delta n} T(a_t)\frac{a_t'}{a_t} dt\right) \geq \exp\left(\int_{\alpha n}^{\Delta n} \frac{dt}{t}\right) = \frac{\Delta}{\alpha},$$

by (3.12).

(ii) Now uniformly for $|x| \in [0, 1)$,

$$\int_{|x|}^1 \frac{ds}{\sqrt{1 - s^2}} \sim (1 - |x|)^{1/2}. \tag{3.18}$$

Hence for $|x| \in [0, 1)$

$$a_n x Q'(a_n x)(1 - |x|)^{1/2} = a_n |x| Q'(a_n |x|)(1 - |x|)^{1/2}$$
$$\leq C a_n |x| Q'(a_n |x|) \frac{2}{\pi} \int_{|x|}^1 \frac{ds}{\sqrt{1 - s^2}}$$
$$\leq C \frac{2}{\pi} \int_{|x|}^1 a_n s Q'(a_n s) \frac{ds}{\sqrt{1 - s^2}} \leq Cn.$$

For $|x| = 1$, (3.16) is trivial. Setting $x = s/a_n$ yields (3.17) $\square$

4. Estimates for the measure μ_n. In this section, we present some estimates for the measure $\mu_n(x)$ that are of independent interest. The methods we use are similar to those in [4], [6], [9].

THEOREM 4.1. *Let $w \in \mathcal{W}$. Then for $n \geq 1$,*

$$\text{(a)} \qquad \max_{x\in[-1,1]} \mu_n(x)\sqrt{1-x^2} \leq C; \tag{4.1}$$

$$\text{(b)} \qquad \max_{x\in[-1,1]} \mu_n(x)/\sqrt{1-x^2} \leq CT(a_n). \tag{4.2}$$

As a corollary, we shall deduce the following.

COROLLARY 4.2. *For $n \geq 1$,*

(4.3)

$$\sqrt{1-x^2}\mu_n(x) \geq C, \quad |x| \leq \frac{a_{\beta n}}{a_n}, \quad n \geq 1$$

We note that under mild additional conditions, we can show that (4.2) and (4.3) are sharp. See, respectively, [5] and [8] for the sharpness of (4.2) and (4.3) in the related situation of Erdös weights. We turn now to the proofs of the above results.

Proof of Theorem 4.1(a). Since μ_n is even, we may consider $x \in (0,1)$. For $n \geq 1$, and s, $x \in (0,1)$, set

$$\Delta_n(s,x) := \frac{a_n s Q'(a_n s) - a_n x Q'(a_n x)}{a_n s - a_n x} \tag{4.5}$$

Then

$$\sqrt{1-x^2}\mu_n(x) = \frac{2a_n}{n\pi^2}(1-x^2)\int_0^1 \frac{\Delta_n(s,x)}{s+x}\frac{ds}{\sqrt{1-s^2}}. \tag{4.6}$$

Note that since $(uQ'(u))' = Q'(u)T(u)$ is increasing in (0,1), $uQ'(u)$ is convex in (0,1). Hence for each fixed x, $\Delta_n(s,x)$ is an increasing function of $s \in (0,1)$.

Case I. $x \in [0, \frac{1}{4}]$. Now suppose that $x \in [0, \frac{1}{4}]$. Then $s \in (2x, 1]$ implies

$$\Delta_n(s,x) \leq \frac{a_n s Q'(a_n s)}{a_n s - a_n s/2} = 2Q'(a_n s).$$

Furthermore, $s \in (0, 2x]$ implies

$$\Delta_n(s,x) \leq \Delta_n(2x,x) \leq \frac{a_n 2x Q'(a_n 2x)}{a_n 2x - a_n x} = 2Q'(a_n 2x).$$

Hence for $0 \leq x \leq \frac{1}{4}$,

$$\begin{aligned}\sqrt{1-x^2}\mu_n(x) &\leq \frac{C_1}{n}(1-|x|)\left[\int_0^{2x} \frac{Q'(a_n 2x)}{s+x}ds + \int_{2x}^1 \frac{2Q'(a_n s)}{s}\frac{ds}{\sqrt{1-s^2}}\right] \\ &\leq \frac{C_2}{n}(1-|x|)\left[2Q'(1/2) + \int_{2x}^{1/2} \frac{Q'(a_n s)}{s}ds + \int_{1/2}^1 \frac{a_n s Q'(a_n s)}{\sqrt{1-s^2}}ds\right] \\ &\leq C,\end{aligned}$$

by Lemma 3.1(v) and the definition of a_n.

Case II. $x \in [\frac{1}{4}, 1)$. Next, suppose $\frac{1}{4} \le x < 1$. Let

$$\delta(x) := \frac{1-x}{4}$$

From (4.6) and as $x + 4\delta(x) \quad 1$, we have

$$\begin{aligned}
\sqrt{1-x^2}\mu_n(x) &\le \frac{C}{n}(1-x)\int_0^1 \frac{\Delta_n(s,x)}{\sqrt{1-s^2}}ds \\
& \frac{C}{n}4\delta(x)\left[\int_0^{x-\delta(x)} + \int_{x-\delta(x)}^{x+\delta(x)} + \int_{x+\delta(x)}^1\right]\frac{\Delta_n(s,x)}{\sqrt{1-s^2}}ds \\
&=: \frac{4C}{n}(I_1 + I_2 + I_3).
\end{aligned}$$

Firstly,

$$\begin{aligned}
I_1 &\le \delta(x)\int_0^{x-\delta(x)} \frac{a_n x Q'(a_n x)}{a_n x - a_n s}\frac{ds}{\sqrt{1-s^2}} \\
&\le C_1\delta(x) x Q'(a_n x)\int_0^{x-\delta(x)} \frac{ds}{(x-s)^{3/2}} \quad (\text{as } 1-s \ge x-s > 0) \\
&\le C_2 Q'(a_n x)\delta(x)^{1/2} \le C_3 n,
\end{aligned}$$

by (3.16). Next, if $s \in [x-\delta(x), x+\delta(x)]$, there exists ξ between s and x such that

$$\begin{aligned}
\Delta_n(s,x) &= \frac{d}{du}(uQ'(u))\Big|_{u=a_n\xi} = T(a_n\xi)Q'(a_n\xi) \\
&\le T(a_n[x+\delta(x)])Q'(a_n[x+\delta(x)]) \\
&\le \delta(x)^{-1}\int_{x+\delta(x)}^{x+2\delta(x)} T(a_n s)Q'(a_n s)ds \\
&= (a_n\delta(x))^{-1}\int_{x+\delta(x)}^{x+2\delta(x)} \frac{d}{ds}(a_n s Q'(a_n s))ds \\
&\le (a_n\delta(x))^{-1}a_n[x+2\delta(x)]Q'(a_n[x+2\delta(x)]) \\
&\le C_4\delta(x)^{-3/2}a_n[x+2\delta(x)]Q'(a_n[x+2\delta(x)])\int_{x+2\delta(x)}^1 \frac{ds}{\sqrt{1-s^2}} \\
&\qquad (\text{observe } 1-[x+2\delta(x)] = 2\delta(x)) \\
&\le C_4\delta(x)^{-3/2}\int_{x+2\delta(x)}^1 \frac{a_n s Q'(a_n s)}{\sqrt{1-s^2}}ds \\
&\le C_5\delta(x)^{-3/2}n.
\end{aligned}$$

Then

$$
\begin{aligned}
I_2 &\le \delta(x) \int_{x-\delta(x)}^{x+\delta(x)} C_5 \delta(x)^{-3/2} n \frac{ds}{\sqrt{1-s^2}} \\
&\le C_5 \delta(x)^{-1/2} n \int_{x-\delta(x)}^{1} \frac{ds}{\sqrt{1-s^2}} \le C_6 n,
\end{aligned}
\tag{4.9}
$$

as $1-(x-\delta(x)) \quad 5\delta(x)$. Finally,

$$
I_3 \le \delta(x) \int_{x+\delta(x)}^{1} \frac{a_n s Q'(a_n s)}{a_n \delta(x)} \frac{ds}{\sqrt{1-s^2}} \le Cn \tag{4.10}
$$

by the definition of a_n. Combining (4.7) to (4.10) yields

$$
\sqrt{1-x^2}\mu_n(x) \le C, \qquad x \in [\tfrac{1}{4}, 1). \qquad \square
$$

Proof of Theorem 4.1(b). With the notation (4.5), and by (4.6), we have for $1 > |x| \ge \frac{1}{2}$,

$$
\begin{aligned}
\mu_n(x)/\sqrt{1-x^2} &\le \frac{C}{n} \int_0^1 \frac{\Delta_n(s,x)}{\sqrt{1-s^2}} ds \\
&\le \frac{C}{n} \int_0^1 \frac{\Delta_n(s,1)}{\sqrt{1-s^2}} ds,
\end{aligned}
\tag{4.11}
$$

since $\Delta_n(s,x)$ is increasing in x for fixed s (as in the proof of Theorem 4.1(a)). let

$$
\begin{aligned}
A_n := a_n \int_0^1 \frac{\Delta_n(s,1)}{\sqrt{1-s^2}} ds \quad & \int_0^1 \frac{a_n Q'(a_n) - a_n s Q'(a_n s)}{(1-s)^{3/2}(1+s)^{1/2}} ds \\
\le & \int_0^1 \frac{a_n Q'(a_n) - a_n s Q'(a_n s)}{(1-s)^{3/2}} ds.
\end{aligned}
$$

Integrating by parts, we have

$$
\begin{aligned}
A_n &\le \frac{2}{(1-s)^{1/2}} (a_n Q'(a_n) - a_n s Q'(a_n s)) \Big|_{s=0}^{s=1} \\
&\quad - \int_0^1 \frac{2}{(1-s)^{1/2}} \frac{d}{ds} (a_n Q'(a_n) - a_n s Q'(a_n s)) ds \\
&\quad 2a_n Q'(a_n) + 2a_n \int_0^1 \frac{T(a_n s) Q'(a_n s)}{(1-s)^{1/2}} ds \\
&\le 4T(a_n) \int_{1/2}^1 \frac{Q'(a_n s)}{\sqrt{1-s}} ds \quad \text{(by the monotonicity of } Q' \text{ and } (1-s)^{-1/2}\text{)} \\
&\le C T(a_n) a_n^{-1} \int_{1/2}^1 \frac{a_n s Q'(a_n s)}{\sqrt{1-s^2}} ds \le C_1 T(a_n) n.
\end{aligned}
$$

Then from (4.11), we obtain for $\frac{1}{2} \leq |x| \leq 1$,

$$\mu_n(x)/\sqrt{1-x^2} \leq C_2 \frac{a_n}{n} A_n \leq C_3 T(a_n).$$

Since (4.1) implies that for $|x| \leq \frac{1}{2}$,

$$\mu_n(x) \leq C_4 \leq C_5 T(a_n),$$

we have our result. $\square$

Proof of Corollary 4.2. For $x^2 \leq 1 - 1/T(a_n)$, (4.1) implies

$$\mu_n(x) \leq \frac{C}{\sqrt{1-x^2}} \leq C_1 T(a_n)^{1/2}.$$

For $1 \geq x^2 \geq 1 - 1/T(a_n)$, inequality (4.2) implies

$$\mu_n(x) \leq CT(a_n)\sqrt{1-x^2} \leq CT(a_n)^{1/2} \qquad \square$$

Finally, we turn to the following.

Proof of Theorem 4.3. Let $0 < \beta < \Delta < 1$. Now from (2.1), and the positivity of

$$\frac{a_n s Q'(a_n s) - a_n x Q'(a_n x)}{a_n s \quad a_n x}$$

we see that for $|x| \leq a_{\beta n}/a_n$,

$$\sqrt{1-x^2}\mu_n(x) \geq \frac{2}{n\pi^2} \int_{a_{\Delta n}/a_n}^{1} \frac{1-x^2}{s^2-x^2} \frac{a_n s Q'(a_n s) - a_n x Q'(a_n x)}{\sqrt{1-s^2}} ds.$$

Now for $s \in [a_{\Delta n}/a_n, 1]$, and $|x| \leq a_{\beta n}/a_n$,

$$\begin{aligned} \frac{a_n s Q'(a_n s) - a_n x Q'(a_n x)}{a_n Q'(a_n)} &= \frac{a_n s Q'(a_n s)}{a_n Q'(a_n)} \left[1 - \frac{a_n x Q'(a_n x)}{a_n s Q'(a_n s)}\right] \\ &\geq s^{T(a_n)} \left[1 - \frac{a_{\beta n} Q'(a_{\beta n})}{a_{\Delta n} Q'(a_{\Delta n})}\right] \quad \text{(by (3.1))} \\ &\geq s^{T(a_n)} \left[\qquad \right] \end{aligned}$$

by (3.15). Furthermore, as $s^2 > x^2$ for $s \in [a_{\Delta n}/a_n, 1]$,

$$1 - x^2 \geq s^2 - x^2 > 0.$$

Hence

$$\sqrt{1-x^2}\mu_n(x) \geq \frac{2}{n\pi^2} \left[1 - \frac{\beta}{\Delta}\right] a_n Q'(a_n) \int_{a_{\Delta n}/a_n}^{1} s^{T(a_n)} \frac{ds}{\sqrt{1-s^2}}$$

Now by (3.13), for $n \geq 1$,

$$\frac{a_{\Delta n}}{a_n} \leq 1 - \frac{C}{T(a_n)}.$$

Hence if δ is small enough,

$$\begin{aligned}\sqrt{1-x^2}\mu_n(x) &\geq \frac{2}{n\pi^2}\left[1-\frac{\beta}{\Delta}\right]a_nQ'(a_n)\int_{1-\delta/T(a_n)}^{1} s^{T(a_n)}\frac{ds}{\sqrt{1-s^2}}\\ &\geq CT(a_n)^{1/2}\left(1-\frac{\delta}{T(a_n)}\right)^{T(a_n)}\int_{1-\delta/T(a_n)}^{1}\frac{ds}{\sqrt{1-s^2}}\\ &\geq C_1,\end{aligned}$$

by (3.9). □

We remark that we used (3.8) only in the last two lines of the above proof, in estimating $a_nQ'(a_n)$ from below.

5. Estimates for the majorisation function U_n. In this section, we estimate the quantity

$$\max_{|t-x|=\epsilon} \exp\left(nU_n\left(\frac{t}{a_n}\right)\right)$$

which appears in Lemma 2.2. Recall that the majorisation function is defined by (2.3).

THEOREM 5.1. *Let $w := e^{-Q} \in \mathcal{W}$. Let $0 < \beta < 1$, and for a fixed $\eta > 0$, let*

$$\epsilon_n := \eta(nT(a_n)^{1/2})^{-1} \qquad n \geq 1$$

Then

$$\max_{|t-x|\leq\epsilon_n} \exp\left(nU_n\left(\frac{t}{a_n}\right)\right) \leq C,$$

uniformly for $n \geq 1$ and for real x satisfying

$$|x| \leq a_{\beta n}$$

Proof. Let $|t-x| \leq \epsilon_n$. Then we can write $t/a_n = a + ib$, where

$$\begin{aligned}|a| \leq \frac{|x|+\epsilon_n}{a_n} &\leq \frac{a_{\beta n}+\eta/[nT(a_n)^{1/2}]}{a_n}\\ &\leq \frac{a_{\beta n}(1+C\eta/T(a_n))}{a_n}\end{aligned}$$

where in the last inequality we used the fact that $n^{-1} \leq C_1T(a_n)^{-1/2}$, which follows from (3.11). Now, if η is small enough, we get from (3.13) that $a_{\beta n}(1+C\eta/T(a_n))/a_n < 1$, and so

$$|x| + \epsilon_n < a_n$$

$$|b| \leq \frac{\epsilon_n}{a_n} \leq \frac{\epsilon_n}{a_1}.$$

Then since $|a| < 1$, $U_n(a) = 0$ (see (2.7)); so we have from (2.3),

$$
\begin{aligned}
U_n(t/a_n) &= U_n(a+ib) - U_n(a) \\
&= \int_{-1}^{1} \log\left|\frac{a+ib-t}{a-t}\right| \qquad\qquad n \qquad\qquad n \\
&\leq \frac{1}{2}\int_{-1}^{1} \log\left[\quad \left(\frac{b}{a-t}\right)^2\right] \mu_n(t)dt \\
&= \frac{1}{2}\int_{-1}^{1} \log\left[1+\left(\frac{|b|}{|a|-t}\right)^2\right] \mu_n(t)dt \quad \text{(as } \mu_n \text{ is even)} \\
&\leq \int_{0}^{1} \log\left[1+\left(\frac{|b|}{|a|-t}\right)^2\right] \mu_n(t)dt \\
&\leq C_1 T(a_n)^{1/2} \int_{0}^{1} \log\left[1+\left(\frac{|b|}{|a| \quad t}\right)^2\right] dt \quad \text{(by Corollary 4.2)} \\
&= C_1 T(a_n)^{1/2} |b| \int_{(|a|-1)/|b|}^{|a|/|b|} \log\left[\qquad\right] \quad \text{(substituting } |a| - t = s|b|\text{)} \\
&\leq C_1 T(a_n)^{1/2} \left(\frac{\epsilon_n}{a_1}\right) \int_{-\infty}^{\infty} \log\left[1+\frac{1}{s^2}\right] \quad \text{(by (5.5))} \\
&< \frac{C_2}{n}
\end{aligned}
$$

by the choice (5.1) of ϵ_n. Hence $\exp(nU_n(t/a_n)) \leq \exp(C_2)$. □

Note that in the above result, we used the bound in Corollary 4.2 for μ_n. Next, we use the bound of Theorem 4.1(a).

THEOREM 5.2. *Let $w \in \mathcal{W}$. Let $n \geq 1$, and for real x satisfying*

$$|x| \leq a_n \left(1 - \frac{4}{n^2 a_1^2}\right)$$

let

$$\epsilon_n(x) := \frac{1}{n}\left(1 \quad \frac{|x|}{a_n}\right)^{1/2}$$

$$\max_{|t-x| \leq \epsilon_n(x)} \exp\left(nU_n\left(\frac{t}{a_n}\right)\right) \leq C. \tag{5.9}$$

Proof. Let x satisfy (5.7), and for $|t \quad x| \leq \epsilon_n(x)$, write $t/a_n = a + ib$. Then

$$
\begin{aligned}
|a| &\le \frac{|x|+\epsilon_n(x)}{a_n} \\
&= 1 - \frac{1-|x|}{a_n} + \frac{\epsilon_n(x)}{a_n} \\
&\le 1 + \epsilon_n(x)\left[-n^2\epsilon_n(x) + \frac{1}{a_1}\right] \quad \text{(by (5.8))} \\
&\le 1 - \frac{n^2\epsilon_n^2(x)}{2},
\end{aligned} \tag{5.10}
$$

since (5.7) and (5.8) ensure that $\epsilon_n(x) \ge 2/(n^2 a_1)$. Furthermore,

$$|b| \le \frac{\epsilon_n(x)}{a_1}. \tag{5.11}$$

Since (2.7) shows that $U_n(a) = 0$, we obtain from (5.6),

$$
\begin{aligned}
U_n\left(\frac{t}{a_n}\right) &= U_n(a+ib) - U_n(a) \\
&\le \int_0^1 \log\left[1 + \left(\frac{|b|}{|a|-t}\right)^2\right]\mu_n(t)dt \\
&\le C\int_0^1 \log\left[1 + \left(\frac{|b|}{|a|-t}\right)^2\right]\frac{dt}{\sqrt{1-t}} \quad \text{(by Theorem 4.1(a))} \\
&= C|b|\int_{(|a|-1)/|b|}^{|a|/|b|} \log\left[1+\frac{1}{s^2}\right]\frac{ds}{\sqrt{1-|a|+s|b|}},
\end{aligned}
$$

by the substitution $|a| - t = s|b|$. Now let

$$\delta := \delta(a) := \frac{1-|a|}{2}. \tag{5.12}$$

We write

$$U_n\left(\frac{t}{a_n}\right) \le C|b|\left(\int_{-2\delta/|b|}^{-\delta/|b|} + \int_{-\delta/|b|}^{|a|/|b|}\right)\log\left[1+\frac{1}{s^2}\right]\frac{ds}{\sqrt{2\delta + s|b|}} =: I_1 + I_2. \tag{5.13}$$

Then

$$
\begin{aligned}
I_1 &= C|b|^{1/2}\int_{-2\delta/|b|}^{-\delta/|b|} \log\left[1+\frac{1}{s^2}\right]\frac{ds}{\sqrt{2\delta/|b| + s}} \\
&\le C|b|^{1/2}\log\left[1+\left(\frac{|b|}{\delta}\right)^2\right]\int_0^{\delta/|b|} u^{-1/2}du
\end{aligned} \tag{5.14}
$$

$$= C|b|^{1/2} \log \left[1 + \left(\frac{|b|}{\delta}\right)^2\right] 2 \left(\frac{\delta}{|b|}\right)^{1/2}$$

$$\leq 4C|b|\delta^{-1/2},$$

where we have used the inequality $\log[1+u^2] \leq 2u,\ u \in [0, \infty)$.

Next,

$$I_2 = C|b| \int_{-\delta/|b|}^{|a|/|b|} \log\left[1 + \frac{1}{s^2}\right] \frac{ds}{\sqrt{2\delta + s|b|}}$$

$$\leq C|b|\delta^{-1/2} \int_{-\delta/|b|}^{|a|/|b|} \log\left[1 + \frac{1}{s^2}\right] ds \tag{5.15}$$

$$\leq C|b|\delta^{-1/2} \int_{-\infty}^{\infty} \log\left[1 + \frac{1}{s^2}\right] ds.$$

So from (5.13) to (5.15) and (5.11),

$$nU_n\left(\frac{t}{a_n}\right) \leq nC_1|b|\delta^{-1/2} \leq nC_1\epsilon_n(x)\delta^{-1/2} \leq C_2,$$

since (5.12) and (5.10) show that

$$\delta = \frac{1 - |a|}{2} \geq \frac{n^2\epsilon_n^2(x)}{4}. \qquad \square$$

Finally, we use the bound of Theorem 4.1(b).

THEOREM 5.3. *Let $w \in \mathcal{W}$ and let $\eta > 0$ be fixed. For $n \geq 1$, and $|x| \leq a_n$, let*

$$\epsilon_n(x) := \eta \min\left\{(nT(a_n))^{-2/3}, \left(nT(a_n)\sqrt{1 - |x|/a_n}\right)^{-1}\right\}. \tag{5.16}$$

Then if η is small enough (independently of x),

$$\max_{|t-x| \leq \epsilon_n(x)} \exp\left(nU_n\left(\frac{t}{a_n}\right)\right) \leq C. \tag{5.17}$$

Proof. Let $|x| \leq a_n$ and for $|t - x| \leq \epsilon_n(x)$, write $t/a_n = a + ib$. Here,

$$|b| \leq \frac{\epsilon_n(x)}{a_n} \leq \frac{\epsilon_n(x)}{a_1}. \tag{5.18}$$

Furthermore, by (3.11),

$$nT(a_n) \geq C_1 T(a_n)^{3/2},$$

so

$$\epsilon_n(x) \leq \eta(nT(a_n))^{-2/3} \leq \eta C_2 T(a_n)^{-1},$$

and hence

$$|x| + \epsilon_n(x) \leq a_n\left(1 + \frac{C_3\eta}{T(a_n)}\right) \leq a_{2n}, \tag{5.19}$$

if η is small enough. In particular, then

$$\frac{|t|}{a_n} \le \frac{|x| + \epsilon_n(x)}{a_n} \le \frac{a_{2n}}{a_n} < 1. \tag{5.20}$$

Thus $U_n(t/a_n)$ is well defined for $|t-x| \le \epsilon_n(x)$. We now consider the cases $|a| \le 1$ and $|a| > 1$ separately.

Case I. $|a| \le 1$. Then $U_n(a) = 0$, and by (5.6),

$$\begin{aligned}
U_n\left(\frac{t}{a_n}\right) &= U_n(a+ib) - U_n(a) \\
&\le \int_0^1 \log\left[1 + \left(\frac{|b|}{|a|-t}\right)^2\right] \mu_n(t)\,dt \\
&\le CT(a_n)\int_0^1 \log\left[1 + \left(\frac{|b|}{|a|-t}\right)^2\right] \sqrt{1-t}\,dt \quad \text{(by Theorem 4.1(b))} \\
&= CT(a_n)|b| \int_{(|a|-1)/|b|}^{|a|/|b|} \log\left[1 + \frac{1}{s^2}\right] (1 - |a| + s|b|)^{1/2} ds \\
&\qquad \text{(substituting } |a| - t = s|b|\text{)} \\
&\le CT(a_n)|b| \int_{-\infty}^{\infty} \log\left[1 + \frac{1}{s^2}\right] \{(1-|a|)^{1/2} + |sb|^{1/2}\}\,ds \\
&\qquad \text{(by the inequality } (u+v)^{1/2} \le |u|^{1/2} + |v|^{1/2},\ u+v \ge 0\text{)} \\
&= CT(a_n)|b|(1-|a|)^{1/2} \int_{-\infty}^{\infty} \log\left[1 + \frac{1}{s^2}\right] ds \\
&\quad + CT(a_n)|b|^{3/2} \int_{-\infty}^{\infty} \log\left[1 + \frac{1}{s^2}\right] |s|^{1/2} ds \qquad (5.21) \\
&\le C_1\left\{T(a_n)\epsilon_n(x)\left(1 - \frac{|x|}{a_n} + \frac{\epsilon_n(x)}{a_n}\right)^{1/2} + T(a_n)\epsilon_n(x)^{3/2}\right\} \\
&\qquad \left(\text{by choice, } |a| \ge \tfrac{|x|}{a_n} - \tfrac{\epsilon_n(x)}{a_n}\right) \\
&\le C_1\left\{T(a_n)\epsilon_n(x)\left(1 - \frac{|x|}{a_n}\right)^{1/2} + \frac{T(a_n)\epsilon_n(x)^{3/2}}{a_n^{1/2}} + T(a_n)\epsilon_n(x)^{3/2}\right\} \\
&\le C_2\left\{T(a_n)\epsilon_n(x)\left(1 - \frac{|x|}{a_n}\right)^{1/2} + T(a_n)\epsilon_n(x)^{3/2}\right\}.
\end{aligned}$$

Then

$$nU_n\left(\frac{t}{a_n}\right) \le C_2\left\{nT(a_n)\epsilon_n(x)\left(1 - \frac{|x|}{a_n}\right)^{1/2} + nT(a_n)\epsilon_n(x)^{3/2}\right\} \le C_3,$$

by the choice (5.16) of $\epsilon_n(x)$.

Case II. $|a| > 1$. Since $U_n(a) < 0$ (see (2.8)), we have

$$U_n\left(\frac{t}{a_n}\right) = U_n(a+ib) - U_n(a) + U_n(a)$$

$$\leq \int_0^1 \log\left[1 + \left(\frac{|b|}{|a|-t}\right)^2\right] \mu_n(t)dt \quad \text{(by (5.6))}$$

$$\leq \int_0^1 \log\left[1 + \left(\frac{|b|}{1-t}\right)^2\right] \mu_n(t)dt,$$

as $|a| - t > 1 - t$, $t \in [0,1]$. The argument of Case I with $a = 1$ then shows that the above integral is bounded above by the right-hand side of (5.21) (with 1 substituted for a there). Thus,

$$nU_n\left(\frac{t}{a_n}\right) \leq nCT(a_n)|b|^{3/2} \int_{-\infty}^{\infty} \log\left[1 + \frac{1}{s^2}\right] |s|^{1/2} ds$$

$$\leq C_1 nT(a_n)\epsilon_n(x)^{3/2} \leq C_2. \qquad \square$$

6. Proofs of Theorems 1.2–1.5.

Proof of Theorem 1.2(i). First, let $0 < \beta < 1$, and let, as in (5.1),

$$\epsilon_n := \eta(nT(a_n)^{1/2})^{-1} \qquad n \geq 1.$$

Then Lemma 2.2 and (5.2) ensure that for $P \in \mathcal{P}_n$, and $|x| \leq a_{\beta n}$,

$$|(Pw)'(x)| \leq \epsilon_n^{-1} e^{\tau_n(x)} \|Pw\|_{L_\infty[-1,1]} C,$$

where, as in (2.13),

$$\tau_n(x) := \begin{cases} Q'(3\epsilon_n)2\epsilon_n, & \text{if } |x| \leq 2\epsilon_n, \\ \left[\frac{Q'(|x|+\epsilon_n)}{|x|} + Q''(|x|+\epsilon_n)\right]2\epsilon_n^2, & \text{if } |x| > 2\epsilon_n. \end{cases}$$

Since $\epsilon_n \to 0$ as $n \to \infty$,

$$Q'(3\epsilon_n)2\epsilon_n \leq C_1, \qquad n \geq 1.$$

Next, from (5.4), $|x| + \epsilon_n < a_n$, if η is small enough. So for $|x| \geq 2\epsilon_n$,

$$Q'(|x|+\epsilon_n)|x|^{-1}\epsilon_n^2 \leq \tfrac{1}{2}Q'(a_n)\epsilon_n \leq CnT(a_n)^{1/2}\epsilon_n$$

by (3.7) and the choice of ϵ_n. Furthermore, by Lemma 3.1(ii), if $|x| \geq \frac{1}{4}$,

$$Q''(|x|+\epsilon_n)\epsilon_n^2 \leq \frac{a_n}{|x|+\epsilon_n} Q''(a_n)\epsilon_n^2 \leq C_4 T(a_n)^{1/2} n^{-1} \leq C_5,$$

by (3.7), (3.11) and the choice of ϵ_n. If $|x| \leq \frac{1}{4}$,

$$Q''(|x|+\epsilon_n)\epsilon_n^2 \leq (|x|+\epsilon_n)Q''(|x|+\epsilon_n)\epsilon_n \leq \tfrac{1}{2}Q''(1/2)\epsilon_n \leq C_6,$$

by Lemma 3.1(ii). Thus we have shown that

$$\tau_n(x) \le C_7, \qquad |x| \le a_{\beta n}.$$

So for $P \in \mathcal{P}_n$, $n \ge 1$,

$$\|(Pw)'\|_{L_\infty[-a_{\beta n}, a_{\beta n}]} \le C_8 n T(a_n)^{1/2} \|Pw\|_{L_\infty[-1,1]} \tag{6.1}$$

Choosing $\beta = \frac{1}{2}$, and replacing n by $2n$, yields for $P \in \mathcal{P}_n \subset \mathcal{P}_{2n}$,

$$\|(Pw)'\|_{L_\infty[-a_n, a_n]} \le C_8 2n T(a_{2n})^{1/2} \|Pw\|_{L_\infty[-1,1]}.$$

Then, as

$$Q'(a_{2n}) = \mathrm{O}(nT(a_{2n})^{1/2}),$$

we obtain

$$\|P'w\|_{L_\infty[-a_n, a_n]} \le C_8 2n T(a_{2n})^{1/2} \|Pw\|_{L_\infty[-1,1]}.$$

The Mhaskar–Saff identity (1.9) then yields Theorem 1.2(i). □

Proof of Theorem 1.2(ii). Suppose that $n \ge 1$, and

$$|x| \le a_n \left(1 - \frac{A}{T(a_{2n})}\right) \tag{6.3}$$

where A is some fixed but large enough positive number. Note that then, by (3.11),

$$|x| \le a_n(1 - 4n^{-2} a_1^{-2}), \tag{6.4}$$

if only A is large enough. Thus (5.7) is satisfied. Let

$$\hat{\epsilon}_n(x) := \eta n^{-1} \left(1 - \frac{|x|}{a_n}\right)^{1/2}$$

where $\eta \in (0,1)$ is fixed and independent of n and x, but small enough. Note then that $\hat{\epsilon}_n(x) \le \epsilon_n(x)$, where $\epsilon_n(x)$ is defined by (5.8). Then Lemma 2.2 and (5.9) ensure that for $P \in \mathcal{P}_n$,

$$|(Pw)'(x)| \le \hat{\epsilon}_n(x)^{-1} e^{\tau_n(x)} \|Pw\|_{L_\infty[-1,1]} C,$$

$$\tau_n(x) := \begin{cases} Q'(3\hat{\epsilon}_n(x)) 2\hat{\epsilon}_n(x), & \text{if } |x| \le 2\hat{\epsilon}_n(x), \\ \left[\frac{Q'(|x| + \hat{\epsilon}_n(x))}{|x|} + Q''(|x| + \hat{\epsilon}_n(x))\right] 2\hat{\epsilon}_n(x)^2, & \text{if } |x| > 2\hat{\epsilon}_n(x). \end{cases}$$

Since $\hat{\epsilon}_n(x) \to 0$ as $n \to \infty$, uniformly for the above range of x, we have

$$\tau_n(x) \le C, \quad \text{if} \quad |x| \le 2\hat{\epsilon}_n(x).$$

Next, if $2\hat{\epsilon}_n(x) \le |x| \le \frac{1}{4}$, then for n large enough,

$$\begin{aligned} \tau_n(x) &= \left[\frac{Q'(|x| + \hat{\epsilon}_n(x))}{|x|} + Q''(|x| + \hat{\epsilon}_n(x))\right] 2\hat{\epsilon}_n(x)^2 \\ &\le 2Q'(|x| + \hat{\epsilon}_n(x))\hat{\epsilon}_n(x) + 2(|x| + \hat{\epsilon}_n(x)) Q''(|x| + \hat{\epsilon}_n(x))\hat{\epsilon}_n(x) \\ &\le \frac{2Q'(1/2)\eta}{n} + \frac{2(1/2)Q''(1/2)\eta}{n} \to 0, \end{aligned}$$

as $n \to \infty$, by Lemma 3.1(ii) If $|x| \geq \frac{1}{4}$, then we estimate $\tau_n(x)$ as follows: First note that by (6.4),

$$\frac{\hat{\epsilon}_n(x)}{(1-|x|/a_n)} = \eta n^{-1}\left(1-\frac{|x|}{a_n}\right)^{-1/2} \leq C_1\eta.$$

Hence,

$$\begin{aligned} 1-\frac{|x|+2\hat{\epsilon}_n(x)}{a_n} &= 1-\frac{|x|}{a_n}-\frac{2\hat{\epsilon}_n(x)}{a_n} \\ &\geq \left(1-\frac{|x|}{a_n}\right)\left(1-\frac{2C_1\eta}{a_1}\right) \\ &\geq \frac{1}{2}\left(1-\frac{|x|}{a_n}\right) \geq \frac{\eta}{2}\hat{\epsilon}_n(x), \end{aligned}$$

if η is small enough. Then as $|x| \geq \frac{1}{4}$, Lemma 3.1(ii) shows that

$$\begin{aligned} Q''(|x|+\hat{\epsilon}_n(x))\hat{\epsilon}_n(x)^2 &\leq \left(\int_{|x|+\hat{\epsilon}_n(x)}^{|x|+2\hat{\epsilon}_n(x)} Q''(t)dt\right)\frac{|x|+2\hat{\epsilon}_n(x)}{|x|+\hat{\epsilon}_n(x)}\hat{\epsilon}_n(x) \\ &\leq 2\hat{\epsilon}_n(x)Q'(|x|+2\hat{\epsilon}_n(x)) \\ &\leq C_2n^{-1}\left\{1-\frac{|x|+2\hat{\epsilon}_n(x)}{a_n}\right\}^{1/2} Q'(|x|+2\hat{\epsilon}_n(x)) \\ &\qquad \text{(by (6.5) and the choice of } \hat{\epsilon}_n(x)) \\ &\leq C, \end{aligned}$$

by (3.17) of Lemma 3.3. Also then,

$$Q'(|x|+\hat{\epsilon}_n(x))|x|^{-1}\hat{\epsilon}_n(x)^2 \leq 4Q'(|x|+2\hat{\epsilon}_n(x))\hat{\epsilon}_n(x) \leq C,$$

as above. So

$$\tau_n(x) \leq C,$$

uniformly for $|x| \leq a_n(1-A/T(a_{2n}))$, and we have

$$|(Pw)'(x)| \leq \hat{\epsilon}_n(x)^{-1}C_1\|Pw\|_{L_\infty[-1,1]} \leq \frac{C_2n}{\sqrt{1-|x|/a_n}}\|Pw\|_{L_\infty[-1,1]},$$

$P \in \mathcal{P}_n$, uniformly for $|x| \leq a_n(1-1/T(a_{2n}))$. On the other hand, if

$$a_n > |x| \geq a_n\left(1-\frac{A}{T(a_{2n})}\right)$$

then (6.2) shows that

$$|(Pw)'(x)| \leq C_3nT(a_{2n})^{1/2}\|Pw\|_{L_\infty[-1,1]} \leq \frac{C_3n}{\sqrt{1-|x|/a_n}}\|Pw\|_{L_\infty[-1,1]}$$

Summarizing, we have shown that

$$\max_{x\in[-a_n,a_n]} \left|(Pw)'(x)\sqrt{1-|x|/a_n}\right| \le C_4 n\|Pw\|_{L_\infty[-1,1]},$$

$P \in \mathcal{P}_n$, $n \ge 1$. Since (3.17) implies that

$$Q'(x)\sqrt{1-|x|/a_n} \le Cn,$$

for $\frac{1}{4} \le |x| \le 1$, and this inequality is trivial for $|x| \le \frac{1}{2}$, we have shown that

$$\max_{x\in[-a_n,a_n]} |P'w|(x)\sqrt{1-|x|/a_n} \le C_4 n\|Pw\|_{L_\infty[-1,1]},$$

$P \in \mathcal{P}_n$, $n \ge 1$, which completes the proof of the theorem. $\square$

Proof of Corollary 1.3. It follows directly from Theorem 1.2(i), (ii), that (1.18) holds for $|x| < a_n$, since

$$\left[\left(1-\frac{|x|}{a_n}\right)^{1/2} + T(a_{2n})^{-1/2}\right]^{-1} \sim \min\left\{\left|1-\frac{|x|}{a_n}\right|^{-1/2}, T(a_{2n})^{1/2}\right\}$$

We can then reformulate (1.18) for $|x| < a_n$ as

$$\left(\left(1-\left(\frac{x}{a_n}\right)^2\right)^2 + T(a_{2n})^{-2}\right)[(P'w)(x)]^4 \le Cn^2\|Pw\|^4_{L_\infty[-1,1]},$$

$|x| \le a_n$, $P \in \mathcal{P}_n$. Now $\psi(x) := \{(1-(x/a_n)^2)^2 + T(a_{2n})^{-2}\}P'(x)^4 \in \mathcal{P}_{4n}$, and by the Mhaskar–Saff identity applied to $w^4 = e^{-4Q}$, (for which $a_{4n}(4Q) = a_n(Q)$), we have

$$\|\psi w^4\|_{L_\infty[-1,1]} = \|\psi w^4\|_{L_\infty[-a_n,a_n]}.$$

Hence (6.9) holds for $x \in [-1,1]$, and so does (1.18). $\square$

Proof of Corollary 1.4(i). It obviously suffices to estimate $T(a_n)$ for $W_{0,\alpha} = e^{-Q}$ given by (1.4). A straightforward calculation shows that

$$T(x) = \frac{2(1+\alpha x^2)}{1-x^2} \sim (1-|x|)^{-1}, \qquad x \in (-1,1).$$

Furthermore,

$$Q'(x) \sim x(1-|x|)^{-\alpha-1}, \qquad x \in (-1,1),$$

and

$$T(x) \sim \frac{Q'(x)}{Q(x)}, \qquad x \text{ near } 1,$$

so by (3.9), for large enough n, $Q'(a_n) \sim nT(a_n)^{1/2}$, which implies

$$(1-a_n)^{-\alpha-1} \sim n(1-a_n)^{-1/2},$$

and hence

$$1-a_n \sim n^{-1/(\alpha+1/2)}$$

Then $T(a_n) \sim n^{1/(\alpha+1/2)}$, and Theorem 1.2(i) yields Corollary 1.4(i) □

Proof of Corollary 1.4(ii). Let

$$\exp_0(x) := x, \quad \exp_\ell(x) := \exp(\exp_{\ell-1}(x)), \quad \ell \geq 1,$$

and

$$F_0(x) := 1, \quad F_\ell(x) := \prod_{j=1}^{\ell} \exp_j(x), \quad \ell \geq 1$$

and

$$Q(x) \quad \exp_k((1-x^2)^{-\alpha})$$

Note that

$$\frac{d}{dx} \exp_\ell(x) = F_\ell(x), \qquad \ell \geq 0,$$

$$\frac{d}{dx} F_\ell(x) = F_\ell(x) \sum_{j=0}^{\ell-1} F_j(x).$$

A straightforward calculation shows that

$$Q'(x) \quad F_k((1-x^2)^{-\alpha}) 2\alpha x (1-x^2)^{-\alpha-1}$$

$$T(x) = \frac{2}{1-x^2} \left\{ \sum_{\ell=0}^{k-1} F_\ell((1-x^2)^{-\alpha}) \alpha x^2 (1-x^2)^{-\alpha} + 1 + \alpha x^2 \right\}$$

Hence for x close to 1

$$Q'(x) \sim F_k((1-x^2)^{-\alpha})(1-x^2)^{-\alpha-1}$$

$$T(x) \sim F_{k-1}((1-x^2)^{-\alpha})(1-x^2)^{-\alpha-1}$$

$$\frac{Q'(x)}{Q(x)}.$$

$$F_k((1-a_n^2)^{-\alpha})(1-a_n^2)^{-\alpha-1} \sim n \left[F_{k-1}((1-a_n^2)^{-\alpha})\right]^{1/2} (1 \quad a_n^2)^{-(\alpha+1)/2}$$

Taking logarithms shows that for n large enough,

$$\exp_{k-1}((1-a_n^2)^{-\alpha}) = \log n - \frac{1}{2} \sum_{j=0}^{k-2} \exp_j((1-a_n^2)^{-\alpha}) - \frac{\alpha+1}{2} \log(1-a_n^2) + O(1),$$

which implies that as $n \to \infty$,

$$\exp_{k-1}((1-a_n^2)^{-\alpha}) = \log n + O\left(\frac{\log\log n}{\log n}\right)$$

From this it readily follows that for $0 \le j \le k-1$, and n large enough,

$$\exp_j((1-a_n^2)^{-\alpha}) = \log_{k-j} n + o(1).$$

Then for n large enough,

$$T(a_n) \sim F_{k-1}((1-a_n^2)^{-\alpha})(1-a_n^2)^{-\alpha-1} \sim \left(\prod_{j=1}^{k-1} \log_{k-j} n \quad (\log_k n)^{(\alpha+1)/\alpha}\right.$$

$$= \left(\prod_{j=1}^{k-1} \log_j n\right)(\log_k n)^{(\alpha+1)/\alpha}.$$

Now Theorem 1.2(i) yields the result. □

Proof of Corollary 1.4(iii). Here $w \quad e^{-Q}$, where $Q(x) = -\alpha\log(1-x^2)$. Then

$$Q'(x) = \frac{2\alpha x}{1-x^2},$$

and a simple calculation shows that

$$T(x) \quad \frac{2}{1-x^2}.$$

Then (1.8) yields

$$n \sim \int_{1/2}^{1} \frac{Q'(a_n t)}{\sqrt{1-t^2}}dt \sim \int_{1/2}^{1} \frac{1}{\sqrt{1-t}(1-a_n t)}dt$$

$$= \int_0^{1/2} \frac{ds}{\sqrt{s}(1-a_n+a_n s)} \sim \int_0^{1-a_n} \frac{ds}{\sqrt{s}(1-a_n)} + \int_{1-a_n}^{1/2} \frac{ds}{s}$$

$$\sim (1-a_n)^{-1/2} + \log\left[\frac{1}{2(1 \quad a_n)}\right]$$

Then we deduce that

$$1 - a_n \sim n^{-2}$$

and hence

$$T(a_n) \sim n^2.$$

Again, Theorem 1.2(i) yields the result. □

Proof of Theorem 1.5(a). From (6.2) and (6.7),

$$|(Pw)'(x)| \le C_5 n \min\{T(a_{2n})^{1/2}, (1-|x|/a_n)^{-1/2}\}\|Pw\|_{L_\infty[-1,1]}$$

for $|x| < a_n$ and $P \in \mathcal{P}_n$. This immediately yields (1.22). $\square$

Proof of Theorem 1.5(b). We remark that (1.23) is implied by (1.22), in a somewhat stronger form, if $|x| \le \frac{1}{2}$. So we may assume that $|x| \ge \frac{1}{2}$. Let

$$\epsilon_n(x) := \eta \min\left\{(nT(a_{2n}))^{-2/3}, \left(nT(a_{2n})\sqrt{1-|x|/a_n}\right)^{-1}\right\}$$

Then by Lemma 2.2 and Theorem 5.3,

$$|(Pw)'(x)| \le \epsilon_n(x)^{-1} e^{\tau_n(x)} \|Pw\|_{L_\infty[-1,1]} C,$$

$P \in \mathcal{P}_n$, $\frac{1}{2} \le |x| \le a_n$, where

$$\begin{aligned}\tau_n(x) &= \left[\frac{Q'(|x|+\epsilon_n(x))}{|x|} + Q''(|x|+\epsilon_n(x))\right] 2\epsilon_n(x)^2 \\ &\le 4[Q'(|x|+\epsilon_n(x)) + Q''(|x|+\epsilon_n(x))]\epsilon_n(x)^2\end{aligned}$$

Now, by (3.11),

$$\epsilon_n(x) \le \eta(nT(a_{2n}))^{-2/3} \le \frac{C\eta}{T(a_{2n})}$$

Then by (3.13),

$$|x| + \epsilon_n(x) \le a_n\left(1 + \frac{C_1\eta}{T(a_{2n})}\right) \le a_{2n}$$

if only η is small enough. Hence the monotonicity of $Q'(u)$ and $uQ''(u)$ yield

$$\begin{aligned}\tau_n(x) &\le C[Q'(a_{2n}) + Q''(a_{2n})]\epsilon_n^2(x) \\ &\le C_1 nT(a_{2n})^{3/2}(nT(a_{2n}))^{-4/3} \quad \text{(by (3.7) and the choice of } \epsilon_n(x)) \\ &= C_1 n^{-1/3} T(a_{2n})^{1/6} \le C_2,\end{aligned}$$

by (3.11). Hence for $|x| < a_n$ and $P \in \mathcal{P}_n$,

$$\begin{aligned}|(Pw)'(x)| &\le \epsilon_n(x)^{-1} C_3 \|Pw\|_{L_\infty[-1,1]} \\ &\le C_4 \max\left\{(nT(a_{2n}))^{2/3}, nT(a_{2n})\left(1-\frac{|x|}{a_n}\right)^{1/2}\right\} \|Pw\|_{L_\infty[-1,1]}\end{aligned}$$

Then (1.23) follows for $|x| < a_n$ and (1.24) follows by continuity. $\square$

7. Proof of Theorem 1.6 and Corollary 1.7.

Proof of Theorem 1.6. Let $P \in \mathcal{P}_{n-1}$ and choose $\xi \in [-a_n, a_n]$ such that

$$|Pw|(\xi) = \|Pw\|_{L_\infty[-1,1]}$$

(7.1) $$\delta := \tfrac{1}{2}(CnT(a_{2n-2})^{1/2})^{-1}$$

where C is as in (1.22). Then for $t \in (\xi-\delta,\xi+\delta)\cap(-1,1)$, there is a $\zeta \in (\xi-\delta,\xi+\delta)\cap(-1,1)$ such that

$$\begin{aligned}|Pw|(t) &= |(Pw)(\xi)+(Pw)'(\zeta)(t-\xi)| \\ &\geq |Pw|(\xi) - \|(Pw)'\|_{L_\infty[-a_n,a_n]}\delta \\ &\geq \|Pw\|_{L_\infty[-1,1]} - \left(\frac{\delta^{-1}}{2}\right)\|Pw\|_{L_\infty[-1,1]}\delta = \frac{1}{2}\|Pw\|_{L_\infty[-1,1]},\end{aligned}$$

by (1.22). So if $x \in (-1,1)$,

$$\int_{-1}^{1}\frac{(Pw)^2(t)dt}{(Pw)^2(x)} \geq \int_{(\xi-\delta,\xi+\delta)\cap(-1,1)}\frac{1}{4}\|Pw\|^2_{L_\infty[-1,1]}dt/(Pw)^2(x) \geq \frac{\delta}{4}.$$

$$\lambda_n(w^2,x)w^{-2}(x) = \inf_{P\in\mathcal{P}_{n-1}}\int_{-1}^{1}(Pw)^2(t)dt/(Pw)^2(x) \geq \frac{\delta}{4}.$$

Taking account of the definition (7.1) of δ, we obtain (1.25). □

Proof of Corollary 1.7. We use a very standard argument (see, for example, [17]) but provide the details for the reader's convenience.

Step 1. Let

$$\rho_n := nT(a_{2n-2})^{1/2}, \qquad n \geq 1.$$

By Theorem 1.6, in the form proved above,

$$(Pw)^2(x) \leq C\rho_n\int_{-1}^{1}(Pw)^2(t)dt \quad \forall x\in[-1,1], \quad P\in\mathcal{P}_{n-1}.$$

Hence

$$\|Pw\|_{L_\infty[-1,1]} \leq C\rho_n^{1/2}\|Pw\|_{L_2[-1,1]}$$

Applying this to $w^k = e^{-kQ}$, and noting that $a_{2kn}(kQ) = a_{2n}(Q)$ yields for $P \in \mathcal{P}_{n-1}$,

$$\|P^kw^k\|_{L_\infty[-1,1]} \leq C_1\rho_n^{1/2}\|P^kw^k\|_{L_2[-1,1]},$$

and hence

$$\|Pw\|_{L_\infty[-1,1]} \leq C\rho_n^{1/(2k)}\|Pw\|_{L_{2k}[-1,1]}.$$

So we have (1.27) for $q=\infty$ and $p=2k$.

Step 2. Let $p>0$, and choose an integer k such that $2k>p$. Then by (7.2),

$$\begin{aligned}\|Pw\|^{2k}_{L_\infty[-1,1]} &\leq C\rho_n\int_{-1}^{1}(Pw)^{2k}(t)dt \\ &\leq C\rho_n\|Pw\|^{2k-p}_{L_\infty[-1,1]}\int_{-1}^{1}|Pw|^p(t)dt;\end{aligned}$$

so if P is not identically zero,

$$\|Pw\|^p_{L_\infty[-1,1]} \le C\rho_n \int_{-1}^{1} |Pw|^p(t)dt.$$

This is still trivially true if P is identically zero. Then (1.27) follows for $q = \infty$ and any $p > 0$.

Step 3. We let $p > 0$, and may assume $q < \infty$. Then

$$\begin{aligned}\|Pw\|^q_{L_q[-1,1]} &= \int_{-1}^{1} |Pw|^{q-p}(t)|Pw|^p(t)dt \\ &\le \|Pw\|^{q-p}_{L_\infty[-1,1]}\|Pw\|^p_{L_p[-1,1]} \\ &\le (C\rho_n)^{(q-p)/q}\|Pw\|^{q-p}_{L_q[-1,1]}\|Pw\|^p_{L_p[-1,1]},\end{aligned}$$

by (1.27) for the case already proven; so if P is not identically zero,

$$\|Pw\|^p_{L_q[-1,1]} \le (C\rho_n)^{(q-p)/q}\|Pw\|^p_{L_p[-1,1]} \qquad \square$$

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