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# $L_p$ EXTENSIONS OF GONCHAR'S INEQUALITY FOR RATIONAL FUNCTIONS

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ABSTRACT. Given a condenser $(E, F)$ in the complex plane, let $C(E, F)$ denote its capacity and let $\mu^* = \mu_E^* - \mu_F^*$ be the (signed) equilibrium distribution for $(E, F)$. Given a finite positive measure $\mu$ on $E \cup F$, let $G(\mu'_E) = \exp\left(\int \log(d\mu/d\mu_E^*)\, d\mu_E^*\right)$ and $G(\mu'_F) = \exp\left(\int \log(d\mu/d\mu_F^*)\, d\mu_F^*\right)$. We show that for $0 < p, q < \infty$ and for any rational function $r_n$ of order $n$

$$(\star) \qquad \|r_n\|_{L_p(d\mu, E)} \|1/r_n\|_{L_q(d\mu, F)} \geq e^{-n/C(E,F)} G^{1/p}(\mu'_E) G^{1/q}(\mu'_F),$$

which extends a classical result due to A. A. Gonchar. For a symmetric condenser we also obtain a sharp lower bound for $\|r_n - \lambda\|_{L_p(d\mu, E\cup F)}$, where $\lambda = \lambda(z)$ is equal to $0$ on $E$ and $1$ on $F$. The question of exactness of $(\star)$ and the relation to certain $n$-widths are also discussed.

Bibliography: 16 titles.

## §1. INTRODUCTION

Let $E$ and $F$ be disjoint closed sets of positive logarithmic capacity in the extended complex plane $\overline{\mathbf{C}}$. The pair $(E, F)$ is called a *condenser*. Put

$$(1.1) \qquad \sigma_n(E, F) := \sup_{\{r_n\}} \frac{\min_F |r_n|}{\max_E |r_n|},$$

where $\{r_n\}$ denotes that the supremum is taken over all rational functions $r_n$ ($\not\equiv 0$) of order $\leq n$. This quantity was introduced by A. A. Gonchar in [Go1] and [Go2]. He proved (cf. [Go3]) that

$$(1.2) \qquad \sigma_n(E, F) \leq e^{n/C(E,F)} \qquad n = 0, 1, 2, \ldots,$$

and that

$$(1.3) \qquad \liminf_{n\to\infty} [\sigma_n(E, F)]^{1/n} \geq e^{1/C(E,F)},$$

where $C(E, F)$ denotes the capacity of the condenser $(E, F)$. This capacity can be defined in several different ways. The one adopted by Gonchar enabled him to prove (1.2) under the additional assumption that one of the sets, say $E$, lies in a connected component of $\overline{\mathbf{C}} \setminus F$. Yet, his method can be modified to deal with the general case (cf. [Go3], Remark 1). Widom [W1] gave a very short proof of (1.2) for an arbitrary condenser. He used an alternative definition of $C(E, F)$ as the Green capacity of $E$ relative to the open set $\overline{\mathbf{C}} \setminus F$. Yet another definition was utilized by Saff and Totik ([ST], Chapter 7), who established a weighted analogue of (1.2). We

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also mention that T. Ganelius [Ga2] has shown that, under some further assumptions on $E$, $F$, inequality (1.3) can be improved to

$$\sigma_n(E, F) \geq \text{const} \cdot e^{n/C(E,F)}, \quad n = 1, 2, \ldots. \tag{1.4}$$

Returning to the definition of $\sigma_n$, observe that it can be rewritten as

$$\sigma_n^{-1}(E, F) = \inf_{\{r_n\}} \{\|r_n\|_{L_\infty(E)} \|1/r_n\|_{L_\infty(F)}\}. \tag{1.5}$$

It thus makes sense to introduce the quantity

$$\delta_n(E, F) = \delta_n(E, F; \|\cdot\|_E, \|\cdot\|_F) := \inf_{\{r_n\}} \{\|r_n\|_E \|1/r_n\|_F\}, \tag{1.6}$$

where $\|\cdot\|_E$, $\|\cdot\|_F$ are given norms on $E$ and $F$, respectively. Natural choices for these norms are $L_p$-norms (with respect to some measure) or the $L_\infty$-norm (with some weight function). For such norms, we will establish the analogue of (1.2). In particular, if $\mu_E^*$, $\mu_F^*$ are the probability measures on $E$, $F$ respectively, such that $\mu^* := \mu_E^* - \mu_F^*$ is the equilibrium distribution on $(E, F)$ (we define this in the next section), then for any $0 < p, q < \infty$ and for any rational function $r_n$ of order $\leq n$, the following inequality holds:

$$\|r_n\|_{L_p(E, d\mu_E^*)} \|1/r_n\|_{L_q(F, d\mu_F^*)} \geq e^{-n/C(E,F)}, \quad n = 0, 1, \ldots. \tag{1.7}$$

Note that since $\mu_E^*$, $\mu_F^*$ are unit measures, inequality (1.7) implies (1.2).

The estimates (1.2), (1.3) have immediate applications to approximation problems. Let $\lambda(z)$ be the function that is equal to 0 on $E$ and 1 on $F$, and set

$$\rho_n(E, F) := \inf_{\{r_n\}} \|\lambda - r_n\|_{L_\infty(E \cup F)}.$$

It is not difficult to show (see [Go3]) that

$$\frac{1}{\sigma_n^{1/2} + 1} \leq \rho_n \leq \frac{1}{\sigma_n^{1/2} - 1}, \quad n = 0, 1, \ldots, \tag{1.8}$$

and consequently, since $\sigma_n \to \infty$, we have

$$\rho_n \sim \sigma_n^{-1/2} = \delta_n^{1/2}(E, F; \|\cdot\|_{L_\infty(E)}, \|\cdot\|_{L_\infty(F)}), \quad \text{as } n \to \infty.$$

Unfortunately, there is no obvious relation between $\rho_n$ and $\delta_n$ for norms other than the sup-norm. For the general $L_p$-norm case, we are able to prove the lower bound for $\rho_n$ only for a symmetric condenser. Our method also applies to the case when $E$ and $F$ have a point in common. As an illustration, we prove that for $0 < p < \infty$ and for any $r_n$ of order $\leq n$, the following inequality holds:

$$\left\{\int_{-1}^0 |r_n(x)|^p\,dx + \int_0^1 |1 - r_n(x)|^p\,dx\right\}^{1/p} \geq c_p n^{1/2p} e^{-\pi\sqrt{n/p}}.$$

This inequality was proved by Vyacheslavov [V] for *real-valued* $r_n$ and for $p \geq 1$.

Another related matter is the Kolmogorov $n$-width. Let $\Omega$ be a domain in $\overline{\mathbb{C}}$ and let $A_\infty(\Omega)$ denote the collection of all functions $f$ that are analytic in $\Omega$ and satisfy $|f| \leq 1$ there. Let $K$ be a compact set in $\Omega$. Given a measure $\mu$ on $K$, the Kolmogorov $n$-width of $A_\infty(\Omega)$ with respect to the norm $\|\cdot\|_{L_p(K;d\mu)}$, $0 < p < \infty$, is defined by

$$d_n(A_\infty(\Omega), \|\cdot\|_{L_p(K;d\mu)}) := \inf_{X_n} \sup_{f \in A_\infty(\Omega)} \inf_{g \in X_n} \|f - g\|_{L_p(K;d\mu)}, \tag{1.10}$$

where $X_n$ runs through all $n$-dimensional subspaces of $L_p(K; d\mu)$. Using the method of Fisher and Micchelli [FM], we can show that

$$d_n(A_\infty(\Omega), \|\cdot\|_{L_p(K;d\mu)}) \le \inf_{\{r_n\}} \{\|r_n\|_{L_p(K;d\mu)} \|1/r_n\|_{L_\infty(\partial\Omega)}\}. \tag{1.11}$$

If $\Omega$ is a disk, the equality holds in (1.11). In this case, the results of O. Parfenov in [P1] and [P2] can be applied to get the strong asymptotics for $\delta_n(K, \partial\Omega)$.

## §2. Some basic facts

A convenient (for our purpose) definition of condenser capacity was given by T. Bagby in [B]. Throughout we assume that $E$, $F$ are disjoint closed subsets of $\overline{\mathbf{C}}$, each having positive logarithmic capacity.

(a) Let $\mu_E$ and $\mu_F$ be Borel probability measures on $E$ and $F$, respectively, and assume that both measures have finite logarithmic energy. Let $\mu = \mu_E - \mu_F$ denote the corresponding signed measure on $E \cup F$. For any such $\mu$, it is known (cf. [H], Theorem 16.4.2) that the integral

$$I(\mu) := \iint \log \frac{1}{|z-t|} \, d\mu(z) \, d\mu(t)$$

exists and is positive. Let

$$V :: \inf_{\mu} I(\mu). \tag{2.1}$$

It turns out that $0 < V < \infty$ since $E$ and $F$ both have positive logarithmic capacity. The condenser capacity $C(E, F)$ is then defined by

$$C(E, F) := V^{-1} \tag{2.2}$$

(some authors use $e^{-V}$ instead of $V^{-1}$ to define $C(E, F)$).

(b) The infimum in (2.1) is attained for the unique signed measure $\mu^* = \mu_E^* - \mu_F^*$, which is called the *equilibrium distribution* for $(E, F)$. The corresponding *equilibrium potential* is then defined by

$$u^*(z) := \int \log \frac{1}{|z-t|} \, d\mu^*(t). \tag{2.3}$$

It is known that $u^*$ exists and is finite for any $z \in \overline{\mathbf{C}}$ Moreover (cf. [B], Theorem 1), there exist finite constants $V_E$, $V_F$ such that

$$V_E \ge 0, \quad V_F \le 0; \tag{2.4}$$

$$V_E - V_F = V \tag{2.5}$$

$$V_F \le u^*(z) \le V_E \quad \text{for all } z \in \overline{\mathbf{C}}$$

$$u^*(z) = V_E \quad \text{quasi-everywhere on } E$$

$$u^*(z) = V_F \quad \text{quasi-everywhere on } F, \tag{2.8}$$

where quasi-everywhere (q.e.) means that the property holds except for a set of zero capacity. We also mention that $\mu^*$ is concentrated on $\partial E \cup \partial F$. Thus, $u^*$ is harmonic in the complement of $\partial E \cup \partial F$. It then follows from (2.7) that $u^*(z) = V_E$ on each component of $\overline{\mathbf{C}} \setminus \partial E$ that does not contain points of $F$. (An analogous statement concerning components of $\overline{\mathbf{C}} \setminus \partial F$ follows from (2.8).)

In particular, if $\infty \notin \partial E$ and if the unbounded component of $\overline{\mathbf{C}}\backslash\partial E$ does not intersect $F$, then $u^*(\infty) = 0$ (see (2.3) and note: $\int \log|z|^{-1}\, d\mu^*(t) = (\log|z|^{-1}) \int d\mu^* = 0$) and therefore, $V_E = 0$.

(c) A condenser $(E, F)$ is called *symmetric* if $E$ lies in the (open) upper half-plane and $F$ is the reflection of $E$ in the real axis. For such condensers, the equilibrium distribution $\mu^*$ is also symmetric, that is (cf. [B], Lemma 9), if $K \subset E$ is any Borel set and $\tilde{K}$ is its reflection in the real axis, then

$$\mu_E^*(K) = \mu_F^*(\tilde{K}). \tag{2.9}$$

It then follows from (2.3) that $u^*(\overline{z}) = -u^*(z)$ and therefore (by (2.7), (2.8))

$$V_E = -V_F = 1/(2C(E, F)). \tag{2.10}$$

(d) Given a (positive) measure $\mu$ on $E \cup F$, we denote by $\mu_E$, $\mu_F$ its restrictions on $E$, $F$, respectively. Let $\mu_E = \mu_E^s + \mu_E^a$ be the canonical decomposition of $\mu_E$ into the singular and the absolutely continuous parts with respect to $\mu_E^*$. We denote by $\mu_E'$ the Radon-Nikodým derivative $d\mu_E^a/d\mu_E^*$ of $\mu_E^a$ with respect to $\mu_E^*$. Let

$$G(\mu_E') := \exp\left(\int (\log \mu_E')\, d\mu_E^*\right) \tag{2.11}$$

be the geometric mean of $\mu_E'$. Similarly, we define $\mu_F'$ and $G(\mu_F')$. If both $G(\mu_E')$ and $G(\mu_F')$ are positive, we say that the measure $\mu$ satisfies the *Szegő condition* with respect to $\mu^*$.

## §3. Gonchar's inequality generalized

**Theorem 3.1.** *Let $(E, F)$ be a condenser and let $\mu$ be a finite (positive) Borel measure on $E \cup F$. Then, for any $0 < p, q < \infty$ and for any rational function $r_n$ of order $\le n$, we have*

$$\|r_n\|_{L_p(d\mu_E)} \|1/r_n\|_{L_q(d\mu_F)} \ge e^{-n/C(E,F)} G^{1/p}(\mu_E') G^{1/q}(\mu_F'), \tag{3.1}$$

*where $\mu_E$, $\mu_F$ denote the restrictions of $\mu$ to $E$, $F$ respectively.*

*Proof.* Let $\mu_E = \mu_E^s + \mu_E^a$. Since $\mu_E$ is positive, so is $\mu_E^s$. Hence,

$$\int |r_n|^p\, d\mu_E \ge \int |r_n|^p \mu_E'\, d\mu_E^*.$$

Assuming $\int (\log \mu_E')\, d\mu_E^* > -\infty$ (otherwise (3.1) is obvious) and applying Jensen's inequality, we obtain

$$\frac{1}{p} \log \int |r_n|^p \mu_E'\, d\mu_E^* \ge \int \log|r_n|\, d\mu_E^* + \frac{1}{p} \log G(\mu_E')$$

and similarly,

$$\frac{1}{q} \log \int |1/r_n|^q \mu_F'\, d\mu_F^* \ge -\int \log|r_n|\, d\mu_F^* + \frac{1}{q} \log G(\mu_F'),$$

where we used the fact that $\mu_E^*$ and $\mu_F^*$ are probability measures. Hence,

$$\log\left\{\|r_n\|_{L_p(d\mu_E)} \|1/r_n\|_{L_q(d\mu_F)}\right\} \ge \int \log|r_n|\, d\mu^* + \frac{1}{p} \log G(\mu_E') + \frac{1}{q} \log G(\mu_F').$$

It remains to show that

$$\int \log|r_n|\, d\mu^* \ge -n/C(E, F).$$

$$r_n(z) = A\prod_{i=1}^{k}(z-\alpha_i)\Big/\prod_{i=1}^{\ell}(z-\beta_i), \tag{3.3}$$

where $\max(k, \ell) \le n$. Since $\int d\mu^* = 0$, we obtain (see (2.3))

$$\int \log|r_n|\, d\mu^* = \sum_{i=1}^{\ell} u^*(\beta_i) - \sum_{i=1}^{k} u^*(\alpha_i).$$

Applying (2.6) and (2.4), we then obtain

$$\int \log|r_n| d\mu^* \ge \ell V_F - k V_E \ge -n(V_E - V_F) \tag{3.4}$$

and (3.2) follows. □

*Remark* 1. One can consider the measures $d\mu_E = w_E^p d\mu_E^*$, $d\mu_F = w_F^q d\mu_F^*$, where $w_E \in L_p(d\mu_E^*)$, $w_F \in L_q(d\mu_F^*)$. We then obtain

$$\|r_n w_E\|_{L_p(d\mu_E^*)}\|(1/r_n)w_F\|_{L_q(d\mu_F^*)} \ge e^{-n/C(E,F)}G(w_E)G(w_F). \tag{3.5}$$

If $w_E \in L_\infty(d\mu_E^*)$ and/or $w_F \in L_\infty(d\mu_F^*)$, then (3.5) holds also for $p = \infty$ and/or $q = \infty$. Furthermore, inequality (3.2) shows that (3.5) holds also for $p = 0$ and/or $q = 0$, where $\|f\|_{L_0}$ means $\exp\left(\int \log|f|\right)$.

*Remark* 2. The inequality (3.1) can be strengthened in the spirit of Remark 1 in [Go3]. For example, let $\infty \notin \partial E$ and assume that the unbounded component of $\overline{\mathbf{C}}\setminus\partial E$ does not intersect $F$. Let $D$ be the union of all components of $\overline{\mathbf{C}}\setminus\partial E$ that do not intersect $F$. Then (see the end of subsection 2(b)) $V_E = 0$ and $u^*(z) = 0$ on $D$. Thus, the proof of Theorem 3.1 shows that for any rational function $r$, we have:

$$\|r\|_{L_p(d\mu_E)}\|1/r\|_{L_q(d\mu_F)} \ge e^{-m/C(E,F)}G^{1/p}(\mu_E')G^{1/q}(\mu_F'), \tag{3.6}$$

where $m$ is the number of poles of $r(z)$ that are not in $D$. In particular, (3.6) holds for $r = P/Q_m$, where $Q_m$ is a polynomial of degree $m$ and $P$ is an arbitrary polynomial.

*Remark* 3. Let $(E, F)$ be a symmetric condenser (with $E$ in the upper half-plane) and let $P_n$ be a polynomial of degree $k \le n$. Then (3.4) and (2.10) give

$$\int \log|P_n|\, d\mu^* \ge -kV_E \ge -nV_E = -n/(2C(E, F)).$$

Since the equilibrium distribution $\mu^*$ is symmetric (see (2.9)) we may rewrite this in the form:

$$\int_E \log\left|\frac{P_n(z)}{P_n(\overline{z})}\right| d\mu_E^* \ge \frac{-n}{2C(E, F)}. \tag{3.7}$$

This inequality remains valid if we replace $\int_E$ by $\int_A$, where $A$ is any Borel subset of $E$. Indeed, write $P_n = Q_m R_{n-m}$, where the zeros of $Q_m$ lie in the open upper half-plane and those of $R_{n-m}$ lie in the closed lower half-plane. Then we have, for any $z$ with $\operatorname{Im} z \ge 0$,

$$\left|\frac{Q_m(z)}{Q_m(\overline{z})}\right| \le 1 \quad \text{and} \quad \left|\frac{P_n(z)}{P_n(\overline{z})}\right| \ge \left|\frac{Q_m(z)}{Q_m(\overline{z})}\right|.$$

Hence,

$$\int_A \log\left|\frac{P_n(z)}{P_n(\overline{z})}\right| d\mu_E^* \geq \int_A \log\left|\frac{Q_m(z)}{Q_m(\overline{z})}\right| d\mu_E^* \tag{3.8}$$
$$\geq \int_E \log\left|\frac{Q_m(z)}{Q_m(\overline{z})}\right| d\mu_E^* \geq \frac{-m}{2C(E, F)}.$$

Since $m \leq n$, we obtain

$$\int_A \log\left|\frac{P_n(z)}{P_n(\overline{z})}\right| d\mu_E^* \geq \frac{-n}{2C(E, F)}, \quad \text{for } A \subseteq E. \tag{3.9}$$

This can be viewed as a kind of Newman's inequality.

Finally, we discuss the sharpness of (3.1). Assuming that $\mu$ satisfies the Szegő condition with respect to $\mu^*$ we obtain from (3.1):

$$\liminf_{n\to\infty} \delta_n^{1/n}(E, F; \|\cdot\|_{L_p(d\mu_E)}, \|\cdot\|_{L_q(d\mu_F)}) \geq e^{-1/C(E,F)}$$

where $\delta_n$ is defined in (1.6). The matching upper bound for $\limsup \delta_n^{1/n}$ follows from (1.3). Hence, the exponent in (3.1) is sharp.

The following example shows that in certain geometric settings the factor $G^{1/p}(\mu_E')G^{1/q}(\mu_F')$ is the best possible.

Let $E$ be the circle $|z| = \rho < 1$ and let $F$ be the unit circle $|z| = 1$. In this case, $e^{-1/C(E,F)} = \rho$ and $d\mu_E^*(\rho e^{i\theta}) = d\mu_F^*(e^{i\theta}) = (1/2\pi)d\theta$. Let $d\mu_E$, $d\mu_F$ be arbitrary Borel probability measures on $E$ and on $F$, respectively. By a result of Parfenov [P1] (see also [LS]), given $0 < p, q < \infty$ there exist sequences $\{B_n^{(1)}\}$, $\{B_n^{(2)}\}$ of Blaschke products of respective orders $n$ such that

$$\|B_n^{(1)}(z)\|_{L_p(d\mu_E)} \sim \rho^n G^{1/p}(\mu_E') \quad \text{and} \quad \|B_n^{(2)}(\rho/z)\|_{L_q(d\mu_F)} \sim \rho^n G^{1/q}(\mu_F'),$$

where $a_n \sim b_n$ means $a_n/b_n \to 1$ as $n \to \infty$.

Define the sequence $\{r_n\}$ of rational functions of order $\leq n$, by

$$r_n(z) := B_{n/2}^{(1)}(z)/B_{n/2}^{(2)}(\rho/z), \quad \text{if } n \text{ is even},$$

$$r_n(z) := zr_{n-1}(z), \quad \text{if } n \text{ is odd}.$$

It is then obvious that

$$\|r_n\|_{L_p(d\mu_E)}\|1/r_n\|_{L_q(d\mu_F)} \sim \rho^n G^{1/p}(\mu_E')G^{1/q}(\mu_F') \quad \text{as } n \to \infty.$$

## §4. Approximation of functions constant on two disjoint sets

Given a condenser $(E, F)$, let $\lambda$ denote the function that equals 0 on $E$ and 1 on $F$. Given $0 < p < \infty$ and given a finite (positive) measure $\mu$ on $E \cup F$, set

$$\rho_n(E, F, \|\cdot\|_{L_p(d\mu)}) := \inf_{\{r_n\}} \|\lambda - r_n\|_{L_p(d\mu)}. \tag{4.1}$$

Gonchar's result (1.3) and the estimate (1.8) imply that

$$\limsup_{n\to\infty} \rho_n^{1/n}(E, F, \|\cdot\|_{L_p(d\mu)}) \leq e^{-1/2C(E,F)}. \tag{4.2}$$

The lower bound (for a symmetric condenser) is given by

**Theorem 4.1.** *Let $(E, F)$ be a symmetric condenser (with $E$ in the upper half-plane) and assume that the finite (positive) measure $\mu$ satisfies the Szegő condition with respect to the equilibrium distribution $\mu^*$ for $(E, F)$. Then, for $0 < p < \infty$, we have*

$$\rho_n(E, F, \|\cdot\|_{L_p(d\mu)}) \geq (1+o(1))[G(\mu'_E)G(\mu'_F)]^{1/2p}e^{-n/2C(E,F)},$$

*where $\mu_E$, $\mu_F$ denote the restrictions of $\mu$ to $E$, $F$, respectively.*

*Proof.* For $n = 1, 2, \dots$, let $r_n$ be a rational function that realizes the infimum in (4.1). It then follows from (4.2) that

$$\int |r_n|^p \mu'_E \, d\mu^*_E + \int |1-r_n|^p \mu'_F \, d\mu^*_F \leq \rho^p_n \leq c\eta^n, \tag{4.4}$$

where $0 < \eta < 1$ and $c > 0$ are independent of $n$. Set

$$E_n := \{z \in E : |r_n(z)| > 1 - 2^{-1/p}\}, \quad F_n := \{z \in F : |1 - r_n(z)| > 1 - 2^{-1/p}\}. \tag{4.5}$$

Then we have by (4.4) that

$$\int_{E_n} \mu'_E \, d\mu^*_E + \int_{F_n} \mu'_F \, d\mu^*_F \leq 2c\eta^n/(2^{1/p} - 1)^p. \tag{4.6}$$

Since $\mu$ satisfies the Szegő condition, (4.6) implies that (as we will show later)

$$\mu^*_E(E_n) + \mu^*_F(F_n) = o(1/n). \tag{4.7}$$

Let $A_n := E_n \cup \tilde{F}_n$ (where $\sim$ denotes the reflection of a set about the real axis). Since $\mu^*$ is symmetric, we obtain

$$\alpha_n := \mu^*_E(A_n) = \mu^*_F(\tilde{A}_n) = o(1/n). \tag{4.8}$$

By the definition of $E_n$, $F_n$ and $A_n$, we obtain (see (4.4)):

$$\rho^p_n \geq \frac{1}{2}\int_{E\setminus A_n} \left|\frac{r_n}{1-r_n}\right|^p \mu'_E \, d\mu^*_E + \frac{1}{2}\int_{F\setminus \tilde{A}_n} \left|\frac{1-r_n}{r_n}\right|^p \mu'_F \, d\mu^*_F$$

and therefore (by the concavity of log):

$$p \log \rho_n \geq \frac{1}{2}\log \int_{E\setminus A_n} \left|\frac{r_n}{1-r_n}\right|^p \mu'_E \, d\mu^*_E + \frac{1}{2}\log \int_{F\setminus \tilde{A}_n} \left|\frac{1-r_n}{r_n}\right|^p \mu'_F \, d\mu^*_F \tag{4.9}$$

For the first integral, we have, by Jensen's inequality (see also (4.8)),

$$\begin{aligned}
&\log \int_{E\setminus A_n} \left|\frac{r_n}{1-r_n}\right|^p \mu'_E \, d\mu^*_E \\
&\geq p\int_{E\setminus A_n} \log\left|\frac{r_n}{1-r_n}\right| \frac{d\mu^*_E}{1-\alpha_n} + \frac{1}{1-\alpha_n}\int_{E\setminus A_n} \log \mu'_E \, d\mu^*_E + \log(1-\alpha_n) \\
&= p\int_{E\setminus A_n} \log\left|\frac{r_n}{1-r_n}\right| \frac{d\mu^*_E}{1-\alpha_n} + \log G(\mu'_E) + o(1).
\end{aligned}$$

(Here we only used the fact that $\alpha_n = o(1)$ A similar inequality holds for the second integral in (4.9). We thus obtain

$$\begin{aligned}
\log \rho_n \geq o(1) &+ \frac{1}{2p}\log[G(\mu'_E)G(\mu'_F)] \\
&+ \frac{1}{2(1-\alpha_n)}\left\{\int_{E\setminus A_n} \log\left|\frac{r_n}{1-r_n}\right| d\mu^*_E + \int_{F\setminus \tilde{A}_n} \log\left|\frac{1-r_n}{r_n}\right| d\mu^*_F\right\}
\end{aligned}$$

By the symmetry, the expression in the braces can be put into the form

$$\int_{E\setminus A_n} \log\left|\frac{r_n(z)}{1-r_n(z)}\,\frac{1-r_n(\overline{z})}{r_n(\overline{z})}\right| d\mu_E^*.$$

Since this last integrand is of the form $\log|P_{2n}(z)/P_{2n}(\overline{z})|$, where $P_{2n}$ is a polynomial of degree $\leq 2n$, it follows from (3.9) that the last term in (4.10) satisfies

$$\frac{1}{2(1-\alpha_n)}\{\cdot\} \geq \frac{1}{2(1-\alpha_n)}\,\frac{-2n}{2C(E\,,F)} = -\frac{n}{2C(E\,,F)} + o(1)$$

(since $\alpha_n = o(1/n)$ by (4.8)), and the proof is complete provided we establish (4.7). Actually we shall prove the following slightly more general statement:

Let $\sigma$ be a finite (positive) measure on a compact set $K$. Let $f \in L_1(K\,; d\sigma)$ and assume also that $\log f \in L_1(K\,; d\sigma)$. Let $\{K_n\}$ be a sequence of subsets of $K$ such that $\sigma(K_n) > 0$ and $\varepsilon_n := \int_{K_n} f\,d\sigma \to 0$ as $n \to \infty$. Then $\sigma(K_n) = o(1/\log(1/\varepsilon_n))$.

Indeed, set $a_n := \int_{K_n} \log f\,d\sigma$. Since $\log f \in L_1$, we have $a_n \geq -a$ for some $a > 0$ independent of $n$. Hence,

$$\begin{aligned}\log\varepsilon_n = \log\int_{K_n} f\,d\sigma &\geq \frac{1}{\sigma(K_n)}\int_{K_n}\log f\,d\sigma + \log\sigma(K_n)\\ &\geq \frac{-a}{\sigma(K_n)} + \log\sigma(K_n).\end{aligned}\tag{4.11}$$

Since $\varepsilon_n \to 0$, it follows from (4.11) that $\sigma(K_n) \to 0$ as $n \to \infty$. By the absolute continuity of integral, we then obtain

$$a_n = \int_{K_n} \log f\,d\sigma \to 0 \quad \text{as } n \to \infty.$$

Returning to inequality (4.11), we see that

$$\log\varepsilon_n \geq \frac{a_n}{\sigma(K_n)} + \log\sigma(K_n)\,, \qquad a_n \to 0\,,$$

and the result follows. □

*Remark.* We assumed that the condenser $(E\,,F)$ is symmetric about the real axis. Given a condenser $(E'\,,F')$ that is symmetric about some straight line or about some circle, we may use a suitable Möbius transformation to map $(E'\,,F')$ onto the condenser $(E\,,F)$ considered above. This transformation preserves rational functions of given order and transfers (in a natural way) the equilibrium distribution on $(E'\,,F')$ to that on $(E\,,F)$. Thus, Theorem 4.1 is valid for any symmetric condenser.

## §5. Approximation of the unit step function on $[-1\,,1]$

In this section, we prove the estimate (1.9). Let

$$\Delta_n^p := \inf_{\{r_n\}}\left\{\int_{-1}^{0}|r_n(x)|^p\,dx + \int_0^1|1-r_n(x)|^p\,dx\right\}\tag{5.1}$$

and let $r_n^*$ denote a function that realizes the infimum. For any $0 < \varepsilon < 1$, we have

$$\Delta_n^p > \int_{-1}^{-\varepsilon}|r_n^*|^p\,dx + \int_\varepsilon^1|1-r_n^*|^p\,dx.\tag{5.2}$$

It is well known that the equilibrium distribution for the condenser

$$(E\,,F) = ([-1\,,-\varepsilon]\,,[\varepsilon\,,1])$$

is given by

$$d\mu_E^*(-x) = d\mu_F^*(x) = \frac{dx}{K'(\varepsilon)\sqrt{(x^2-\varepsilon^2)(1-x^2)}}, \quad \varepsilon \le x \le 1,$$

and that $1/C(E, F) = 2\pi K(\varepsilon)/K'(\varepsilon)$, where $K(\varepsilon)$, $K'(\varepsilon)$ denote the complete elliptic integrals for moduli $\varepsilon$ and $\sqrt{1-\varepsilon^2}$, respectively. We also have, as $\varepsilon \to 0$,

$$K(\varepsilon) = \pi/2 + o(1), \quad K'(\varepsilon) = \log(1/\varepsilon) + O(1).$$

Using these facts we could proceed as in §4 by estimating the right-hand side of (5.2) from below and choosing $\varepsilon$ in an "optimal" way. But it will be simpler to utilize the measure $dx/(x\log(1/\varepsilon))$ (a unit measure on $[\varepsilon, 1]$) instead of $d\mu^*(x)_E$ and to apply the classical Newman's inequality (cf. [N])

$$\text{(5.3)} \qquad \int_a^b \log\left|\frac{P_n(x)}{P_n(-x)}\right| \frac{dx}{x} \ge -\frac{\pi^2 n}{2}, \quad 0 < a < b.$$

Notice that (5.3) implies (in the same way as (3.7) implies (3.8)) that

$$\text{(5.4)} \qquad \int_K \log\left|\frac{P_n(x)}{P_n(-x)}\right| \frac{dx}{x} \ge -\frac{\pi^2 n}{2}, \quad K \subset (0, \infty).$$

Returning to (5.2), let us introduce the exceptional set

$$A_n := \{x \in [\varepsilon, 1] : |1 - r_n^*(x)|^p > \tfrac{1}{2} \text{ or } |r_n^*(-x)|^p > \tfrac{1}{2}\}.$$

From (5.2), we see that $\operatorname{meas}(A_n) := \int_{A_n} dx \le 2\Delta_n^p$. Set (compare with (4.8))

$$\alpha_n := \int_{A_n} \frac{dx}{x\log(1/\varepsilon)}$$

and observe that

$$\text{(5.5)} \qquad \alpha_n \le \int_\varepsilon^{\varepsilon+2\Delta_n^p} \frac{dx}{x\log(1/\varepsilon)} = \log\left(1 + \frac{2\Delta_n^p}{\varepsilon}\right) \Big/ \log\frac{1}{\varepsilon}$$

Proceeding as in §4 (with $dx/(x\log(1/\varepsilon))$ instead of $d\mu_E^*$ and with $d\mu_E$ $x\log(1/\varepsilon)dx$) we obtain

$$\text{(5.6)} \qquad \begin{aligned} \log\Delta_n \ge{}& \frac{1}{2(1-\alpha_n)} \int_{[\varepsilon, 1]\setminus A_n} \log\left|\frac{P_{2n}(x)}{P_{2n}(-x)}\right| \frac{dx}{x\log(1/\varepsilon)} \\ &+ \frac{1}{p(1-\alpha_n)} \int_{[\varepsilon, 1]\setminus A_n} \log\left(x\log\frac{1}{\varepsilon}\right) \frac{dx}{x\log(1/\varepsilon)} + \frac{1}{p}\log(1-\alpha_n). \end{aligned}$$

Applying (5.4) and observing that

$$\int_{[\varepsilon, 1]\setminus A_n} \log x \frac{dx}{x} \ge \int_\varepsilon^1 \frac{\log x}{x}\, dx = -\frac{1}{2}\log^2\frac{1}{\varepsilon}$$

we get from (5.6) that

$$\text{(5.7)} \qquad \log\Delta_n \ge -\frac{1}{2(1-\alpha_n)}\left\{\frac{\pi^2 n}{\log(1/\varepsilon)} + \frac{1}{p}\log\frac{1}{\varepsilon}\right\} + \frac{1}{p}\log\log\frac{1}{\varepsilon} + \frac{1}{p}\log(1-\alpha_n)$$

Now we choose

$$\text{(5.8)} \qquad \varepsilon := \Delta_n^p$$

We then see from (5.5) that

$$\text{(5.9)} \qquad \alpha_n \le c/\log(1/\Delta_n)$$

Using (5.8) and (5.9), we obtain from (5.7) that

$$\log \Delta_n \geq -\frac{1}{2}\left\{\frac{\pi^2 n}{p\log(1/\Delta_n)} + \log\frac{1}{\Delta_n}\right\} - c_1\frac{n}{\log^2(1/\Delta_n)} + \frac{1}{p}\log\left(p\log\frac{1}{\Delta_n}\right) + O(1). \tag{5.10}$$

To obtain the desired lower bound, define $\tau_n$ by

$$\Delta_n =: \tau_n n^{1/2p} e^{-\pi\sqrt{n/p}} \tag{5.11}$$

and assume that $\tau_n \to 0$ as $n \to \infty$ This will lead to a contradiction. Indeed, (5.1 implies that

$$\log\frac{1}{\Delta_n} \geq \pi\sqrt{\frac{n}{p}} - \frac{}{2p}\log n \quad (\text{for } n \text{ large}).$$

Hence (5.10) gives (after subtracting $\frac{1}{2}\log\Delta_n$ from both sides)

$$\frac{1}{2}\log\Delta_n \geq -\frac{1}{2}\pi\sqrt{\frac{n}{p}} - \frac{1}{4p}\log n + \frac{1}{2p}\log n + O(1).$$

Replacing here $\Delta_n$ by (5.11), we obtain

$$\tfrac{1}{2}\log\tau_n \geq O(1)$$

which contradicts the assumption $\tau_n \to 0$.

## §6. Estimates for $n$-widths

We first prove the estimate .11).

**Theorem 6.1.** *Let $\Omega$ be a domain in $\overline{\mathbf{C}}$ and let $A_\infty(\Omega)$ denote the collection of all functions $f$ that are analytic in $\Omega$ and satisfy $|f| \leq 1$ there. Let $K$ be a compact set in $\Omega$ and let $\mu$ be a finite (positive) measure on $K$. Then, for $0 < p < \infty$, we have*

$$d_n(A_\infty(\Omega), \|\cdot\|_{L_p(K;d\mu)}) \leq \inf_{\{r_n\}} \|r_n\|_{L_p(K;d\mu)} \|1/r_n\|_{L_\infty(\partial\Omega)}, \tag{6.1}$$

*where the infimum is taken over all rational functions of order $\leq n$.*

*Proof.* Assume first that $\Omega$ is bounded by a finite number of disjoint analytic simple closed curves. Let $g(z, \xi)$ be the Green function for $\Omega$ with singularity at $\xi$. Then the conjugate function $\tilde{g}(z, \xi)$ is multiple-valued, but the function

$$\varphi(z, \xi) := \frac{\partial}{\partial z}(g(z, \xi) + i\tilde{g}(z, \xi))$$

is analytic and single-valued in $\Omega$. The following facts are well known:

(a) $\varphi(z, \xi)$ has a simple pole at $\xi$ with the residue 1.

(b) $\varphi(z, \xi)$ is continuous in both arguments.

(c)
$$\frac{1}{2\pi}\int_{\partial\Omega}|\varphi(z, \xi)||dz| = \frac{1}{2\pi}\int_{\partial\Omega}\frac{\partial g(z, \xi)}{\partial\nu_z}|dz| = 1$$

($\nu_z$ denotes the inward normal to $\partial\Omega$ at $z \in \partial\Omega$).

In order to prove (6.1) it suffices to consider rational functions with simple zeros, and we may assume that they do not lie on $\partial\Omega$. Let $R(z)$ be such a function and let $z_1, \dots, z_m$ $(m \le n)$ denote those zeros of $R$ that belong to $\Omega$ For any $f$ that is analytic in $\overline{\Omega}$, we have by (a), (b) that

$$\frac{1}{2\pi i}\int_{\partial\Omega}\frac{R(\xi)}{R(z)}f(z)\varphi(z,\xi)\,dz = f(\xi) + R(\xi)\sum_{i=1}^{m}\frac{f(z_i)\varphi(z_i,\xi)}{R'(z_i)}.$$

If $f| \le 1$ in $\overline{\Omega}$, then by (c) we get

$$\left|\frac{1}{2\pi i}\int_{\partial\Omega}\frac{R(\xi)}{R(z)}f(z)\varphi(z,\xi)\,dz\right| \le |R(\xi)|\cdot\|1/R\|_{L_\infty(\partial\Omega)}\frac{1}{2\pi}\int_{\partial\Omega}|\varphi(z,\xi)||dz|$$
$$= |R(\xi)|\cdot\|1/R\|_{L_\infty(\partial\Omega)}.$$

Thus, (6.2) gives that the distance in $L_p(K; d\mu)$ from $f(\xi)$ to span $\{R(\xi)\varphi(z_i,\xi)\}_{i=1}^m$ does not exceed $\|R\|_{L_p(K;d\mu)}\|1/R\|_{L_\infty(\partial\Omega)}$. Since any $f \in A_\infty(\Omega)$ can be approximated (uniformly on $K$) by those $f \in A_\infty(\Omega)$ that are analytic in $\overline{\Omega}$, (6.1) follows.

It remains to remove the restriction we put on $\Omega$. This is done in a standard way (see the proof of Theorem 4 in [FM]). □

Assume now that $\Omega$ is the unit disk $\Delta := \{z : |z| < 1\}$. For this case, Fisher and Micchelli proved ([FM], Theorem 1) that

$$d_n(A_\infty(\Delta), L_p(K; d\mu)) = \inf_{\{B_n\}}\|B_n\|_{L_p(K;d\mu)}, \tag{6.3}$$

where the infimum is taken over all Blaschke products of order $\le n$. Since $\|1/B_n\|_{L_\infty(\partial\Omega)} = 1$, we obtain from (6.1), (6.3) that

$$d_n(A_\infty(\Delta), L_p(K; d\mu)) = \inf_{\{r_n\}}\|r_n\|_{L_p(K;d\mu)}\|1/r_n\|_{L_\infty(\partial\Omega)}. \tag{6.4}$$

This equality provides another example of the exactness of the generalized Gonchar inequality. Let $p = 2$ and assume that $K$ is a simple closed analytic curve. Let $d\mu = w(z)|dz|$, where $w(z)$ is a continuous positive function on $K$ and $|dz|$ is the arc length. Let $A_2(\Delta)$ denote the unit ball of the Hardy space $H^2(\Delta)$.

For this case, Parfenov found the strong asymptotics for the $n$-width. Although stated in different terms, the result reads (see [P2])

$$d_n(A_2(\Delta), L_2(K; d\mu)) \sim e^{-n/C(K,\partial\Delta)}G(\mu')^{1/2}, \tag{6.5}$$

where $\mu'$ denotes, as before, the derivative $d\mu/d\mu^*$. Since $A_2(\Delta) \supset A_\infty(\Delta)$, we have

$$d_n(A_\infty(\Delta), L_2(K; d\mu)) \le d_n(A_2(\Delta), L_2(K; d\mu)).$$

This inequality, together with (3.1) (for $p = 2$, $q = \infty$) and (6.4) implies

$$\inf_{\{r_n\}}\|r_n\|_{L_p(K;d\mu)}\|1/r_n\|_{L_\infty(\partial D)} \sim e^{-n/C(K,\partial D)}G(\mu')^{1/2}.$$

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