**L_p EXTENSIONS OF GONCHAR'S INEQUALITY FOR RATIONAL FUNCTIONS**

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**Abstract.** Given a condenser \((E, F)\) in the complex plane, let \(C(E, F)\) denote its capacity and let \(\mu^* = \mu^*_E - \mu^*_F\) be the (signed) equilibrium distribution for \((E, F)\). Given a finite positive measure \(\mu\) on \(E \cup F\), let \(G(\mu_E) = \exp \left( \int \log(d\mu/d\mu^*_E) d\mu^*_E \right)\) and \(G(\mu_F) = \exp \left( \int \log(d\mu/d\mu^*_F) d\mu^*_F \right)\). We show that for \(0 < p, q < \infty\) and for any rational function \(r_n\) of order \(n\)

\[
\|r_n\|_{L_p(d\mu, E)} \|1/r_n\|_{L_q(d\mu, F)} \leq e^{-n/C(E, F)} G^{1/p}(\mu_E) G^{1/q}(\mu_F),
\]

which extends a classical result due to A. A. Gonchar. For a symmetric condenser we also obtain a sharp lower bound for \(\|r_n - \lambda\|_{L_p(d\mu, E \cup F)}\), where \(\lambda = \lambda(z)\) is equal to 0 on \(E\) and 1 on \(F\). The question of exactness of (\(*\)) and the relation to certain \(n\)-widths are also discussed.

**Bibliography:** 16 titles.

§1. INTRODUCTION

Let \(E\) and \(F\) be disjoint closed sets of positive logarithmic capacity in the extended complex plane \(\overline{C}\). The pair \((E, F)\) is called a condenser. Put

\[
\sigma_n(E, F) := \sup_{\{r_n\}} \max_E |r_n| / \min_F |r_n|,
\]

where \(\{r_n\}\) denotes that the supremum is taken over all rational functions \(r_n \neq 0\) of order \(\leq n\). This quantity was introduced by A. A. Gonchar in [Go1] and [Go2]. He proved (cf. [Go3]) that

\[
\sigma_n(E, F) \leq e^{n/C(E, F)}, \quad n = 0, 1, 2, \ldots ,
\]

and that

\[
\liminf_{n \to \infty} \sigma_n(E, F) \leq e^{1/C(E, F)},
\]

where \(C(E, F)\) denotes the capacity of the condenser \((E, F)\). This capacity can be defined in several different ways. The one adopted by Gonchar enabled him to prove (1.2) under the additional assumption that one of the sets, say \(E\), lies in a connected component of \(\overline{C \setminus F}\). Yet, his method can be modified to deal with the general case (cf. [Go3], Remark 1). Widom [W1] gave a very short proof of (1.2) for an arbitrary condenser. He used an alternative definition of \(C(E, F)\) as the Green capacity of \(E\) relative to the open set \(\overline{C \setminus F}\). Yet another definition was utilized by Saff and Totik ([ST], Chapter 7), who established a weighted analogue of (1.2). We
also mention that T. Ganelius [Ga2] has shown that, under some further assumptions on \( E, F \), inequality (1.3) can be improved to

\[
\sigma_n(E, F) \geq \text{const} \cdot e^{n/C(E, F)}, \quad n = 1, 2, \ldots.
\]

Returning to the definition of \( \sigma_n \), observe that it can be rewritten as

\[
\sigma_n^{-1}(E, F) = \inf \{ \| r_n \|_{L_\infty(E)} \| 1/r_n \|_{L_\infty(F)} \}.
\]

It thus makes sense to introduce the quantity

\[
\delta_n(E, F) = \delta_n(E, F; \| \cdot \|_E, \| \cdot \|_F) := \inf \{ \| r_n \|_E \| 1/r_n \|_F \},
\]

where \( \| \cdot \|_E, \| \cdot \|_F \) are given norms on \( E \) and \( F \), respectively. Natural choices for these norms are \( L_p \)-norms (with respect to some measure) or the \( L_\infty \)-norm (with some weight function). For such norms, we will establish the analogue of (1.2). In particular, if \( \mu^*_E, \mu^*_F \) are the probability measures on \( E, F \) respectively, such that \( \mu^* := \mu^*_E - \mu^*_F \) is the equilibrium distribution on \( (E, F) \) (we define this in the next section), then for any \( 0 < p, q < \infty \) and for any rational function \( r_n \) of order \( \leq n \), the following inequality holds:

\[
\| r_n \|_{L_p(E, d\mu^*_E)} \| 1/r_n \|_{L_q(F, d\mu^*_F)} \geq e^{-n/C(E, F)}, \quad n = 0, 1, \ldots.
\]

Note that since \( \mu^*_E, \mu^*_F \) are unit measures, inequality (1.7) implies (1.2).

The estimates (1.2), (1.3) have immediate applications to approximation problems. Let \( \lambda(z) \) be the function that is equal to 0 on \( E \) and 1 on \( F \), and set

\[
\rho_n(E, F) := \inf \{ \| \lambda - r_n \|_{L_\infty(E \cup F)} \}.
\]

It is not difficult to show (see [Go3]) that

\[
\frac{1}{\sigma_n^{1/2} + 1} \leq \rho_n \leq \frac{1}{\sigma_n^{1/2} - 1}, \quad n = 0, 1, \ldots,
\]

and consequently, since \( \sigma_n \to \infty \), we have

\[
\rho_n \sim \sigma_n^{-1/2} = \delta_n^{1/2}(E, F; \| \cdot \|_{L_\infty(E)}, \| \cdot \|_{L_\infty(F)}), \quad \text{as} \ n \to \infty.
\]

Unfortunately, there is no obvious relation between \( \rho_n \) and \( \delta_n \) for norms other than the sup-norm. For the general \( L_p \)-norm case, we are able to prove the lower bound for \( \rho_n \) only for a symmetric condenser. Our method also applies to the case when \( E \) and \( F \) have a point in common. As an illustration, we prove that for \( 0 < p < \infty \) and for any \( r_n \) of order \( \leq n \), the following inequality holds:

\[
\left\{ \int_{-1}^{0} |r_n(x)|^p \, dx + \int_{0}^{1} |1 - r_n(x)|^p \, dx \right\}^{1/p} \geq c_p n^{1/2p} e^{-\pi \sqrt{n}/p}
\]

This inequality was proved by Vyacheslavov [V] for real-valued \( r_n \) and for \( p \geq 1 \).

Another related matter is the Kolmogorov \( n \)-width. Let \( \Omega \) be a domain in \( \overline{C} \) and let \( A_\infty(\Omega) \) denote the collection of all functions \( f \) that are analytic in \( \Omega \) and satisfy \( |f| \leq 1 \) there. Let \( K \) be a compact set in \( \Omega \). Given a measure \( \mu \) on \( K \), the Kolmogorov \( n \)-width of \( A_\infty(\Omega) \) with respect to the norm \( \| \cdot \|_{L_p(K; d\mu)} \), \( 0 < p < \infty \), is defined by

\[
d_n(A_\infty(\Omega), \| \cdot \|_{L_p(K; d\mu)}) := \inf \sup \inf \{ \| f - g \|_{L_p(K; d\mu)} \}. \quad \text{for} \ x_n \in A_\infty(\Omega) \text{ and } g \in X_n.
\]
where $X_n$ runs through all $n$-dimensional subspaces of $L_p(K; d\mu)$. Using the method of Fisher and Micchelli [FM], we can show that

$$d_n(A_\infty(\Omega), \| \cdot \|_{L_p(K; d\mu)}) \leq \inf_{\{r_n\}} \{ \|r_n\|_{L_p(K; d\mu)} \|1/r_n\|_{L_\infty(\partial\Omega)} \}.$$  

If $\Omega$ is a disk, the equality holds in (1.11). In this case, the results of O. Parfenov in [P1] and [P2] can be applied to get the strong asymptotics for $\delta_n(K, \partial\Omega)$.

§2. SOME BASIC FACTS

A convenient (for our purpose) definition of condenser capacity was given by T. Bagby in [B]. Throughout we assume that $E, F$ are disjoint closed subsets of $\overline{C}$, each having positive logarithmic capacity.

(a) Let $\mu_E$ and $\mu_F$ be Borel probability measures on $E$ and $F$, respectively, and assume that both measures have finite logarithmic energy. Let $\mu = \mu_E - \mu_F$ denote the corresponding signed measure on $E \cup F$. For any such $\mu$, it is known (cf. [H], Theorem 16.4.2) that the integral

$$I(\mu) := \int \int \log \frac{1}{|z - t|} \, d\mu(z) \, d\mu(t)$$

exists and is positive. Let

$$V := \inf_{\mu} I(\mu).$$

It turns out that $0 < V < \infty$ since $E$ and $F$ both have positive logarithmic capacity. The condenser capacity $C(E, F)$ is then defined by

$$C(E, F) := V^{-1}$$

(some authors use $e^{-V}$ instead of $V^{-1}$ to define $C(E, F)$).

(b) The infimum in (2.1) is attained for the unique signed measure $\mu^* = \mu_E^* - \mu_F^*$, which is called the equilibrium distribution for $(E, F)$. The corresponding equilibrium potential is then defined by

$$u^*(z) := \int \log \frac{1}{|z - t|} \, d\mu^*(t).$$

It is known that $u^*$ exists and is finite for any $z \in \overline{C}$. Moreover (cf. [B], Theorem 1), there exist finite constants $V_E, V_F$ such that

$$V_E \geq 0, \quad V_F \leq 0;$$

$$V_F - V_E = V$$

$$V_F \leq u^*(z) \leq V_E \quad \text{for all} \quad z \in \overline{C}$$

$$u^*(z) = V_E \quad \text{quasi-everywhere on} \quad E$$

$$u^*(z) = V_F \quad \text{quasi-everywhere on} \quad F,$$

where quasi-everywhere (q.e.) means that the property holds except for a set of zero capacity. We also mention that $\mu^*$ is concentrated on $\partial E \cup \partial F$. Thus, $u^*$ is harmonic in the complement of $\partial E \cup \partial F$. It then follows from (2.7) that $u^*(z) = V_E$ on each component of $\overline{C} \setminus \partial E$ that does not contain points of $F$. (An analogous statement concerning components of $\overline{C} \setminus \partial F$ follows from (2.8).)
In particular, if \( \infty \notin \partial E \) and if the unbounded component of \( \overline{C \setminus \partial E} \) does not intersect \( F \), then \( u^*(\infty) = 0 \) (see (2.3) and note: \( \int \log |z|^{-1} \, d\mu^*(t) = (\log |z|^{-1}) \int d\mu^* = 0 \) and therefore, \( V_E = 0 \).

(c) A condenser \( (E, F) \) is called symmetric if \( E \) lies in the (open) upper half-plane and \( F \) is the reflection of \( E \) in the real axis. For such condensers, the equilibrium distribution \( \mu^* \) is also symmetric, that is (cf. [B], Lemma 9), if \( K \subset E \) is any Borel set and \( K \) is its reflection in the real axis, then

\[
\mu^*_E(K) = \mu^*_F(K).
\]

It then follows from (2.3) that \( u^*(\overline{z}) = -u^*(z) \) and therefore (by (2.7), (2.8))

\[
V_E = -V_F = 1/(2C(E, F)).
\]

(d) Given a (positive) measure \( \mu \) on \( E \cup F \), we denote by \( \mu_E \), \( \mu_F \) its restrictions on \( E \), \( F \), respectively. Let \( \mu_E = \mu^*_E + \mu^a_E \) be the canonical decomposition of \( \mu_E \) into the singular and the absolutely continuous parts with respect to \( \mu^*_E \). We denote by \( \mu^c_E \) the Radon-Nikodym derivative \( d\mu^c_E \) of \( \mu^a_E \) with respect to \( \mu^*_E \). Let

\[
G(\mu_E) := \exp \left( \int (\log \mu^c_E) \, d\mu^c_E \right)
\]

be the geometric mean of \( \mu^c_E \). Similarly, we define \( \mu^c_F \) and \( G(\mu^c_F) \). If both \( G(\mu^c_E) \) and \( G(\mu^c_F) \) are positive, we say that the measure \( \mu \) satisfies the Szegő condition with respect to \( \mu^* \).

§3. Gonchar’s inequality generalized

Theorem 3.1. Let \( (E, F) \) be a condenser and let \( \mu \) be a finite (positive) Borel measure on \( E \cup F \). Then, for any \( 0 < p, q < \infty \) and for any rational function \( r_n \) of order \( \leq n \), we have

\[
\| r_n \|_{L_p(d\mu_E)} \| r_n \|_{L_q(d\mu_F)} \geq e^{-n/C(E, F)} G^{1/p}(\mu^c_E) G^{1/q}(\mu^c_F),
\]

where \( \mu_E, \mu_F \) denote the restrictions of \( \mu \) to \( E, F \) respectively.

Proof. Let \( \mu_E = \mu^s_E + \mu^c_E \). Since \( \mu_E \) is positive, so is \( \mu^c_E \). Hence,

\[
\int |r_n|^p \, d\mu_E \geq \int |r_n|^p \, d\mu^c_E.
\]

Assuming \( \int (\log \mu^c_E) \, d\mu^c_E > -\infty \) (otherwise (3.1) is obvious) and applying Jensen’s inequality, we obtain

\[
\frac{1}{p} \log \int |r_n|^p \, d\mu^c_E \geq \int \log |r_n| \, d\mu^c_E + \frac{1}{p} \log G(\mu^c_E)
\]

and similarly,

\[
\frac{1}{q} \log \int |1/r_n|^q \, d\mu^c_F \geq -\int \log |r_n| \, d\mu^c_F + \frac{1}{q} \log G(\mu^c_F),
\]

where we used the fact that \( \mu^*_E \) and \( \mu^*_F \) are probability measures. Hence,

\[
\log \left\{ \| r_n \|_{L_p(d\mu_E)} \| 1/r_n \|_{L_q(d\mu_F)} \right\} \geq \int \log |r_n| \, d\mu^* + \frac{1}{p} \log G(\mu^c_E) + \frac{1}{q} \log G(\mu^c_F).
\]

It remains to show that

\[
\int \log |r_n| \, d\mu^* \geq -n/C(E, F).
\]
(3.3) \[ r_n(z) = A \prod_{i=1}^{k} (z - \alpha_i) \prod_{i=1}^{t} (z - \beta_i), \]

where \( \max(k, \ell) \leq n \). Since \( \int d\mu^* = 0 \), we obtain (see (2.3))

\[ \int \log |r_n| \, d\mu^* = \sum_{i=1}^{\ell} u^*(\beta_i) - \sum_{i=1}^{k} u^*(\alpha_i). \]

Applying (2.6) and (2.4), we then obtain

(3.4) \[ \int \log |r_n| \, d\mu^* \geq \ell V_F - k V_E \geq -n(V_E - V_F) \]

and (3.2) follows. \( \square \)

**Remark 1.** One can consider the measures \( d\mu_E = w_E^k \, d\mu_E^k \), \( d\mu_F = w_F^k \, d\mu_F^k \), where \( \mu_E \in L_p(d\mu_E^k) \), \( \mu_F \in L_q(d\mu_F^k) \). We then obtain

(3.5) \[ \|r_n w_E\|_{L_p(d\mu_E^k)} \|1/r_n w_F\|_{L_q(d\mu_F^k)} \geq e^{-n/C(E,F)} G(\mu_E) G(\mu_F). \]

If \( w_E \in L_\infty(d\mu_E^k) \) and/or \( w_F \in L_\infty(d\mu_F^k) \), then (3.5) holds also for \( p = \infty \) and/or \( q = \infty \). Furthermore, inequality (3.2) shows that (3.5) holds also for \( p = 0 \) and/or \( q = 0 \), where \( \|f\|_{L_0} \) means \( \sqrt{\int \log |f|^2} \).

**Remark 2.** The inequality (3.1) can be strengthened in the spirit of Remark 1 in [Go3]. For example, let \( \infty \notin \partial E \) and assume that the unbounded component of \( \overline{C \setminus \partial E} \) does not intersect \( F \). Let \( D \) be the union of all components of \( \overline{C \setminus \partial E} \) that do not intersect \( F \). Then (see the end of subsection 2(b)) \( V_E = 0 \) and \( u^*(z) = 0 \) on \( D \). Thus, the proof of Theorem 3.1 shows that for any rational function \( r \), we have:

(3.6) \[ \|r\|_{L_p(d\mu_E)} \|1/r\|_{L_q(d\mu_F)} \geq e^{-m/C(E,F)} G^{1/p}(\mu_E) G^{1/q}(\mu_F), \]

where \( m \) is the number of poles of \( r(z) \) that are not in \( D \). In particular, (3.6) holds for \( r = P/Q_m \), where \( Q_m \) is a polynomial of degree \( m \) and \( P \) is an arbitrary polynomial.

**Remark 3.** Let \((E, F)\) be a symmetric condenser (with \( E \) in the upper half-plane) and let \( P_n \) be a polynomial of degree \( k \leq n \). Then (3.4) and (2.10) give

\[ \int \log |P_n| \, d\mu^* \geq -k V_E \geq -n V_E = -n/(2C(E,F)). \]

Since the equilibrium distribution \( \mu^* \) is symmetric (see (2.9)) we may rewrite this in the form:

(3.7) \[ \int_E \log \left| \frac{P_n(z)}{P_n(\overline{z})} \right| \, d\mu_E^* \geq -n \frac{1}{2C(E,F)}. \]

This inequality remains valid if we replace \( \int_E \) by \( \int_A \), where \( A \) is any Borel subset of \( E \). Indeed, write \( P_n = Q_m R_{n-m} \), where the zeros of \( Q_m \) lie in the open upper half-plane and those of \( R_{n-m} \) lie in the closed lower half-plane. Then we have, for any \( z \) with \( \text{Im} z \geq 0 \),

\[ \left| \frac{Q_m(z)}{Q_m(\overline{z})} \right| \leq 1 \quad \text{and} \quad \frac{|P_n(z)|}{|P_n(\overline{z})|} \geq \frac{|Q_m(z)|}{|Q_m(\overline{z})|}. \]
Hence,

\[
\int_A \log \left| \frac{P_n(z)}{P_n(\overline{z})} \right| \, d\mu_E^* \geq \int_A \log \left| \frac{Q_m(z)}{Q_m(\overline{z})} \right| \, d\mu_E^* \\
\geq \int_E \log \left| \frac{Q_m(z)}{Q_m(\overline{z})} \right| \, d\mu_E^* \geq \frac{-m}{2C(E, F)}.
\]

Since \( m \leq n \), we obtain

\[
\int_A \log \left| \frac{P_n(z)}{P_n(\overline{z})} \right| \, d\mu_E^* \geq \frac{-n}{2C(E, F)}, \quad \text{for } A \subseteq E.
\]

This can be viewed as a kind of Newman's inequality.

Finally, we discuss the sharpness of (3.1). Assuming that \( \mu \) satisfies the Szegö condition with respect to \( \mu^* \) we obtain from (3.1):

\[
\liminf_{n \to \infty} \delta_n^{1/n}(E, F; \| \cdot \|_{L_p(d\mu_E^*)}, \| \cdot \|_{L_q(d\mu_F^*)}) \geq e^{-1/C(E, F)}
\]

where \( \delta_n \) is defined in (1.6). The matching upper bound for \( \limsup \delta_n^{1/n} \) follows from (1.3). Hence, the exponent in (3.1) is sharp.

The following example shows that in certain geometric settings the factor \( G^{1/p}(\mu_E^*)G^{1/q}(\mu_F^*) \) is the best possible.

Let \( E \) be the circle \( |z| = \rho < 1 \) and let \( F \) be the unit circle \( |z| = 1 \). In this case, \( e^{-1/C(E, F)} = \rho \) and \( d\mu_E^*(\rho e^{i\theta}) = d\mu_F^*(e^{i\theta}) = (1/2\pi) d\theta \). Let \( d\mu_E, d\mu_F \) be arbitrary Borel probability measures on \( E \) and on \( F \), respectively. By a result of Parfenov [P1] (see also [LS]), given \( 0 < \rho, q < \infty \) there exist sequences \( \{B_n^{(1)}\}, \{B_n^{(2)}\} \) of Blaschke products of respective orders \( n \) such that

\[
\|B_n^{(1)}(z)\|_{L_p(d\mu_E)} \sim \rho^n G^{1/p}(\mu_E^*) \quad \text{and} \quad \|B_n^{(2)}(\rho/z)\|_{L_q(d\mu_F)} \sim \rho^n G^{1/q}(\mu_F^*),
\]

where \( a_n \sim b_n \) means \( a_n/b_n \to 1 \) as \( n \to \infty \).

Define the sequence \( \{r_n\} \) of rational functions of order \( \leq n \), by

\[
r_n(z) := B_{n/2}^{(1)}(z)/B_{n/2}^{(2)}(\rho/z), \quad \text{if } n \text{ is even},
\]

\[
r_n(z) := zr_{n-1}(z), \quad \text{if } n \text{ is odd}.
\]

It is then obvious that

\[
\|r_n\|_{L_p(d\mu_E)} \|1/r_n\|_{L_q(d\mu_F)} \sim \rho^n G^{1/p}(\mu_E^*) G^{1/q}(\mu_F^*) \quad \text{as } n \to \infty.
\]

§4. APPROXIMATION OF FUNCTIONS CONSTANT ON TWO DISJOINT SETS

Given a condenser \( (E, F) \), let \( \lambda \) denote the function that equals 0 on \( E \) and 1 on \( F \). Given \( 0 < p < \infty \) and given a finite (positive) measure \( \mu \) on \( E \cup F \), set

\[
\rho_n(E, F, \| \cdot \|_{L_p(d\mu)}) := \inf_{\{r_n\}} \| \lambda - r_n \|_{L_p(d\mu)}.
\]

Gonchar's result (1.3) and the estimate (1.8) imply that

\[
\limsup_{n \to \infty} \rho_n^{1/n}(E, F, \| \cdot \|_{L_p(d\mu)}) \leq e^{-1/2C(E, F)}.
\]

The lower bound (for a symmetric condenser) is given by
Theorem 4.1. Let \((E, F)\) be a symmetric condenser (with \(E\) in the upper half-plane) and assume that the finite (positive) measure \(\mu\) satisfies the Szegő condition with respect to the equilibrium distribution \(\mu^*\) for \((E, F)\). Then, for \(0 < p < \infty\), we have

\[
\rho_n(E, F, \|L_p(\mu)\|) \geq (1 + o(1)) [G(\mu_E)G(\mu_F)]^{1/2} e^{-n/2C(E, F)},
\]

where \(\mu_E, \mu_F\) denote the restrictions of \(\mu\) to \(E, F\), respectively.

Proof. For \(n = 1, 2, \ldots\), let \(r_n\) be a rational function that realizes the infimum in (4.1). It then follows from (4.2) that

\[
\int |r_n|^p \mu_E^* d\mu_E^* + \int |1 - r_n|^p \mu_F^* d\mu_F^* \leq \rho_n^p \leq c\eta^n,
\]

where \(0 < \eta < 1\) and \(c > 0\) are independent of \(n\). Set

\[(4.5) \quad E_n := \{z \in E : |r_n(z)| \geq 1 - 2^{-1/p}\}, \quad F_n := \{z \in F : |1 - r_n(z)| \geq 1 - 2^{-1/p}\}.
\]

Then we have by (4.4) that

\[
(4.6) \quad \int_{E_n} \mu_E^* d\mu_E^* + \int_{F_n} \mu_F^* d\mu_F^* \leq 2c\eta^n/(2^{1/p} - 1)^p.
\]

Since \(\mu\) satisfies the Szegő condition, (4.6) implies that (as we will show later)

\[
(4.7) \quad \mu_E^*(E_n) + \mu_F^*(F_n) = o(1/n).
\]

Let \(A_n := E_n \cup F_n\) (where \(\sim\) denotes the reflection of a set about the real axis). Since \(\mu^*\) is symmetric, we obtain

\[
(4.8) \quad \alpha_n := \mu_E^*(A_n) = \mu_F^*(A_n) = o(1/n).
\]

By the definition of \(E_n, F_n\) and \(A_n\), we obtain (see (4.4)):

\[
\rho_n^p \geq \frac{1}{2} \int_{E \setminus A_n} \left|\frac{r_n}{1 - r_n}\right|^p \mu_E^* d\mu_E^* + \frac{1}{2} \int_{E \setminus A_n} \left|\frac{1 - r_n}{r_n}\right|^p \mu_F^* d\mu_F^*.
\]

and therefore (by the concavity of log):

\[
(4.9) \quad p \log \rho_n \geq \frac{1}{2} \log \int_{E \setminus A_n} \left|\frac{r_n}{1 - r_n}\right|^p \mu_E^* d\mu_E^* + \frac{1}{2} \log \int_{F \setminus A_n} \left|\frac{1 - r_n}{r_n}\right|^p \mu_F^* d\mu_F^*.
\]

For the first integral, we have, by Jensen’s inequality (see also (4.8)),

\[
\log \int_{E \setminus A_n} \left|\frac{r_n}{1 - r_n}\right|^p \mu_E^* d\mu_E^* \\
\geq p \int_{E \setminus A_n} \log \left|\frac{r_n}{1 - r_n}\right| \frac{d\mu_E^*}{1 - \alpha_n} + \frac{1}{1 - \alpha_n} \int_{E \setminus A_n} \log \mu_E^* d\mu_E^* + \log(1 - \alpha_n)
\]

\[
= p \int_{E \setminus A_n} \log \left|\frac{r_n}{1 - \alpha_n}\right| \frac{d\mu_E^*}{1 - \alpha_n} + \log G(\mu_E^*) + o(1).
\]

(Here we only used the fact that \(\alpha_n = o(1)\). A similar inequality holds for the second integral in (4.9). We thus obtain

\[
\log \rho_n \geq o(1) + \frac{1}{2p} \log[\mu_E^* G(\mu_F^*)]
\]

\[
+ \frac{1}{2(1 - \alpha_n)} \left\{ \int_{E \setminus A_n} \log \left|\frac{r_n}{1 - r_n}\right| d\mu_E^* + \int_{F \setminus A_n} \log \left|\frac{1 - r_n}{r_n}\right| d\mu_F^* \right\}
\]

\[
\sim (1 + o(1)) [G(\mu_E)G(\mu_F)]^{1/2} e^{-n/2C(E, F)}.
\]
By the symmetry, the expression in the braces can be put into the form
\[ \int_{E \setminus A_n} \log \left| \frac{r_n(z)}{1 - r_n(z)} \right| \left| \frac{1 - r_n(\overline{z})}{r_n(\overline{z})} \right| d\mu_n^* \cdot \]
Since this last integrand is of the form \( \log |P_{2n}(z)/P_{2n}(\overline{z})| \), where \( P_{2n} \) is a polynomial of degree \( \leq 2n \), it follows from (3.9) that the last term in (4.10) satisfies
\[ \frac{1}{2(1 - \alpha_n)} \cdot \left( \frac{-2n}{2C(E, F)} \right) = -\frac{n}{2C(E, F)} + o(1) \]
(since \( \alpha_n = o(1/n) \) by (4.8)), and the proof is complete provided we establish (4.7).
Actually we shall prove the following slightly more general statement:
Let \( \sigma \) be a finite (positive) measure on a compact set \( K \). Let \( f \in L_1(K; d\sigma) \) and assume also that \( \log f \in L_1(K; d\sigma) \). Let \( \{K_n\} \) be a sequence of subsets of \( K \) such that \( \sigma(K_n) > 0 \) and \( \varepsilon_n := \int_{K_n} f d\sigma \to 0 \) as \( n \to \infty \). Then \( \sigma(K_n) = o(1/\log(1/\varepsilon_n)) \).
Indeed, set \( a_n := \int_{K_n} \log f d\sigma \). Since \( \log f \in L_1 \), we have \( a_n \geq -a \) for some \( a > 0 \) independent of \( n \). Hence,
\[ \log \varepsilon_n = \log \int_{K_n} f d\sigma \geq \frac{1}{\sigma(K_n)} \int_{K_n} \log f d\sigma + \log \sigma(K_n) \]
(4.11)
\[ \geq \frac{-a}{\sigma(K_n)} + \log \sigma(K_n). \]
Since \( \varepsilon_n \to 0 \), it follows from (4.11) that \( \sigma(K_n) \to 0 \) as \( n \to \infty \). By the absolute continuity of integral, we then obtain
\[ a_n = \int_{K_n} \log f d\sigma \to 0 \quad \text{as} \quad n \to \infty. \]
Returning to inequality (4.11), we see that
\[ \log \varepsilon_n \geq \frac{a_n}{\sigma(K_n)} + \log \sigma(K_n), \quad a_n \to 0, \]
and the result follows.

Remark. We assumed that the condenser \((E, F)\) is symmetric about the real axis. Given a condenser \((E', F')\) that is symmetric about some straight line or about some circle, we may use a suitable Möbius transformation to map \((E', F')\) onto the condenser \((E, F)\) considered above. This transformation preserves rational functions of given order and transfers (in a natural way) the equilibrium distribution on \((E', F')\) to that on \((E, F)\). Thus, Theorem 4.1 is valid for any symmetric condenser.

§5. APPROXIMATION OF THE UNIT STEP FUNCTION ON \([-1, 1] \]

In this section, we prove the estimate (1.9). Let
\[ \Delta_n^p := \inf \left\{ \int_{-1}^{0} |r_n(x)|^p dx + \int_{0}^{1} |1 - r_n(x)|^p dx \right\} \]
and let \( r_n^* \) denote a function that realizes the infimum. For any \( 0 < \varepsilon < 1 \), we have
\[ \Delta_n^p \geq \int_{-\varepsilon}^{-\varepsilon} |r_n^*(x)|^p dx + \int_{\varepsilon}^{1} |1 - r_n^*(x)|^p dx. \]
It is well known that the equilibrium distribution for the condenser
\[ (E, F) = ([-1, -\varepsilon], [\varepsilon, 1]) \]
is given by
\[ d\mu^*_E(-x) = d\mu^*_E(x) = \frac{dx}{K'(\varepsilon)\sqrt{(x^2 - \varepsilon^2)(1 - x^2)}} , \quad 0 < x < 1 , \]
and that \(1/C(\varepsilon, F) = 2\pi K(\varepsilon)/K'(\varepsilon)\), where \(K(\varepsilon), K'(\varepsilon)\) denote the complete elliptic integrals for moduli \(\varepsilon\) and \(\sqrt{1 - \varepsilon^2}\), respectively. We also have, as \(\varepsilon \to 0\),
\[ K(\varepsilon) = \pi/2 + o(1) , \quad K'(\varepsilon) = \log(1/\varepsilon) + O(1) . \]
Using these facts we could proceed as in §4 by estimating the right-hand side of (5.2) from below and choosing \(\varepsilon\) in an “optimal” way. But it will be simpler to utilize the measure \(dx/(x \log(1/\varepsilon))\) (a unit measure on \([\varepsilon, 1]\)) instead of \(d\mu^*(x)_E\) and to apply the classical Newman’s inequality (cf. [N])
\[(5.3) \quad \int_a^b \left| \frac{P_n(x)}{P_n(-x)} \right| \frac{dx}{x} \geq -\frac{\pi^2n}{2} , \quad 0 < a < b.\]
Notice that (5.3) implies (in the same way as (3.7) implies (3.8)) that
\[(5.4) \quad \int_K \log \left| \frac{P_n(x)}{P_n(-x)} \right| \frac{dx}{x} \geq -\frac{\pi^2n}{2} , \quad K \subset (0, \infty). \]
Returning to (5.2), let us introduce the exceptional set
\[ A_n := \{ x \in [\varepsilon, 1] : |1 - r^*_n(x)|^p > \frac{1}{2} \text{ or } |r^*_n(-x)|^p > \frac{1}{2} \}. \]
From (5.2), we see that \(\text{meas}(A_n) := \int_{A_n} dx \leq 2\Delta^p_n\). Set (compare with (4.8))
\[ \alpha_n := \int_{A_n} \frac{dx}{x \log(1/\varepsilon)} , \]
and observe that
\[(5.5) \quad \alpha_n \leq \int_{\varepsilon}^{\varepsilon + 2\Delta^p_n} \frac{dx}{x \log(1/\varepsilon)} = \log \left( 1 + \frac{2\Delta^p_n}{\varepsilon} \right) / \log \frac{1}{\varepsilon} . \]
Proceeding as in §4 (with \(dx/(x \log(1/\varepsilon))\) instead of \(d\mu^*_E\) and with \(d\mu_E x \log(1/\varepsilon)dx\)) we obtain
\[(5.6) \quad \log \Delta_n \geq \frac{1}{2(1 - \alpha_n)} \int_{[\varepsilon, 1] \setminus A_n} \log \left| \frac{P_{2n}(x)}{P_{2n}(-x)} \right| \frac{dx}{x \log(1/\varepsilon)} + \frac{1}{p(1 - \alpha_n)} \int_{[\varepsilon, 1] \setminus A_n} \log \left( x \log \frac{1}{\varepsilon} \right) \frac{dx}{x \log(1/\varepsilon)} + \frac{1}{p} \log(1 - \alpha_n) . \]
Applying (5.4) and observing that
\[ \int_{[\varepsilon, 1] \setminus A_n} \log x \frac{dx}{x} \geq \int_{\varepsilon}^{1} \frac{\log x}{x} \frac{dx}{x} = -\frac{1}{2} \log^2 \frac{1}{\varepsilon} , \]
we get from (5.6) that
\[(5.7) \quad \log \Delta_n \geq -\frac{\pi^2n}{2(1 - \alpha_n)} \left\{ \frac{\log(1/\varepsilon)}{\log(1/\varepsilon)} + \frac{1}{p} \log \frac{1}{\varepsilon} \right\} + \frac{1}{p} \log \log \frac{1}{\varepsilon} + \frac{1}{p} \log(1 - \alpha_n) . \]
Now we choose
\[(5.8) \quad \varepsilon := \Delta^p_n , \]
We then see from (5.5) that
\[(5.9) \quad \alpha_n \leq c/ \log(1/\Delta_n) . \]
Using (5.8) and (5.9), we obtain from (5.7) that
\[
\log \Delta_n \geq -\frac{1}{2} \left\{ \frac{\pi^2 n}{p \log(1/\Delta_n)} + \log \frac{1}{\Delta_n} \right\} - c_1 \frac{n}{\log^2(1/\Delta_n)} + \frac{1}{p} \log \left( p \log \frac{1}{\Delta_n} \right) + O(1)
\]  
(5.10)

To obtain the desired lower bound, define \( \tau_n \) by

\[
\Delta_n = \tau_n n^{1/2p} e^{-\pi \sqrt{n/p}}
\]  
(5.11)

and assume that \( \tau_n \to 0 \) as \( n \to \infty \). This will lead to a contradiction. Indeed, (5.1) implies that

\[
\log \frac{1}{\Delta_n} \geq \frac{n}{p} \log n - \frac{1}{2p} \log n \quad \text{(for \( n \) large)}.
\]

Hence (5.10) gives (after subtracting \( \frac{1}{2} \log \Delta_n \) from both sides)

\[
\frac{1}{2} \log \Delta_n \geq -\frac{1}{2} \sqrt{\frac{n}{p}} - \frac{1}{4p} \log n + \frac{1}{2p} \log n + O(1).
\]

Replacing here \( \Delta_n \) by (5.11), we obtain

\[
\frac{1}{2} \log \tau_n \geq O(1)
\]

which contradicts the assumption \( \tau_n \to 0 \).

§6. ESTIMATES FOR \( n \)-WIDTHS

We first prove the estimate (5.11).

**Theorem 6.1.** Let \( \Omega \) be a domain in \( \overline{\mathbb{C}} \) and let \( A_\infty(\Omega) \) denote the collection of all functions \( f \) that are analytic in \( \Omega \) and satisfy \( |f| \leq 1 \) there. Let \( K \) be a compact set in \( \Omega \) and let \( \mu \) be a finite (positive) measure on \( K \). Then, for \( 0 < p < \infty \), we have

\[
d_n(A_\infty(\Omega), \| \cdot \|_{L_p(K; d\mu)}) \leq \inf_{\{r_n\}} \| r_n \|_{L_p(K; d\mu)} / r_n \|_{L_\infty(\partial \Omega)},
\]

(6.1)

where the infimum is taken over all rational functions of order \( \leq n \).

**Proof.** Assume first that \( \Omega \) is bounded by a finite number of disjoint analytic simple closed curves. Let \( g(z, \xi) \) be the Green function for \( \Omega \) with singularity at \( \xi \). Then the conjugate function \( \bar{g}(z, \xi) \) is multiple-valued, but the function

\[
\varphi(z, \xi) := \frac{\partial}{\partial z} (g(z, \xi) + i\bar{g}(z, \xi))
\]

is analytic and single-valued in \( \Omega \). The following facts are well known:

(a) \( \varphi(z, \xi) \) has a simple pole at \( \xi \) with the residue 1.

(b) \( \varphi(z, \xi) \) is continuous in both arguments.

(c)

\[
\frac{1}{2\pi} \int_{\partial \Omega} |\varphi(z, \xi)| dz = \frac{1}{2\pi} \int_{\partial \Omega} \left| \frac{\partial g(z, \xi)}{\partial \nu_z} \right| dz = 1
\]

(\( \nu_z \) denotes the inward normal to \( \partial \Omega \) at \( z \in \partial \Omega \)).

In order to prove (6.1) it suffices to consider rational functions with simple zeros, and we may assume that they do not lie on \( \partial \Omega \). Let \( R(z) \) be such a function and
let $z_1, \ldots, z_m$ ($m \leq n$) denote those zeros of $R$ that belong to $\Omega$. For any $f$ that is analytic in $\overline{\Omega}$, we have by (a), (b) that

$$\frac{1}{2\pi i} \int_{\partial \Omega} \frac{R(\zeta)}{R(z)} f(z) \varphi(z, \zeta) \, dz = f(\zeta) + R(\zeta) \sum_{i=1}^{m} \frac{f(z_i)\varphi(z_i, \zeta)}{R'(z_i)}.$$

If $f \in L^p(\overline{\Omega})$, then by (c) we get

$$\left| \frac{1}{2\pi i} \int_{\partial \Omega} \frac{R(\zeta)}{R(z)} f(z) \varphi(z, \zeta) \, dz \right| \leq |R(\zeta)| \cdot \|1/R\|_{L^p(\partial \Omega)} \frac{1}{2\pi} \int_{\partial \Omega} |\varphi(z, \zeta)| \, |dz| = |R(\zeta)| \cdot \|1/R\|_{L^p(\partial \Omega)}.$$

Thus, (6.2) gives that the distance in $L^p(K; d\mu)$ from $f(\zeta)$ to span $\{R(\zeta)\varphi(z_i, \zeta)\}_{i=1}^{m}$ does not exceed $\|R\|_{L^p(K; d\mu)} \cdot 1/R\|_{L^p(\partial \Omega)}$. Since any $f \in A_\infty(\Omega)$ can be approximated (uniformly on $K$) by those $f \in A_\infty(\Omega)$ that are analytic in $\overline{\Omega}$, (6.1) follows.

It remains to remove the restriction we put on $\Omega$. This is done in a standard way (see the proof of Theorem 4 in [FM]).

Assume now that $\Omega$ is the unit disk $\Delta := \{z : |z| < 1\}$. For this case, Fisher and Micchelli proved ([FM], Theorem 1) that

$$d_n(A_\infty(\Delta), L^p(K; d\mu)) = \inf_{\{B_n\}} \|B_n\|_{L^p(K; d\mu)},$$

where the infimum is taken over all Blaschke products of order $\leq n$. Since $\|1/B_n\|_{L^\infty(\partial \Omega)} = 1$, we obtain from (6.1), (6.3) that

$$d_n(A_\infty(\Delta), L^p(K; d\mu)) = \inf_{\{r_n\}} \|r_n\|_{L^p(K; d\mu)} \cdot 1/r_n \|1/r_n\|_{L^\infty(\partial \Omega)}.$$

This equality provides another example of the exactness of the generalized Gonchar inequality. Let $p = 2$ and assume that $K$ is a simple closed analytic curve. Let $d\mu = w(z) \, |dz|$, where $w(z)$ is a continuous positive function on $K$ and $|dz|$ is the arc length. Let $A_2(\Delta)$ denote the unit ball of the Hardy space $H^2(\Delta)$.

For this case, Parfenov found the strong asymptotics for the $n$-width. Although stated in different terms, the result reads (see [P2])

$$d_n(A_\infty(\Delta), L^2(K; d\mu)) \sim e^{-n/C(K, \partial \Delta)} G(\mu')^{1/2},$$

where $\mu'$ denotes, as before, the derivative $d\mu/d\mu^*$. Since $A_2(\Delta) \supset A_\infty(\Delta)$, we have

$$d_n(A_\infty(\Delta), L^2(K; d\mu)) \leq d_n(A_2(\Delta), L^2(K; d\mu))$$

This inequality, together with (3.1) (for $p = 2$, $q = \infty$) and (6.4) implies

$$\inf_{\{r_n\}} \|r_n\|_{L^p(K; d\mu)} \cdot 1/r_n \|1/r_n\|_{L^\infty(\partial \Delta)} \sim e^{-n/C(K, \partial \Delta)} G(\mu')^{1/2}.$$

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