

# On the Denseness of Weighted Incomplete Approximations

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**ABSTRACT** For a given weight function  $w(x)$  on an interval  $[a, b]$ , we study the generalized Weierstrass problem of determining the class of functions  $f \in C[a, b]$  that are uniform limits of weighted polynomials of the form  $\{w^n(x)p_n(x)\}_1^\infty$ , where  $p_n$  is a polynomial of degree at most  $n$ . For a special class of weights, we show that the problem can be solved by knowing the denseness interval of the alternation points for the associated Chebyshev polynomials.

## 1 Introduction

In the asymptotic analysis of orthogonal polynomials with respect to an exponential weight of the form  $w(x) = \exp(-|x|^\alpha)$ ,  $\alpha > 1$ , on  $\mathbb{R} = (-\infty, \infty)$ , an important step is to determine the class of functions  $f$  continuous on  $\mathbb{R}$  that are uniform limits of weighted polynomials  $\{w^n p_n\}$ , where  $p_n \in \Pi_n$  (the class of polynomials of degree  $\leq n$ ), and the power  $n$  of  $w$  matches the (maximum) degree of the polynomial. For these so-called Freud weights, this problem was solved by Lubinsky and Saff [9] using techniques from potential theory. The analogous problem for weighted polynomials of the form  $\{x^{n\lambda} p_n(x)\}$  on  $[0, 1]$ , which are called *incomplete polynomials*, was raised by G.G. Lorentz and was resolved independently by Saff and Varga [12] and by M. v. Golitschek [3]. Further extensions to Jacobi type weights were obtained by He and Li [6] and He [5].

The above investigations are special cases of the following:

*Generalized Weierstrass Problem:* Given a closed set  $E \subset \mathbb{R}$  and a weight  $w : E \rightarrow [0, \infty)$ , determine necessary and sufficient conditions on  $f$  such that  $f$  is the uniform limit on  $E$  of a sequence of weighted polynomials  $\{w^n p_n\}$ ,  $p_n \in \Pi_n$ , as  $n \rightarrow \infty$ .

For the case when  $E$  is an interval and  $w(x) = e^{-Q(x)}$ , with  $Q(x)$  con-

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vex on  $E$ , a plausible solution can be described as follows. From potential theoretic considerations, it is known (cf. [10]), that there exists a unique smallest compact interval  $S_w$  such that for every  $n \geq 1$  and every  $p_n \in \Pi_n$ ,

$$\|w^n p_n\|_E = \|w^n p_n\|_{S_w},$$

where  $\|\cdot\|_A$  denotes the sup norm over the set  $A$ . Moreover, if one considers *weighted Chebyshev polynomials*  $\hat{T}_n(x) = w^n(x)(x^n + \dots)$  that are defined by the extremal property

$$\|\hat{T}_n\|_E = \inf_{p_{n-1} \in \Pi_{n-1}} \|w^n(x)(x^n + p_{n-1}(x))\|_E,$$

then the alternation (extreme) points of  $\hat{T}_n$  are dense in  $S_w$ . Based on the above mentioned special cases, the second author has previously made the following

**Conjecture.** If  $E \subset \mathbb{R}$  is a compact interval and  $w(x) = e^{-Q(x)}$ , with  $Q(x)$  convex on  $E$ , then  $f \in C(E)$  is the uniform limit on  $E$  of a sequence of the form  $\{w^n p_n\}_1^\infty$ ,  $p_n \in \Pi_n$ , if and only if  $f$  vanishes identically on  $E \setminus S_w$ . (In case  $E$  is unbounded, additional assumptions need to be imposed on  $Q(x)$  as  $|x| \rightarrow \infty$ ,  $x \in E$ .)

The aim of the present paper is to show that for a special class of weights  $w$ , a proof of the above conjecture follows from the denseness property of the alternation points of the weighted Chebyshev polynomials. Thus we avoid much of the "hard analysis" involved with the potential theoretic arguments used in [9]. However, our technique requires strong assumptions on the weight  $w$  and so falls short of proving the general conjecture.

## 2 An Approximation Lemma

Let

$$H_n := \text{span}\{g_0, \dots, g_n\}, \quad g_i \in C[a, b]$$

be a Chebyshev system on  $[a, b]$ . Define  $T_n$ , the normalized Chebyshev polynomial for  $H_n$  on  $[a, b]$ , by

$$T_n := T_{n,[a,b]} = \sum_{i=0}^n c_i g_i,$$

where the  $c_i$  are chosen so that  $\|T_n\|_{[a,b]} = 1$  and so that  $T_n$  has exactly  $n$  zeros  $x_1 < \dots < x_n$  in  $(a, b)$  and oscillates  $n + 1$  times between  $\pm 1$  on  $[a, b]$ . So defined,  $T_n$  exists and is unique up to multiplication by  $-1$ . (See [7, p. 72].) With  $x_0 := a$  and  $x_{n+1} := b$  we define the *mesh* of  $T_n$  by

$$M_n(T_n : [a, b]) := \max_{1 \leq i \leq n+1} |x_i - x_{i-1}|$$

and the mesh of  $T_n$  restricted to an interval  $I := [\alpha, \beta] \subset [a, b]$  by

$$M_n|_I := \max_{k \leq i \leq j+1} |x_i - x_{i-1}|,$$

where  $x_{k-1} := \alpha$ ,  $x_{j+1} := \beta$  and  $x_k < \dots < x_j$  are all the zeros of  $T_n$  in  $(\alpha, \beta)$ .

Lemma 1. Assume that

$$H_n = \text{span}\{g_0, \dots, g_n\}$$

is a Chebyshev system on  $[a, b]$  with associated Chebyshev polynomials  $T_n$ .

a) Suppose that each  $g_i \in C^1[\alpha, \beta]$  and  $H'_n := \text{span}\{g'_0, \dots, g'_n\}$  is a Chebyshev system on  $[\alpha, \beta] \subset [a, b]$ . If  $f \in C[\alpha, \beta]$ , then there exists  $h_n^* \in H_n^* := \text{span}\{1, g_0, \dots, g_n\}$  such that

$$\|h_n^* - f\|_{[\alpha, \beta]} \leq D\omega_f(\sqrt{\delta_n}),$$

where  $\delta_n := M_n(T_n : [a, b])|_{[\alpha, \beta]}$ . (Here  $D$  is a constant that depends only on  $a$  and  $b$ , and  $\omega_f$  is the modulus of continuity).

b) Suppose further that  $f \in C[\alpha, \beta]$ , that  $I$  is a closed interval contained in  $[\alpha, \beta]$ , and that  $f$  is constant on  $[\alpha, \beta] \setminus I$ . Then there exists  $h_n^* \in H_n^*$  such that

$$\|h_n^* - f\|_{[\alpha, \beta]} \leq D'\omega_f(\sqrt{\delta'_n})$$

where  $\delta'_n := M_n(T_n : [a, b])|_I$  and  $D'$  depends only on  $a$  and  $b$ .

Proof. The proof of Lemma 1 follows [1] closely where a similar result is proved for Markov systems, but is reworked for current purposes. Note that  $H'_n$  a Chebyshev system on  $[\alpha, \beta]$  implies that  $H_n^*$  is a Chebyshev system on  $[\alpha, \beta]$ . Suppose  $S_n \in H_n^*$  is the best uniform approximation from  $H_n^*$  to  $F$  on  $[\alpha, c] \cup [d, \beta]$ , where

$$F(x) := \begin{cases} 0, & x \in [\alpha, c] \\ 1, & x \in [d, \beta] \end{cases}$$

Then we claim the following:

- A)  $S_n$  is monotone on  $[c, d]$ .
- B)  $\|S_n - F\|_{[\alpha, c] \cup [d, \beta]} \leq 10\delta_n / (d - c)$ .

Let  $\eta := n + 2$  be the size of the Chebyshev system  $H_n^*$ . Since  $S_n$  is a best approximant to  $F$ , there exist  $\eta + 1$  points where the maximum error,  $\epsilon_n$ , occurs with alternating sign. Suppose  $m + 1$  of these points  $y_0 < \dots < y_m$  lie in  $[\alpha, c]$  and  $\eta - m$  of these points  $y_{m+1} < \dots < y_\eta$  lie in  $[d, \beta]$ . Then  $S'_n$  has at least  $m - 1$  zeros in  $[\alpha, c]$  (one at each alternation point in  $[\alpha, c]$  except possibly at the endpoints  $\alpha$  and  $c$ ). Likewise  $S'_n$  has at least  $\eta - m - 2$  zeros

in  $(d, \beta)$ . So  $S'_n$  has at least  $\eta - 3$  zeros in  $(\alpha, c) \cup (d, \beta)$ . Note that this count excludes  $y_m$  and  $y_{m+1}$ . Thus  $S'_n$  has at most one more zero in  $(\alpha, \beta)$  unless  $S'_n$  vanishes identically (which is not possible for  $\eta > 1$ ). Now suppose  $S'_n$  has a zero (with sign change) on  $(c, d)$ . Then since there is at most one zero of  $S'_n$  in  $(c, d)$  it cannot be the case that both  $y_m = c$  and  $y_{m+1} = d$  with both  $S'_n(c) \neq 0$  and  $S'_n(d) \neq 0$ . (Otherwise  $\text{sign}(S_n(c) - f(c)) = \text{sign}(S_n(d) - f(d))$  as a consideration of the two cases shows.) But if  $y_m \neq c$  or  $y_{m+1} \neq d$  or  $S'_n(c) = 0$  or  $S'_n(d) = 0$ , we have accounted for all the zeros of  $S'_n$  by accounting for the (possibly) one additional zero (either  $S'_n$  vanishes at  $c$  or  $d$  or one of  $y_m$  or  $y_{m+1}$  is an interior alternation point where  $S'_n$  vanishes). Thus  $S'_n$  has no zeros with sign change in  $(c, d)$  and claim (A) is proved.

For claim (B) we make the following observation. Let

$$\epsilon_n := \|F - S_n\|_{[\alpha, c] \cup [d, \beta]}$$

Then

$$D_n := \epsilon_n T_n - S_n$$

has at least  $m$  zeros on  $[\alpha, c]$  and

$$D_n^* := D_n + 1 = 1 + \epsilon_n T_n - S_n$$

has at least  $\eta - m - 1$  zeros on  $[d, \beta]$  (counting the possibility of double zeros). Thus  $D_n^*$  has at least  $\eta - 3$  zeros on  $[\alpha, c] \cup [d, \beta]$ . Suppose  $T_n$  has at least 4 alternations on an interval  $[\delta, \gamma] \subset (c, d)$  and suppose that

$$S_n(\gamma) - S_n(\delta) \leq \epsilon_n$$

Then, because of part (A) and the oscillation of  $T_n$  on  $[\delta, \gamma]$ ,

$$D_n + \frac{S_n(\gamma) + S_n(\delta)}{2} = \epsilon_n T_n - \left[ S_n - \frac{S_n(\gamma) + S_n(\delta)}{2} \right]$$

has at least 3 zeros on  $[\alpha, \beta]$  and hence

$$D'_n = \left( D_n + \frac{S_n(\gamma) + S_n(\delta)}{2} \right)$$

has at least 2 zeros on  $[\delta, \gamma]$ . This, however, gives  $D'_n \in H'_n$  a total of at least  $\eta - 1 = n + 1$  zeros which is impossible. In particular,

$$S_n(\gamma) - S_n(\delta) > \epsilon_n$$

on any interval  $[\delta, \gamma] \subset (c, d)$  where  $T_n$  has at least 4 alternations. Thus

$$S_n(d) - S_n(c) \geq \frac{(d-c)}{5\delta_n} \epsilon_n$$

However, since  $S_n$  is a best approximation,

$$S_n(d) - S_n(c) \leq 1 + 2\epsilon_n$$

and we deduce claim (B) on comparing these last two inequalities and noting that  $\epsilon_n \leq 1/2$ .

The proof of (a) is now a routine argument which for simplicity we present only on the interval  $[\alpha, \beta] := [0, 1]$ . Let

$$V(x) = \sum_{i=1}^{m-1} \left( f\left(\frac{i+1}{m}\right) - f\left(\frac{i}{m}\right) \right) S_i(x) + f(0),$$

where  $S_i(x) \in H_n^*$  is the best approximant to

$$F_i(x) := \begin{cases} 0, & x \in [0, \frac{i}{m}] \\ 1, & x \in [\frac{i+1}{m}, 1] \end{cases}$$

(as in claims (A) and (B)). Then with  $(d - c) = 1/m$  we deduce that

$$|V(x) - f(x)| \leq m^2 10\delta_n \omega_f\left(\frac{1}{m}\right) + \omega_f\left(\frac{1}{m}\right)$$

and with  $m := 1/\sqrt{\delta_n}$

$$|V(x) - f(x)| \leq D\omega_f(\sqrt{\delta_n}).$$

The proof of part (b) is an obvious modification of the proof of part (a). ■

### 3 Weighted Incomplete Approximants for Special Weights

We restrict our attention to systems of the form

$$H_n = \text{span}\{w^n - 1, w^n x, \dots, w^n x^n\}$$

where  $w := w(x) \geq 0$ ,  $x \in [a, b]$ . Then for a large class of weights  $w$  we are guaranteed the existence of a support set  $S_w$  where all the zeros of all the associated Chebyshev polynomials lie. Moreover, whenever  $H_n$  satisfies the conditions of Lemma 1 we will be able to conclude that  $\{H_n\}$  is dense in the continuous functions that vanish off of  $S_w$ . Denseness, for such  $f$ , in this context means that there exists  $f_n \in H_n$ ,  $\lim_{n \rightarrow \infty} f_n = f$ . The basic result we need is (essentially) Corollary 2.5 due to Mhaskar and Saff [10].

**Theorem 1.** Let  $\Sigma := [a, b]$  with  $a, b$  possibly infinite. Let  $w(x) = e^{-Q(x)}$  where,  $Q(x)$  is continuous on  $[a, b]$ , convex on  $(a, b)$  and where  $w(x) \cdot |x| \rightarrow 0$  as  $|x| \rightarrow \infty$  (when  $\Sigma$  is unbounded). There exists a smallest compact interval  $S_w \subset \Sigma$  with the following properties.

- a) The Chebyshev polynomials for  $H_n$  have all their zeros in  $S_w$ .
- b) The zeros are dense in  $S_w$  in the sense that  $M_n|_{S_w} \rightarrow 0$  as  $n \rightarrow \infty$ .
- c) If  $p_n \in \Pi_n$ , then  $\|w^n p_n\|_{S_w} = \|w^n p_n\|_{\Sigma}$ ,  $n = 0, 1, \dots$ .
- d) If  $p_n \in \Pi_n$  and  $A$  is a compact subset of  $\Sigma \setminus S_w$ , then

$$\|w^n p_n\|_A = o(\|w^n p_n\|_{S_w}), \quad \text{as } n \rightarrow \infty.$$

The interval  $S_w$  is known (cf. [10]) to be the support of the unique probability measure  $\mu_w$  that minimizes the generalized energy integral

$$I[\mu] := \iint \log[|z - t|w(z)w(t)]^{-1} d\mu(z)d\mu(t)$$

over all probability measures supported on  $[a, b]$ . Moreover, for the case when  $Q(x)$  is convex on  $[a, b]$ , the endpoints of the support set  $S_w = [c^*, d^*]$  can be obtained by maximizing the so-called  $F$ -functional

$$F(c, d) = \log\left(\frac{d-c}{4}\right) - \frac{1}{\pi} \int_c^d \frac{Q(x)dx}{\sqrt{(d-x)(x-c)}},$$

over all pairs  $(c, d)$  with  $a \leq c < d \leq b$ . This maximum will be attained precisely when  $c = c^*$  and  $d = d^*$ , i.e. at the endpoints of  $S_w$ .

**Lemma 2.** a) Suppose  $w$  satisfies the conditions of Theorem 1 on  $[a, b]$  and  $S_w^c = [a, b] \setminus S_w$  is nonempty. Let  $H_n := \text{span}\{w^n \cdot 1, w^n x, \dots, w^n x^n\}$ . Suppose that

$$H_n^* := \text{span}\{1, w^n \cdot 1, w^n x, \dots, w^n x^n\}$$

and

$$H_n' := \text{span}\{(w^n \cdot 1)', (w^n x)', \dots, (w^n x^n)'\}$$

are both Chebyshev systems on the interval  $[a, b]$ , for all  $n$ . Then for every  $f \in C[a, b]$  that vanishes identically on  $S_w^c$  (a collection we denote by  $C_0[S_w]$ ), there exists a sequence  $p_n \in \Pi_n$ , with

$$\lim_{n \rightarrow \infty} \|w^n p_n - f\|_{[a, b]} = 0.$$

(This is referred to as  $\{H_n\}$  being dense in  $C_0[S_w]$ ).

b) Suppose  $H_n^*$  and  $H_n'$  are Chebyshev systems on  $S_w$  (but not necessarily on  $[a, b]$ ). Suppose the other assumptions of (a) hold. Then there exists a sequence  $p_n \in \Pi_n$  with

$$\lim_{n \rightarrow \infty} \|w^n p_n - f\|_{S_w} = 0$$

and

$$\lim_{n \rightarrow \infty} \|w^n p_n - f\|_A = 0.$$

where  $A$  is any compact subset of  $[a, b] \setminus S_w$ .

Proof. By Lemma 1 and Theorem 1,  $f \in C_0[S_w]$  is uniformly the limit of elements  $h_n^* \in H_n^*$  on  $[a, b]$ . We now show that  $f$  is actually the limit of elements  $q_n \in H_n$ . If  $h_n^* \rightarrow f$ ,  $h_n^* \in H_n^*$ , then we may write

$$h_n^*(x) := a_n + q_n(x), \quad q_n \in H_n, \quad a_n \in \mathbb{R}.$$

If  $|a_n| \rightarrow \infty$ , then  $q_n(x)/a_n \rightarrow -1$  uniformly on  $[a, b]$  and we may approximate constants from  $\{H_n\}$ . If  $|a_n| \not\rightarrow \infty$ , then there exists  $\{a_{n_k}\}$  with  $a_{n_k} \rightarrow c \neq \pm\infty$ . In this case  $\|q_{n_k}\|_{S_w}$  is uniformly bounded and by Theorem 1, part (d), if  $A$  is a compact subset of  $\Sigma \setminus S_w$ , then  $\|q_{n_k}\|_A \rightarrow 0$ . From this and the assumption that  $f \equiv 0$  on  $S_w^c$  we deduce that  $a_{n_k} \rightarrow 0$  and we are done. ■

We wish now to record classes of weights which satisfy the conditions of Lemma 2, part (a), because for these weights we can conclude that the weighted incomplete approximants are dense exactly in  $C_0[S_w]$ .

**Lemma 3.** *Suppose  $w \in C^\infty[a, b]$ ,  $w(x) \geq 0$ . If  $\text{span}\{1, w^n \cdot 1, \dots, w^n x^m\}$  is a Chebyshev system for all positive integers  $n$  and  $m$ , then  $\text{span}\{(w^n 1)', \dots, (w^n x^m)'\}$  is also a Chebyshev system.*

Proof. See [7, p. 378].

**Lemma 4.** *Suppose either*

- a)  $1/w$  is totally monotone on  $[a, b]$  or
- b)  $1/w(x) = \sum_{n=0}^\infty a_n(x-a)^n$ ,  $a_n \geq 0$ , is convergent on  $[a, b]$ , where  $a \geq 0$ .

Then  $w$  satisfies the conditions of Lemma 3.

Proof. To show that  $w$  satisfies the conditions of Lemma 2 it suffices to show that a non-vanishing linear form

$$L_m(x) := \frac{1}{w^n(x)} \sum_{i=0}^m b_i x^i$$

has at most  $m+1$  zeros. This follows, in both cases, on differentiating  $m+1$  times and observing that  $(L_m(x))^{(m+1)}$  has no sign changes in  $[a, b]$ . ■

This gives us the following result.

**Theorem 2.** *Suppose  $w$  satisfies the conditions of Theorem 1 and that either*

- a)  $w^{-1}$  is totally monotone on  $[a, b]$  or
- b)  $w^{-1}$  has a power series expansion at  $a$ , convergent on  $[a, b]$ , with non-negative coefficients or (equivalently to (b))

b')  $w^{-1}$  has all derivatives strictly positive on  $(a, b]$ .

Then  $\{w^n p_n\}$ ,  $p_n \in \Pi_n$ , is dense in  $C_0[S_w]$ .

Observe that weights of the following form work on any interval  $[a, b] \subset [0, \infty)$ .

a)  $\exp(-x^p)$ ,  $p$  a positive integer;

b)  $x^\theta$ ,  $\theta > 0$ ;

c)  $\exp(-x^\delta)$ ,  $\delta \in (0, 1)$ .

For (c) above, the convexity condition of Theorem 1, doesn't hold. However in the case  $\Sigma := [a, b] \subset [0, \infty)$ , we can replace the convexity of  $Q = \log(1/w)$  by the condition that  $xQ'(x)$  is strictly increasing on  $(a, b)$ .

We remark that for the generalized Weierstrass problem von Golitschek, Lorentz, and Makovoz (cf. [4]) have simultaneously but independently obtained results similar to Lemma 2.

## 4 The Sublinear and Superlinear Cases

For a sequence of positive numbers  $\{\lambda_n\}_1^\infty$ , we consider weighted spaces

$$H_n(w, \lambda_n) := \{w^{n\lambda_n} p_n \mid p_n \in \Pi_n\}$$

We expect the following to happen for "decent" nonconstant weights. If  $\lambda_n \rightarrow \infty$ , the approximation should be impossible. If  $\lambda_n \rightarrow 0$ , then the whole interval becomes the interval of approximation. If  $\lambda_n \rightarrow c > 0$ , then the approximants should live on the set  $S_{w^c}$  associated with  $w^c$ . We prove the following.

**Theorem 3.** *Assume  $w \in C[a, b]$ ,  $w \geq 0$  and  $w$  is nonconstant on  $[a, b]$ . Suppose that  $\lambda_n \rightarrow \infty$  as  $n \rightarrow \infty$ . If there exist  $w^{n\lambda_n} p_n \in H_n(w, \lambda_n)$  such that  $w^{n\lambda_n} p_n \rightarrow f$  as  $n \rightarrow \infty$  uniformly on  $[a, b]$ , then  $f \equiv 0$ .*

**Proof.** Suppose  $w^{n\lambda_n} p_n \in H_n(w, \lambda_n)$  converges uniformly to  $f > 0$  on  $[\alpha, \beta] \subset [a, b]$ . Since  $w$  is not constant on  $[\alpha, \beta]$ , there exist intervals  $I_1$  and  $I_2$  contained in  $[\alpha, \beta]$  with

$$\begin{aligned} 0 < c_1 \leq w(x) \leq c_2, \quad x \in I_1, \\ 0 < c_3 \leq w(x) \leq c_4 < c_1, \quad x \in I_2, \end{aligned}$$

for some positive constants  $c_1, c_2, c_3, c_4$ . Now from the convergence of  $w^{n\lambda_n} p_n$  to a strictly positive limit on  $[\alpha, \beta]$  we deduce the existence of positive constants  $d_1, d_2, d_3, d_4$  so that

$$\frac{d_2}{c_1^{n\lambda_n}} \leq \|p_n(x)\|_{I_1} \leq \frac{d_1}{c_3^{n\lambda_n}}, \quad n \geq N_1 \tag{4.1}$$

and

$$\frac{d_4}{c_4^{n\lambda_n}} \leq \|p_n(x)\|_{I_2} \leq \frac{d_3}{c_3^{n\lambda_n}}, \quad n \geq N_2.$$

From (3.1) and Bernstein's inequality we have

$$\|p_n(x)\|_{[a,b]} \leq \frac{d_5^n}{c_1^{n\lambda_n}}, \quad n \geq N_1$$

for some constant  $d_5$ . However with (3.2) and the facts that  $\lambda_n \rightarrow \infty$  and  $c_4 < c_1$  this leads to the contradiction that

$$\|p_n(x)\|_{[a,b]} \leq \frac{d_5^n}{c_1^{n\lambda_n}} < \frac{d_4}{c_3^{n\lambda_n}} \leq \|p_n(x)\|_{I_2}$$

for some large  $n$ . □

**Theorem 4.** *Suppose that for  $n$  large  $w^{\lambda_n}$  as well as  $w$  satisfy the conditions of Lemma 2.*

- a) *If  $\lim_{n \rightarrow \infty} \lambda_n = \theta > 0$ , then  $\{w^{n\lambda_n} p_n\}$  is dense in  $C_0[S^\theta]$ , where  $S^\theta$  is the support associated with  $w^\theta$ .*
- b) *If  $\lim_{n \rightarrow \infty} \lambda_n = 0$ , then  $\{1, w^{n\lambda_n} p_n\}$  is dense in  $C[a, b]$ .*

*Proof.* Part (a) requires knowing that  $S_w$  only depends on the  $n^{\text{th}}$  root asymptotic (cf. [11]) and the rest follows as before.

For part (b) we show that the zeros of the Chebyshev polynomials fill out  $[a, b]$  and apply Lemma 1. For this purpose consider functions of the form

$$F_n(x) := \left[ w^{(m+k)\lambda_n}(x)(b-x)^m(x-a)^k \right]^{n/(m+k)},$$

where the integer  $m+k$  divides  $n$ . Observe that since  $(F_n)^{(m+k)/n}$  converges to  $(b-x)^m(x-a)^k$  uniformly,  $F_n$  behaves like a  $\delta$ -function. In particular, given  $I \subset [a, b]$  it is possible to construct  $q_n \in \{w^{n\lambda_n} p_n\}$ ,  $\forall n \geq N_\epsilon$  so that

$$\|q_n(x)\|_{[a,b]-I} \leq \epsilon, \quad \max_{x \in I} q_n(x) \geq 2 \quad \text{and}$$

$$\min_{x \in I} q_n(x) \leq -2.$$

In fact, such  $q_n$  can be constructed having many oscillations of magnitude  $\geq 2$ . It now follows, that for  $n \geq N_\epsilon$  the Chebyshev polynomial,  $T_n$ , for  $\{w^{n\lambda_n} p_n\}$  has a zero in  $I$ ; otherwise  $T_n$  would have too many zeros. Thus,  $M_n|_{[a,b]} \rightarrow 0$  and we can apply Lemma 1, to get denseness. □

## 5 Remarks

1. The condition in Lemma 2 that  $H'_n$  be Chebyshev can be weakened to the following condition (as is apparent from the proof of Lemma 1).

Condition. Suppose  $S_n \in H_n^*$  is the best approximation to  $f$  on  $[\alpha, c] \cup [d, \beta] \subset S_w$  where

$$f(x) := \begin{cases} 0, & x \in [\alpha, c] \\ 1, & x \in [d, \beta] \end{cases}$$

Then  $S'_n$  has at most  $n + 1$  zeros in the interval  $(a', b')$  where  $a'$  is the first alternation point and  $b'$  is the last alternation point of error.

This always holds for the Jacobi weights  $x^u(1-x)^v$ ,  $u, v > 0$  because the zeros at 0 and 1 imply the above condition and we deduce that Lemma 2, part (b) holds for these weights. In this case the interval  $S_w$  can be given explicitly (cf. [7]):

$$S_w = [\sin(\tau_1 - \tau_2), \sin(\tau_1 + \tau_2)]$$

and  $\tau_1 = \arcsin\left(\frac{u+v}{1+u+v}\right)$  and  $\tau_2 = \arcsin\left(\frac{v-u}{1+u+v}\right)$ . Similar extensions to the generalized Jacobi weights  $(x - \alpha_1)^{u_1}(x - \alpha_2)^{u_2} \dots (x - \alpha_n)^{u_n}$  exist for similar reasons.

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