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CONSTRUCTIVE APPROXIMATION

Distribution of Interpolation Points of Best L_2-Approximants (nth Partial Sums of Fourier Series)

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Abstract. For a continuous 2π-periodic real-valued function f, we investigate the asymptotic behavior of the zeros of the error $f(\theta) - s_n(\theta)$, where $s_n(\theta)$ is the nth Fourier section. We prove that there is a subsequence $\{n_k\}$ for which such zeros (interpolation points) are uniformly distributed on $[-\pi, \pi]$. This extends previous results of Saff and Shekhtman. Moreover, results dealing with the maximal distance between consecutive zeros of $f - s_{n_k}$ are obtained. The technique of proof involves coefficient estimates for lacunary trigonometric polynomials in terms of its L_q-norm on a subinterval.

1. Introduction

Let $C^*[-\pi, \pi]$ denote the collection of all real-valued, continuous 2π-periodic functions. For $f \in C^*[-\pi, \pi]$ we denote by $s_n(\theta) = s_n(f, \theta)$ the nth partial sum of the classical Fourier series expansion for f; that is,

$$s_n(\theta) := \sum_{k=-n}^{n} c_k e^{ik\theta}, \tag{1.1}$$

where

$$c_k = c_k(f) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-ik\theta}\, d\theta, \qquad k = 0, 1, \ldots. \tag{1.2}$$

Since f is real valued we have $c_{-k} = \overline{c_k}$ so that

$$s_n(\theta) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos k\theta + b_k \sin k\theta), \tag{1.3}$$

where

$$a_k = 2 \operatorname{Re} c_k \qquad \text{and} \qquad b_k = -2 \operatorname{Im} c_k, \qquad k = 0, 1, \ldots.$$

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As is well known (see [Z]), the Fourier section $s_n(\theta)$ is the unique trigonometric polynomial of degree at most n of least squares (best L_2) approximation to f, i.e.,

$$E_n^2(f) := \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta) - s_n(\theta)|^2\, d\theta \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta) - t_n(\theta)|^2\, d\theta \tag{1.4}$$

for every $t_n \in \mathcal{T}_n$, where $\mathcal{T}_n$ denotes the collection of all trigonometric polynomials of degree at most n. Moreover, $f - s_n$ is orthogonal to $\mathcal{T}_n$ in the sense that

$$\int_{-\pi}^{\pi} (f(\theta) - s_n(\theta))t_n(\theta)\, d\theta = 0, \qquad \forall t_n \in \mathcal{T}_n. \tag{1.5}$$

The aim of this paper is to study the points $\theta \in [-\pi, \pi)$ where $s_n(\theta)$ interpolates $f(\theta)$. It is easy to see from (1.5) that for each $n = 0, 1, \dots,$ there are at least $2n + 1$ points

$$-\pi \le \theta_1^{(n)} < \theta_2^{(n)} < \cdots < \theta_{2n+1}^{(n)} < \pi$$

such that $s_n(\theta_j^{(n)}) = f(\theta_j^{(n)})$, $j = 1, 2, \dots, 2n + 1$. (In fact, $2n + 1$ such points exist where $f - s_n$ changes sign.) The question naturally arises (and was posed independently by A. Kroó and V. Tikhomiroff) as to the denseness and distribution of such interpolation points $\theta_j^{(n)}$.

The issue of denseness was recently resolved by E. B. Saff and B. Shekhtman [SS] who proved the following:

Theorem 1.1. *Let $F \in C^*[-\pi, \pi]$. Then for each subinterval $[a, b] \subset [-\pi, \pi)$ there exists a subsequence of integers $\{n_j\}$ such that $f(\theta) - s_{n_j}(\theta)$ changes sign on $[a, b]$.*

Furthermore, for the analogous case of Legendre expansions on $[-1, 1]$, it is shown in [SS] that the assertion of Theorem 1.1 does not, in general, hold for all sufficiently large integers n.

While Theorem 1.1 settles the issue of denseness of interpolation points, it does not resolve several important related questions. First, does the subsequence s_{n_j} of Theorem 1.1 necessarily depend on the selected subinterval $[a, b]$ or does there exist a "universal" subsequence s_{m_k} with the property that $f - s_{m_k}$ vanishes on every subinterval $[a, b]$ for all $k \ge k_{[a,b]}$? (In [SS] it is shown that this is nearly true in the sense that there exists a subsequence $\{m_k\}$ of integers such that either $f - s_{m_k}$ or $f - s_{m_k - 1}$ has a zero in any interval $[a, b]$ for all $k \ge k_{[a,b]}$.) More importantly, there remains the deeper question of the *distribution* of interpolation points. The classical result of Kadec [Ka] asserts that for the case of best *uniform* approximation to f on $[-\pi, \pi]$ by trigonometric polynomials, there exists a subsequence for which the corresponding interpolation points are uniformly distributed. A similar result for best L_1-approximation was obtained by Kroó and Peherstorfer [KP].

In this paper we settle the above questions for the Fourier sections. For example, as a consequence of Theorem 2.1, it follows that there exists a subsequence $\{s_{n_\nu}\}$ with the property that for any interval $[a, b] \subset [-\pi, \pi)$,

$$\liminf_{\nu \to \infty} \frac{\#\text{ of zeros of } (f - s_{n_\nu}) \text{ in } [a, b]}{2n_\nu} \geq \frac{b-a}{2\pi}. \tag{1.6}$$

The proof of (1.6) is much more complicated than the analogous results for L_∞ or L_1. It utilizes estimates for the coefficients of lacunary trigonometric polynomials that are related to the power-sum inequalities investigated by P. Turán [Tu].

The outline of the paper is as follows. In Section 2 we state our main results. The essential lemmas needed for their proofs are given in Section 3. The remaining sections contain the proofs of our theorems.

2. Statements of Main Results

As in the introduction, we denote by s_n the nth Fourier section of $f \in C^*[-\pi, \pi]$ and denote by $\{c_n\}_{-\infty}^{\infty}$ its Fourier coefficients.

Our main results are the following:

Theorem 2.1. *Let $0 < \delta < (\frac{2}{3})(2 - \sqrt{3})$. Then for each $f \in C^*[-\pi, \pi]$ there exists a subsequence $\{n_\nu\}_{\nu=1}^{\infty}$ of positive integers such that for any positive integer $k \leq 2n_\nu$ and any interval Δ with length $|\Delta| = k\pi/n_\nu + n_\nu^{-\delta}$, the difference $f - s_{n_\nu}$ has at least k zeros in Δ.*

Note that for any interval $\Delta = [a, b] \subset [-\pi, \pi)$, selecting

$$k = k_\nu := [(n_\nu|\Delta| - n_\nu^{1-\delta})/\pi]$$

in Theorem 2.1 yields the inequality (1.6).

Theorem 2.2. *Let $0 < \delta < (3 - \sqrt{5})/2$. Then there exists a subsequence $\{n_\nu\}_{\nu=1}^{\infty}$ such that the maximal distance between two consecutive zeros of $f - s_{n_\nu}$ is less than $n_\nu^{-\delta}$.*

Remark. Theorems 2.1 and 2.2 are L_2-analogues of the L_∞ results of Kadec [Ka] and Tashev [Ta], respectively.

Corollary 2.3. *For any $0 < \delta < (\frac{2}{3})(2 - \sqrt{3})$, there exist a subsequence $\{n_\nu\}_{\nu=1}^{\infty}$ and points $\{x_j^{(\nu)}\}_{j=1,\ldots,2n_\nu,\, \nu=1,2,\ldots}$ such that $f - s_{n_\nu}$ vanishes at $x_j^{(\nu)}, j = 1, 2, \ldots, 2n_\nu$, and $|x_j^{(\nu)} - j\pi/n_\nu| < n_\nu^{-\delta}, j = 1, 2, \ldots, 2n_\nu;\ \nu = 1, 2, \ldots$.*

The next result shows that subsequences $\{n_k\}$ for which $f - s_{n_k}$ is zero-free in an interval of fixed length (independent of k) must be *lacunary*. This essentially answers a question raised in [SS]. (A related result has also been obtained by A. Kroó [Kr].)

Theorem 2.4. *Let $f \in C^*[-\pi, \pi]$. If there exist a $\delta > 0$, a subsequence $\{n_k\}_{k=1}^{\infty}$, and a sequence of intervals $\{\Delta_k\}_{k=1}^{\infty}$ with the properties:*

(i) $|\Delta_k| \geq \delta$ *for* $k = 1, 2, \ldots$; *and*
(ii) $f - s_{n_k}$ *has no zeros in* Δ_k *for* $k = 1, 2, \ldots$;

then

$$\limsup_{k \to \infty} \frac{n_{k+1}}{n_k} > 1.$$

Remark. In the above theorems, the requirement that f be continuous on $[-\pi, \pi]$ can be relaxed. For example, it is enough to assume that $f \in L_2[-\pi, \pi]$. In this case, the assertions of the above theorems hold where "zeros" is replaced by "sign changes." By asserting that $f - s_n$ has a sign change at θ_0 we mean that for each $\varepsilon > 0$ neither the inequality $f - s_n \geq 0$ nor the inequality $f - s_n \leq 0$ holds a.e. on $(\theta_0 - \varepsilon, \theta_0 + \varepsilon)$.

3. Lemmas for Theorems 2.1 and 2.2

In this section we establish the essential lemmas needed for the proofs of Theorems 2.1 and 2.2. Throughout we let s_n denote the nth partial sum of the Fourier expansion for f and $E_n = E_n(f)$ denote the L_2 error as defined in (1.4). We denote by $\|f\|_{C(A)}$ the L_∞ norm of f on A.

Lemma 3.1. *If $f \in C^*[-\pi, \pi]$ and $f - s_n$ has no zeros on $[a, b]$, $b - a \geq 2\varepsilon > 0$, then*

$$\int_{a_1}^{b_1} |f(t) - s_n(t)|\, dt \leq E_n(f) \exp(-\kappa_0 \varepsilon n),$$

where $a_1 := (3a + b)/4$, $b_1 := (a + 3b)/4$, and $\kappa_0 > 0$ is an absolute constant.

The proof of Lemma 3.1 is essentially given in [SS], where algebraic rather than trigonometric polynomials are considered. Because the modifications are minor, we omit the details.

Lemma 3.2. *Let $\varepsilon > 0$, $n \geq 4$, and $0 \leq k \leq n$. Then there exists an even trigonometric polynomial P of degree n and $k + 1$ points*

$$0 = t_0 < t_1 < \cdots < t_k < (k\pi/n) + \varepsilon$$

such that $\|P\|_{C[(k\pi/n)+\varepsilon, \pi]} \leq 1$ and, for $\nu = 0, 1, \ldots, k$,

$$|P(t_\nu)| \geq \tfrac{1}{2} \exp(\tfrac{1}{33}\varepsilon^{3/2} n), \qquad (-1)^\nu P(t_\nu) > 0.$$

Proof. It is readily seen that the lemma is valid in the following four cases:

(i) $0 < \varepsilon \le 2\pi/n$. (Then we have $(\frac{1}{2})\exp(\varepsilon^{3/2}n/33) < 1$.)
(ii) $(k\pi/n) + \varepsilon \ge \pi$.
(iii) $2\pi/n < \varepsilon < \pi$ and $k \ge n-2$. (Then $(k\pi/n) + \varepsilon \ge \pi$.)
(iv) $k = 0$. (Then the lemma follows from the proof of Lemma 3.1.)

Let $2\pi/n < \varepsilon < \pi$, $1 \le k < n-2$, and $(k\pi/n) + \varepsilon < \pi$. Choose m such that $1 \le m \le n-1$ and

$$\frac{\varepsilon}{6} < \frac{k\pi m}{n(n-m)} \le \frac{\varepsilon}{2}.$$

This is possible because

$$3\,\frac{m}{n-m} \le \frac{m+1}{n-m-1}, \qquad \text{for} \quad m = 1, 2, \ldots, n-2; \quad n \ge 4,$$

$$\frac{k\pi}{n(n-1)} < \frac{\pi}{n} \qquad \text{and} \qquad \frac{k\pi(n-1)}{n} > \frac{\pi}{2}.$$

Set

$$\delta := \frac{k\pi m}{n(n-m)} \qquad \text{and} \qquad a := \frac{k\pi}{n-m} + \delta.$$

We have $\varepsilon/6 < \delta \le \varepsilon/2$,

$$\frac{k\pi}{n-m} - \frac{k\pi}{n} = \frac{k\pi m}{n(n-m)} = \delta$$

and hence

$$a = \frac{k\pi}{n} + 2\delta \le \frac{k\pi}{n} + \varepsilon < \pi.$$

Also, we have

$$m = \frac{\delta n^2}{\pi k + \delta n} \ge \frac{\delta}{\pi + \delta}\, n \ge \frac{\varepsilon n}{6(\pi + \pi/2)} \ge \frac{\varepsilon n}{30}.$$

Set

$$Q(x) := T_m\!\left(\frac{1 - \cos a + 2\cos x}{1 + \cos a}\right),$$

where $T_m(x) = \cos(m \arccos x)$, $|x| \le 1$, is the Chebyshev polynomial.

Clearly, Q is an even trigonometric polynomial of degree m and

$$\|Q\|_{C[a,\pi]} = 1.$$

On the other hand, with ch denoting the hyperbolic cosine, we have for $x \in [0, a - \delta]$

$$\begin{aligned} Q(x) &= T_m\left(\frac{1 - \cos a + 2\cos x}{1 + \cos a}\right) \geq T_m(\operatorname{ch}\sqrt{a^2 - x^2}) \\ &\geq \tfrac{1}{2}(\operatorname{ch}\sqrt{a^2 - x^2} + \sqrt{\operatorname{ch}^2\sqrt{a^2 - x^2} - 1})^m = \tfrac{1}{2}\exp(m\sqrt{a^2 - x^2}) \\ &\geq \tfrac{1}{2}\exp(m\sqrt{a^2 - (a-\delta)^2}) \geq \tfrac{1}{2}\exp(\sqrt{\delta a m^2}) \\ &\geq \tfrac{1}{2}\exp\left(\sqrt{\frac{\varepsilon}{6}\frac{k\pi m}{n(n-m)}}\,nm\right) \geq \tfrac{1}{2}\exp\left(\sqrt{\left(\frac{\varepsilon}{6}\right)^2\frac{\varepsilon}{30}}\,n^2\right) \\ &\geq \tfrac{1}{2}\exp(\tfrac{1}{33}\varepsilon^{3/2}n), \end{aligned}$$

where we have used the following inequality:

$$\frac{1 - \cos a + 2\cos x}{1 + \cos a} \geq \operatorname{ch}\sqrt{a^2 - x^2}, \qquad 0 \leq x \leq a \leq \pi$$

(see [Ka]).

Set $P(x) := Q(x)\cos(n - m)x$. Clearly, P is an even trigonometric polynomial of degree n and

$$\|P\|_{C[k\pi/n + \varepsilon, \pi]} \leq \|P\|_{C[a,\pi]} = 1.$$

Set $t_\nu := (\nu\pi)/(n - m)$. We have

$$(-1)^\nu P(t_\nu) = Q(t_\nu) \geq \tfrac{1}{2}\exp(\tfrac{1}{33}\varepsilon^{3/2}n)$$

for $\nu = 0, 1, \ldots, k(t_\nu \in [0, a - \delta])$, which proves the lemma. ∎

Lemma 3.3. *Let $\Delta = [a, b]$, $|\Delta| < 2\pi$, and $a \leq t_1 < \cdots < t_l = b$, $l < 2n$. Then, there exists a trigonometric polynomial P of degree n, and points*

$$b \leq x_1 < \cdots < x_{2n-l+1} \leq a + 2\pi,$$

with the following properties:

(i) $P(t_k) = 0$ *for* $k = 1, 2, \ldots, l$ *and* $P(t) \neq 0$ *for* $t \in \Delta \backslash \{t_1, \ldots, t_l\}$; *and*
(ii) $\|P\|_{C[b, a+2\pi]} = 1$ *and* $(-1)^\nu P(x_\nu) = 1$ *for* $\nu = 1, \ldots, 2n - l + 1$.

The proof of this lemma is similar to the proof of the Chebyshev equioscilation theorem (see [Ti]) and is omitted.

Lemma 3.4. *Let the assumptions of Lemma 3.3 hold and let Δ be an interval of length $|\Delta| \geq (l\pi/n) + \varepsilon$, where $0 < \varepsilon < 2\pi$. Then the polynomial P of Lemma 3.3 has the following additional property*:

$$\|P\|_{C(\Delta)} \geq \tfrac{1}{2}\exp(\tfrac{1}{264}\varepsilon^{3/2}n).$$

Proof. If $0 < \varepsilon \leq 4\pi/n$, then $(\frac{1}{2})\exp((1/264)\varepsilon^{3/2}n) < 1$ and the lemma is valid since, if $\|P\|_{C(\Delta)} < 1 - \delta$ for some $\delta > 0$, then $(1 - \delta)\cos(nx + \alpha) - P(x)$ has more than $2n$ zeros in $[-\pi, \pi)$ for a suitable choice of α.

Now let $\varepsilon > 4\pi/n$. Choose an integer k such that $l < 2k \leq l + 2$ and set

$$\varepsilon_1 := \frac{1}{2}\left(\varepsilon - \frac{2\pi}{n}\right),$$

so that $\varepsilon/4 < \varepsilon_1 < \varepsilon/2$. Then we have

$$\frac{2k\pi}{n} + 2\varepsilon_1 \leq \frac{l\pi}{n} + \varepsilon.$$

Now, Lemma 3.2 implies that there exists a polynomial Q with $2k + 1$ sign alternation points $u_1, \ldots, u_{2k+1}$ in Δ such that

$$|Q(u_\nu)| \geq \tfrac{1}{2}\exp(\tfrac{1}{33}\varepsilon_1^{3/2}n) \qquad \text{and} \qquad \|Q\|_{C([-\pi,\pi)\setminus\Delta)} = 1.$$

Now, we claim that

$$\|P\|_{C(\Delta)} \geq \tfrac{1}{2}\exp(\tfrac{1}{33}\varepsilon_1^{3/2}n).$$

Indeed, assume to the contrary that

$$\|P\|_{C(\Delta)} < \frac{1-\delta}{2}\exp(\tfrac{1}{33}\varepsilon_1^{3/2}n) \qquad \text{for some } \delta \quad (1 > \delta > 0).$$

Then the polynomial $(1 - \delta)Q(x) - P(x)$ has at least $2k$ zeros in Δ and $2n - l$ zeros in $[-\pi, \pi)\setminus\Delta$, i.e., this polynomial has at least $2k + 2n - l > 2n$ different zeros in $[-\pi, \pi)$. But this is a contradiction. Therefore,

$$\|P\|_{C(\Delta)} \geq \tfrac{1}{2}\exp(\tfrac{1}{33}\varepsilon_1^{3/2}n) \geq \tfrac{1}{2}\exp(\tfrac{1}{264}\varepsilon^{3/2}n). \qquad \blacksquare$$

Lemma 3.5. *Let $f \in C^*[-\pi, \pi]$ and let k and n be integers with $0 \leq k \leq 2n$. Let Δ be an interval such that $2\pi > |\Delta| \geq (k\pi/n) + \varepsilon$ for some ε, $0 < \varepsilon < 1$. Assume that $f - s_n$ possesses no more than k zeros in Δ. Then there exists a set Ω with the following three properties*:

(i) *$\Omega \subset \Delta$ and Ω is a union of no more than $2n$ disjoint intervals*;
(ii) *$|\Omega| \geq \kappa_1\varepsilon^{3/2}$; and*
(iii) *$\int_\Omega |f(x) - s_n(x)|\,dx \leq E_n(f)\exp(-\kappa_2\varepsilon^{3/2}n)$;*

where κ_1 $(0 < \kappa_1 < 1)$ and κ_2 $(0 < \kappa_2 < 1)$ are absolute constants.

Proof. If $\varepsilon^{3/2}n < 528 \ln 2$, then the lemma follows by choosing Ω to be a subinterval of Δ of length $|\Omega| = |\Delta|/(3\pi)$. So assume that $\varepsilon^{3/2}n \geq 528 \ln 2$. Let $\Delta = [a, b]$ and $a \leq t_1 < t_2 < \cdots < t_l \leq b$, $l \leq k$, be the zeros of $f - s_n$ in Δ and consider the polynomial P of Lemma 3.4. We have

$$\|P\|_{C(\Delta)} \geq \tfrac{1}{2}\exp(\tfrac{1}{264}\varepsilon^{3/2}n) =: A.$$

The proof of the lemma is based on the following result due to T. Erdelyi [E]:

If p is a trigonometric polynomial of degree n and

$$|\{t \in [-\pi, \pi): |p(t)| \le 1\}| \ge 2\pi - s \qquad (0 < s < \pi/2),$$

then

$$\|p\|_{C[-\pi, \pi]} \le \exp(\kappa_3 ns),$$

where $\kappa_3 > 1$ is an absolute constant.

Using this statement for the polynomial $P/\sqrt{A}$ and setting

$$\Omega := \{t \in \Delta: |P(t)/\sqrt{A}| \ge 1\}$$

we get that in the case when $|\Omega| < \pi/2$

$$|\Omega| \ge \frac{1}{2\kappa_3 n} \ln A = \frac{1}{2\kappa_2 n} (\tfrac{1}{264}\varepsilon^{3/2}n - \ln 2) \ge \frac{\varepsilon^{3/2}}{1056\kappa_3}.$$

On the other hand, if $|\Omega| \ge \pi/2$, then the last inequality trivially holds.

Now, we prove the property (iii) of Ω. Since P is a trigonometric polynomial of degree n, we have

$$\int_{[-\pi, \pi]} (f(x) - s_n(x))P(x)\, dx = 0$$

and hence

$$\int_\Delta (f(x) - s_n(x))P(x)\, dx = -\int_{[-\pi, \pi]\setminus\Delta} (f(x) - s_n(x))P(x)\, dx$$

(without loss of generality we may assume that $\Delta \subset [-\pi, \pi]$).

Also, since $\|P\|_{C([-\pi, \pi]\setminus\Delta)} = 1$, it follows that

$$\left|\int_{[-\pi, \pi]\setminus\Delta} (f - s_n)P\, dx\right| \le \int_{[-\pi, \pi]\setminus\Delta} |f - s_n|\, dx$$
$$\le \int_{[-\pi, \pi]} |f - s_n|\, dx \le 2\pi E_n.$$

Now, we estimate $\int_\Delta (f - s_n)P\, dx$. Since $f - s_n$ and P have the same zeros in Δ, the sign of $(f - s_n)P$ does not change in Δ. Therefore,

$$\left|\int_\Delta (f - s_n)P\, dx\right| = \int_\Delta |(f - s_n)P|\, dx$$
$$\ge \int_\Omega |(f - s_n)P|\, dx \ge \sqrt{A} \int_\Omega |f - s_n|\, dx.$$

Hence

$$\int_\Omega |f - s_n|\, dx \le \frac{2\pi}{\sqrt{A}} E_n \le E_n \exp(-(1/1056)\varepsilon^{3/2}n).$$

Note that the property (i) of Ω obviously holds, because P is a trigonometric polynomial of degree n. The lemma is proved. ■

Lemma 3.6. *Let*

$$P(x) = \sum_{j=1}^{m} c_j e^{in_j x},$$

where c_j are complex numbers, n_j are integers, and $n_j \neq n_k$ for $j \neq k$. Let Δ be an interval, $|\Delta| \leq 2\pi$, and $1 \leq q \leq \infty$. Then

$$\|P\|_{L_q(\Delta)} \geq (\kappa_4 |\Delta|)^{m+1/q} \max_{1 \leq j \leq m} |c_j|, \tag{3.1}$$

where κ_4 $(0 < \kappa_4 < 1/2\pi)$ is an absolute constant.

Proof. The proof of this lemma is based on the following estimate of Turán (see [Tu, Theorem 6.6, p. 69]). Let $g(v) := \sum_{j=1}^{m} b_j z_j^v$, where b_j, z_j are complex numbers and $\min_{1 \leq j \leq m} |z_j| = 1$ and let $k \geq m$. Then the inequality

$$\sum_{v=k+1}^{k+m} |g(v)|^2 \geq \frac{1}{m} \left(\frac{m}{2e(k+m+1)} \right)^{2m} \sum_{v=0}^{m} |g(v)|^2 \tag{3.2}$$

holds.

Estimate (3.2) and the well-known relations between l_1 and l_2 norms imply

$$\sum_{v=k+1}^{k+m} |g(v)| \geq \frac{1}{\sqrt{m(m+1)}} \left(\frac{m}{2e(k+m+1)} \right)^{m} \sum_{v=0}^{m} |g(v)|. \tag{3.3}$$

Now, using (3.3), we will prove that for any intervals Δ_1, Δ_2 with $|\Delta_1| = |\Delta_2| = |\Delta|$ and $\Delta_1 \cap \Delta_2 = \varnothing$ we have

$$\int_{\Delta_1} |P(x)|\, dx \geq (\kappa_4 |\Delta|)^m \int_{\Delta_2} |P(x)|\, dx, \tag{3.4}$$

where $0 < \kappa_4 < \frac{1}{2}$ is an absolute constant.

Indeed, without loss of generality, we may assume that $\Delta_2 = (0, \delta)$, $\Delta_1 = (a, a+\delta)$ where $\delta := |\Delta|$ and $\delta \leq a \leq \delta + 2\pi$. Applying estimate (3.3) with

$$g(v) = \sum_{j=1}^{m} b_j z_j^v, \qquad z_j = \exp\left(\frac{in_j\delta}{m+1}\right) \quad (|z_j| = 1), \qquad b_j = c_j \exp(in_j t),$$

we get

$$\sum_{v=k+1}^{k+m} \left| P\left(\frac{\delta v}{m+1} + t\right) \right| \geq \frac{1}{m+1} \left(\frac{m}{2e(k+m+1)} \right)^m \sum_{v=0}^{m} \left| P\left(\frac{\delta v}{m+1} + t\right) \right|$$

for $k \geq m$.

Integrating with respect to t over the interval $(0, \delta/(m+1))$ we get (for $k \geq m$)

$$\int_{\delta(k+1)/(m+1)}^{\delta(k+m+1)/(m+1)} |P(x)|\, dx \geq \frac{1}{m+1}\left(\frac{m}{2e(k+m+1)}\right)^m \int_0^\delta |P(x)|\, dx. \tag{3.5}$$

Choose $l = (a(m+1))/\delta$ and $k = [l]$. Clearly, $l \geq m+1$ and hence $k \geq m+1$. Then (3.5) implies

$$\int_{\delta l/(m+1)}^{\delta(l+m+1)/(m+1)} |P(x)|\, dx \geq \frac{1}{m+1}\left(\frac{m}{2e(l+m+1)}\right)^m \int_0^\delta |P(x)|\, dx.$$

This gives

$$\int_a^{a+\delta} |P(x)|\, dx \geq \frac{1}{(m+1)(1+1/m)^m}\left(\frac{\delta}{2e(a+\delta)}\right)^m \int_0^\delta |P(x)|\, dx,$$

which implies (3.4), since $\delta \leq 2\pi$, $a \leq 4\pi$. (Similar arguments are used in the proof of Theorem 19.1 in [Tu].)

It remains to show that (3.4) implies (3.1). Choose an integer k such that $k\delta \leq 2\pi < (k+1)\delta$. With no loss of generality we may assume that $\Delta = (0, \delta)$. Denote $\Delta_\nu = (\nu\delta, (\nu+1)\delta)$, $\nu = 0, 1, \ldots, k$. Estimate (3.4) implies

$$\int_\Delta |P(x)|\, dx \geq (\kappa_4|\Delta|)^m \int_{\Delta_\nu} |P(x)|\, dx.$$

Summation with respect to ν from 0 to k yields

$$(k+1)\int_\Delta |P(x)|\, dx \geq (\kappa_4|\Delta|)^m \int_{-\pi}^{\pi} |P(x)|\, dx,$$

and hence

$$\int_\Delta |P(x)|\, dx \geq \frac{\delta}{4\pi}(\kappa_4|\Delta|)^m \int_{-\pi}^{\pi} |P(x)|\, dx \geq (\kappa_4|\Delta|)^{m+1} \int_{-\pi}^{\pi} |P(x)|\, dx. \tag{3.6}$$

Suppose $|c_{j_0}| = \max_{1\leq j\leq m} |c_j|$. Then we have

$$\int_{-\pi}^{\pi} |P(x)|\, dx = \int_{-\pi}^{\pi} |P(x)e^{-in_{j_0}x}|\, dx \geq \left|\int_{-\pi}^{\pi} P(x)e^{-in_{j_0}x}\, dx\right|$$
$$= \left|\int_{-\pi}^{\pi} c_{j_0}\, dx\right| = 2\pi|c_{j_0}|.$$

Hence

$$\int_{-\pi}^{\pi} |P(x)|\, dx \geq 2\pi \max_{1\leq j\leq m} |c_j|.$$

This estimate and (3.6) imply (3.1). ∎

Lemma 3.7. *Let $P(x) = \sum_{j=1}^{N} c_j e^{in_j x}$, where c_j are complex numbers, n_j are real numbers, and $n_{j+1} \geq n_j + 1$ for $j = 1, 2, \ldots, N-1$. Let Δ be an interval and $|\Delta| \leq \frac{1}{2}$. Assume that for some integers μ and t ($\mu \geq 1$) and real number m we have*

$$m \geq \frac{\kappa_5 \mu \ln(1/|\Delta|)}{|\Delta|}, \qquad \kappa_5 := 980 \ln(1/\kappa_4),$$

where κ_4 is the constant from Lemma 3.6,

$$\max_{1 \leq j \leq N} |c_j| = \max_{t+1 \leq j \leq t+\mu} |c_j|$$

and

$$\text{(3.7)} \qquad \begin{cases} n_{t+1} - n_t \geq m & \text{provided } \ t > 0, \\ n_{t+\mu+1} - n_{t+\mu} \geq m & \text{provided } \ t + \mu < N. \end{cases}$$

Then

$$\|P\|_{L_1(\Delta)} \geq (\kappa_6 |\Delta|)^{2\mu+1} \max_{1 \leq j \leq N} |c_j|,$$

where $\kappa_6 > 0$ is an absolute constant.

Proof. We may assume that $\Delta = [-a, a]$ because any translation of Δ does not alter $|c_j|$. Assume that $t > 0$ and $t + \mu < N$. We have

$$P(x) = \sum_{j=t+1}^{t+\mu} c_j e^{in_j x} + \left(\sum_{j=1}^{t} c_j e^{in_j x} + \sum_{j=t+\mu+1}^{N} c_j e^{in_j x} \right) =: P_1(x) + P_2(x).$$

Set $l := [1 + 3\mu \ln(4/(\kappa_4 |\Delta|))]$, where κ_4 ($0 < \kappa_4 < \frac{1}{2}$) is the constant from Lemma 3.6.

Consider the following B-spline

$$B(x) = (l+1)(-1)^{l+1}[y_0, y_1, \ldots, y_{l+1}](x - \cdot)_+^l, \qquad y_j = \cos\left(\frac{l+1-j}{l+1}\pi\right).$$

Set $\varphi(x) := 2 \int_{-\infty}^{x} B(2t+1)\,dt$ for $x \leq 0$ and $\varphi(x) = \varphi(-x)$ for $x > 0$.

The function φ has the following properties (see [Scho] or [Schu, Theorem 4.34, p. 139]):

φ is an even function, supp $\varphi = [-1, 1]$;
φ is increasing in $[-1, 0]$, $\varphi(-1/2) = 1/2$, $\varphi(0) = 1$; and
$\varphi^{(l)}$ is absolutely continuous and

$$\|\varphi^{(l+1)}\|_{C(-\infty,\infty)} = l!\, 2^{2l}.$$

Set $\psi(x) = \varphi(x/a)$. We have supp $\psi = \Delta$. Clearly, we have

$$\begin{aligned}\int_\Delta |P(x)|\,dx &\geq \int_\Delta \left|\psi(x)\frac{\overline{P_1(x)}}{\|P_1\|_{C(\Delta)}}P(x)\right|dx \\ &\geq \frac{1}{\|P_1\|_{C(\Delta)}}\left|\int_\Delta \psi(x)\overline{P_1(x)}P(x)\,dx\right| \\ &= \frac{1}{\|P_1\|_{C(\Delta)}}\left(\int_\Delta \psi(x)|P_1(x)|^2\,dx - \left|\int_\Delta \psi(x)\overline{P_1(x)}P_2(x)\,dx\right|\right).\end{aligned}$$

Applying Lemma 3.6 to P_1 with $q = 2$ and using $\psi(x) \geq \frac{1}{2}$ for $x \in [-a/2, a/2]$ we get

$$\begin{aligned}\int_\Delta \psi(x)|P_1(x)|^2\,dx &\geq \frac{1}{2}\int_{-a/2}^{a/2}|P_1(x)|^2\,dx \\ &\geq \frac{1}{2}\left(\frac{\kappa_4|\Delta|}{2}\right)^{2\mu+1}\max_j |c_j|^2.\end{aligned} \tag{3.8}$$

By changing the variable $x \to ax$ we find

(3.9)

$$\begin{aligned}\left|\int_\Delta \psi(x)\overline{P_1(x)}P_2(x)\,dx\right| &= a\left|\int_{-1}^{1}\varphi(x)\overline{P_1(ax)}P_2(ax)\,dx\right| \\ &= \frac{|\Delta|}{2}\left|\sum_{j=t+1}^{t+\mu}\left(\sum_{\nu=1}^{t} + \sum_{\nu=t+\mu+1}^{N}\right)\bar{c}_j c_\nu \int_{-1}^{1}\varphi(x)e^{ia(n_\nu - n_j)x}\,dx\right|.\end{aligned}$$

Using the properties of φ, we get

$$\begin{aligned}\gamma(u) := \int_{-1}^{1}\varphi(x)e^{iux}\,dx &= \int_{-1}^{1}\varphi(x)\cos ux\,dx \\ &= \frac{1}{u}\int_{-1}^{1}\varphi(x)d\sin ux = -\frac{1}{u}\int_{-1}^{1}\varphi'(x)\sin ux\,dx \\ &= -\frac{1}{u^2}\int_{-1}^{1}\varphi''(x)\cos ux\,dx = \cdots \\ &= \begin{pmatrix}+1\\ \text{or}\\ -1\end{pmatrix}\frac{1}{u^{l+1}}\int_{-1}^{1}\varphi^{(l+1)}(x)\begin{pmatrix}\cos ux\\ \text{or}\\ \sin ux\end{pmatrix}dx.\end{aligned}$$

Hence

$$|\gamma(u)| \leq \frac{2\|\varphi^{(l+1)}\|_{C[-1,1]}}{|u|^{l+1}} \leq \frac{l!\,2^{2l+1}}{|u|^{l+1}} < \frac{10(2l)^l}{|u|^{l+1}} \tag{3.10}$$

for any integer $l \geq 4$ and $u \neq 0$.

Using (3.7), (3.9), and (3.10) we get

$$\begin{aligned}\left|\int_\Delta \psi(x)\overline{P_1(x)}P_2(x)\,dx\right| &\le \frac{|\Delta|}{2}\max_j |c_j|^2 \sum_{j=t+1}^{t+\mu}\left(\sum_{v=1}^{t} + \sum_{v=t+\mu+1}^{N}\right)|\gamma(a(n_v - n_j))| \\ &\le \frac{|\Delta|}{2}\left(\max_j |c_j|^2\right)2\mu \sum_{v=0}^{\infty}\frac{10(2l)^l}{a^{l+1}(m+v)^{l+1}} \\ &\le \mu|\Delta|\frac{10(2l)^l}{a^{l+1}l(m-1)^l}\max_j |c_j|^2 \\ &\le \frac{3\mu(5l)^l}{|\Delta|^l m^l}\max_j |c_j|^2.\end{aligned}$$

These estimates and (3.8) imply

$$\begin{aligned}\int_\Delta |P(x)|\,dx &\ge \frac{1}{\|P_1\|_{C(\Delta)}}\left(\frac{1}{2}\left(\frac{\kappa_4|\Delta|}{2}\right)^{2\mu+1} - \frac{3\mu(5l)^l}{|\Delta|^l m^l}\right)\max_j |c_j|^2 \\ &\ge \frac{1}{4\mu}\left(\frac{\kappa_4|\Delta|}{2}\right)^{2\mu+1}\max_j |c_j|,\end{aligned}$$

where we have used the facts that $\|P_1\|_{C(\Delta)} \le \mu \max_j |c_j|$ and

$$\begin{aligned}m &\ge \frac{980\ln(1/\kappa_4)\mu\ln(1/|\Delta|)}{|\Delta|} \ge \frac{10l(4/\kappa_4)^{3\mu/l}}{|\Delta|^{1+(3\mu/l)}} \\ &\ge \frac{12^{1/l}\mu^{1/l}5l}{(\kappa_4/2)^{(2\mu+1)/l}|\Delta|^{(2\mu+l+1)/l}},\end{aligned}$$

by our assumption, and since

$$3\mu\ln\left(\frac{4}{\kappa_4|\Delta|}\right) < l < 6\mu\ln\left(\frac{4}{\kappa_4|\Delta|}\right),$$

Lemma 3.7 is proved. ∎

Lemma 3.8.

(i) *Let the series* $\sum_{n=1}^{\infty} b_n < \infty$ *be convergent,* $b_n \ge 0$, *and* $r_n := \sum_{k=n+1}^{\infty} b_k > 0$, *for* $n = 1, 2, \ldots$. *Let* $\sum_{n=1}^{\infty} a_n < \infty$, $a_n > 0$, $n = 1, 2, \ldots$. *Then, there exists a subsequence* $\{n_v\}_1^\infty$ *such that* $(b_{n_v+1}/r_{n_v}) > a_{n_v}$ *for every* $v = 1, 2, \ldots$.

(ii) *Let* $\varepsilon > 0$ *and* $\{c_n\}_{n=-\infty}^{\infty}$ *be the Fourier coefficients of* $f \in L_2[-\pi, \pi]$, f *not a trigonometric polynomial. Then, there exists a subsequence* $\{n_k\}_1^\infty$ *such that*

$$\frac{|c_{n_k}|}{E_{n_k-1}(f)} \ge \frac{1}{n_k^{(1/2+\varepsilon)}}, \qquad k = 1, 2, \ldots.$$

Proof. Suppose that (i) does not hold. Then there exists an N such that for every $n \geq N$

$$\frac{b_{n+1}}{r_n} \leq a_n.$$

Since $a_n \to 0$ as $n \to \infty$ we may assume that $a_n < \frac{1}{2}$ when $n \geq N$. Then for $n \geq N$

$$\frac{1}{2} < 1 - a_n \leq 1 - \frac{b_{n+1}}{r_n} = 1 - \frac{r_n - r_{n+1}}{r_n} = \frac{r_{n+1}}{r_n} \leq 1. \tag{3.11}$$

Since $\sum_1^\infty a_n$ is convergent, so is $\prod_{n=N}^\infty (1 - a_n)$. But the product

$$\prod_{n=N}^{N+M} \frac{r_{n+1}}{r_n} = \frac{r_{N+M+1}}{r_N} \to 0 \quad \text{as} \quad M \to \infty,$$

which implies by (3.11) that $\prod_{n=N}^\infty (1 - a_n)$ is divergent. This contradiction proves (i).

To prove (ii) we apply (i) to the series

$$\sum_{n=1}^\infty b_n = \sum_{n=1}^\infty |c_n|^2, \quad r_n = \tfrac{1}{2}E_n(f)^2 = \sum_{k=n+1}^\infty |c_n|^2 \quad \text{and} \quad a_n = \frac{2}{(n+1)^{1+2\varepsilon}}.$$

The lemma is proved. ■

4. Proofs of Theorems 2.1 and 2.2

We will now proceed with the

Proof of Theorem 2.1. Assume to the contrary that there exist a λ, $0 < \lambda < 2 - \sqrt{3}$, and an $N_1 > 1$ such that for every $n > N_1$ there exist k_n, $1 \leq k_n \leq 2n$, and an interval Δ_n such that $|\Delta_n| \geq (k_n \pi/n) + n^{-(2/3)\lambda}$ and $f - s_n$ has less than k_n zeros on Δ_n. Applying Lemma 3.5 with $\varepsilon = n^{-(2/3)\lambda}$ we conclude that for any $n > N_1$ there exists a set $G_n \subset (-\pi, \pi)$ with measure $|G_n| \geq \kappa_1 n^{-\lambda}$ which is a union of no more than $2n$ intervals and such that

$$\int_{G_n} |f(t) - s_n(t)|\, dt \leq E_n(f) \exp(-\kappa_2 n^{1-\lambda}), \tag{4.1}$$

where $E_n(f)$ is defined in (1.4).

Let $r = r(n)$ be an integer such that $\pi/2^{r-1} > \kappa_1/(8n^{1+\lambda}) \geq \pi/2^r$. Then the intervals $((\pi/2^r)(j-1), (\pi/2^r)j)$, $j = -2^r + 1, \ldots, 2^r$, have common length $\pi/2^r > \kappa_1/(16n^{1+\lambda})$ and the closure of their union is $[-\pi, \pi]$.

Denote by U_n the union of the intervals of the above type that are subsets of G_n. Notice that there are at least

$$\frac{\kappa_1 2^r}{n^\lambda \pi} - 2(2n) \geq 8n - 4n = 4n$$

subintervals that form U_n. Therefore,

$$|U_n| \geq 4n \frac{\kappa_1}{16n^{1+\lambda}} = \frac{\kappa_1}{4} n^{-\lambda}.$$

The following two propositions are immediate consequences of the above definitions:

Proposition 4.1. *Let $v > n > N_1$ and suppose that $U_n \cap U_v \neq \varnothing$. Then at least one of the intervals that form U_v is a subset of U_n. If Δ denotes such an interval, then*

$$\Delta \subset G_n \cap G_v \qquad \text{and} \qquad \tfrac{1}{2} > |\Delta| \geq \frac{\kappa_1}{16v^{1+\lambda}}.$$

Proposition 4.2. *If $N_1 < n_1 < n_2 < \cdots < n_\mu \leq N$ and $U_{n_j} \cap U_{n_l} = \varnothing$ for all $j \neq l$, then*

$$\mu \leq \frac{8\pi}{\kappa_1} N^\lambda. \tag{4.2}$$

Indeed, (4.2) follows from the inequality

$$2\pi \geq \left| \bigcup_{j=1}^{\mu} U_{n_j} \right| = \sum_{j=1}^{\mu} |U_{n_j}| \geq \mu \frac{\kappa_1}{4} N^{-\lambda}.$$

Now let $h(n)$ be a decreasing function of n that satisfies

$$h(n) \geq \exp(-\kappa_2 n^{1-\lambda}), \tag{4.3}$$

and set

$$Q(h) := \{n: |c_{n+1}| \geq h(n)E_n\}, \qquad Q_{a,b}(h) := \{n \in Q(h): a \leq n < b\}, \tag{4.4}$$

where the c_n's are the Fourier coefficients for f and $E_n = E_n(f)$. Our goal is to prove that $Q_{a,\infty}(h)$ is empty for some h and a. To this end, we estimate the number of elements of $Q_{n_1,n_2}(h)$ for different real numbers n_1, n_2 and functions h.

Proposition 4.3. *Let $N_1 < n \leq v_1 < v_2 \leq n^\rho$, $\rho > 1$. If $0 < \xi < 1 - \lambda$ and*

$$|Q_{v_1,v_2}(h)| \leq n^\xi,$$

where h is decreasing and satisfies (4.3), *then*

$$|Q_{v_1,v_2}(gh)| \leq \frac{8\pi}{\kappa_1} n^{\rho\lambda}, \tag{4.5}$$

where

$$g(n) = g(n, \rho, \lambda, \xi) := \left(\frac{16}{\kappa_4 \kappa_1} n^{\rho(\lambda+1)} \right)^{2n^\xi + 1} n^\rho \tag{4.6}$$

and κ_4 is the constant of Lemma 3.6.

Proof. Define

$$A := \{j \in Q_{v_1, v_2}(h) \colon \exists v, j < v < v_2 \text{ and } U_j \cap U_v \neq \varnothing\}.$$

Let $j \in A$ and v be the corresponding number from the definition. From Proposition 4.1 we obtain that there exists an interval $\Delta \subset G_j \cap G_v$, with $\frac{1}{2} > |\Delta| \geq \kappa_1/(16v^{1+\lambda})$. Hence from (4.1) we get

$$\int_\Delta |f(t) - s_j(t)|\, dt \leq E_j \exp(-\kappa_2 j^{1-\lambda})$$

and

$$\int_\Delta |f(t) - s_v(t)|\, dt \leq E_v \exp(-\kappa_2 v^{1-\lambda}) \leq E_j \exp(-\kappa_2 j^{1-\lambda}).$$

The triangle inequality then gives

$$\begin{aligned} \|s_v - s_j\|_{L_1(\Delta)} &\leq \|f - s_v\|_{L_1(\Delta)} + \|f - s_j\|_{L_1(\Delta)} \\ &\leq 2E_j \exp(-\kappa_2 j^{1-\lambda}). \end{aligned} \tag{4.7}$$

Set $D_1 := \{l \in \mathbf{Z} \colon j + 1 \leq |l| \leq v \text{ and } |l| - 1 \notin Q(h)\}$ and

$$D_2 := \{l \in \mathbf{Z} \colon j + 1 \leq |l| \leq v \text{ and } |l| - 1 \in Q(h)\}$$

and put

$$P_1(t) := \sum_{l \in D_1} c_l e^{ilt} \qquad \text{and} \qquad P_2(t) := \sum_{l \in D_2} c_l e^{ilt}.$$

Then $s_v - s_j = P_1 + P_2$, and so

$$\|P_2\|_{L_1(\Delta)} \leq \|s_v - s_j\|_{L_1(\Delta)} + \|P_1\|_{L_1(\Delta)}. \tag{4.8}$$

Recall that $|\Delta| < \frac{1}{2}$. Then from the definitions of $Q(h)$ and D_1 it follows that

$$\|P_1\|_{L_1(\Delta)} \leq |\Delta| \sum_{l \in D_1} |c_l| \leq (v - j)h(j)E_j.$$

Applying (4.7), (4.8), and (4.3) we get

$$\|P_2\|_{L_1(\Delta)} \leq (v - j + 2)h(j)E_j. \tag{4.9}$$

On the other hand, Lemma 3.6 asserts that for

$$m := 2|Q_{v_1, v_2}(h)| \geq 2|Q_{j, v}(h)| \geq |D_2|$$

we have

$$\|P_2\|_{L_1(\Delta)} \geq (\kappa_4 |\Delta|)^{m+1} \max_{l \in Q_{j,v}(h)} |c_{l+1}| \geq (\kappa_4 |\Delta|)^{m+1} |c_{j+1}|. \tag{4.10}$$

Consequently, using (4.9) and the lower estimate for $|\Delta|$, we obtain

$$|c_{j+1}| \leq \left(\frac{1}{\kappa_4|\Delta|}\right)^{m+1} (v - j + 2)h(j)E_j \tag{4.11}$$

$$\leq \left(\frac{16}{\kappa_4\kappa_1} v^{1+\lambda}\right)^{m+1} (v - j + 2)h(j)E_j.$$

Furthermore, since $m \leq 2n^{\xi}$, $v \leq n^{\rho}$ satisfy $v - j + 2 \leq n^{\rho}$, and $\kappa_1 < 1$, $\kappa_4 < 1$ we get

$$|c_{j+1}| \leq g(n)h(j)E_j < g(j)h(j)E_j.$$

We have thus proved that if $j \in A$, then $j \notin Q_{v_1, v_2}(gh)$. Therefore

$$A \cap Q_{v_1, v_2}(gh) = \varnothing.$$

Let $B := Q_{v_1, v_2}(h)\backslash A$. The definition of A gives that for every two elements $j \neq v$ of B we have $U_j \cap U_v = \varnothing$ and so (4.2) of Proposition 4.2 yields

$$|B| \leq \frac{8\pi}{\kappa_1} v_2^{\lambda} \leq \frac{8\pi}{\kappa_1} n^{\rho\lambda}. \tag{4.12}$$

Since $g(n) \geq 1$, we have $Q_{v_1, v_2}(gh) \subset Q_{v_1, v_2}(h)$ and hence $Q_{v_1, v_2}(gh) \subset B$. Therefore

$$|Q_{v_1, v_2}(gh)| \leq |B| \leq \frac{8\pi}{\kappa_1} n^{\rho\lambda},$$

which completes the proof. ∎

Proposition 4.4. *Let $\rho > 1$ satisfy $\rho\lambda < 1 - \lambda$. Then there exists*

$$N_2 = N_2(\rho, \lambda) \geq N_1$$

such that for every $n \geq N_2$

$$|Q_{n, n^{\rho}}(\psi^3)| \leq \frac{8\pi}{\kappa_1} n^{\rho\lambda}, \tag{4.13}$$

where $\psi(n) := \exp(-(\frac{1}{6})\kappa_2 n^{1-\lambda})$.

Proof. Let $\xi := (\frac{1}{2})(\rho\lambda + 1 - \lambda)$. Then $\rho\lambda < \xi < 1 - \lambda$. Set $\alpha := n^{\xi}$, and

$$\beta := [(\kappa_1/(8\pi))n^{\xi - \rho\lambda}]$$

and let $q = q(\rho, \lambda)$ be the least integer such that $\alpha\beta^q \geq n^{\rho}$. Set $h(n) := \exp(-\kappa_2 n^{1-\lambda})$. Since

$$g(n) = \left(\frac{16}{\kappa_4\kappa_1} n^{\rho(\lambda+1)}\right)^{2n^{\xi}+1} n^{\rho}$$

and $\xi < 1 - \lambda$ we can find $N = N(\rho, \lambda)$ such that the functions hg^j are decreasing for $n \geq N$ and $j = 0, 1, \ldots, q$. Obviously,

$$|Q_{n+j\alpha, n+(j+1)\alpha}(h)| \leq \alpha = n^\xi.$$

By applying Proposition 4.3 we get for $j = 0, 1, \ldots, \beta - 1$

$$|Q_{n+j\alpha, n+(j+1)\alpha}(gh)| \leq \frac{8\pi}{\kappa_1} n^{\rho\lambda} \leq \frac{n^\xi}{\beta}.$$

Therefore, $|Q_{n,n+\alpha\beta}(gh)| \leq n^\xi$.

Again applying Proposition 4.3 we get

$$|Q_{n,n+\alpha\beta}(g^2h)| \leq \frac{8\pi}{\kappa_1} n^{\rho\lambda} \leq \frac{n^\xi}{\beta},$$

and similarly (provided $\alpha\beta \leq n^\rho$)

$$|Q_{n+j\alpha\beta, n+(j+1)\alpha\beta}(g^2h)| \leq \frac{n^\xi}{\beta} \qquad \text{for} \quad j = 0, 1, \ldots, \beta - 1.$$

Hence

$$|Q_{n,n+\alpha\beta^2}(g^2h)| \leq n^\xi$$

which implies that

$$|Q_{n,n+\alpha\beta^2}(g^3h)| \leq \frac{8\pi}{\kappa_1} n^{\rho\lambda}$$

provided $\alpha\beta^2 \leq n^\rho$.

In the same manner we obtain

$$|Q_{n,n+\alpha\beta^s}(g^{s+1}h)| \leq \frac{8\pi}{\kappa_1} n^{\rho\lambda} \qquad \text{for} \quad s = 3, 4, \ldots, q - 1.$$

Furthermore, for $j = 1, \ldots, \beta - 1$,

$$|Q_{n+j\alpha\beta^{q-1}, n+(j+1)\alpha\beta^{q-1}}(g^qh)| \leq \frac{8\pi}{\kappa_1} n^{\rho\lambda} \leq \frac{n^\xi}{\beta},$$

provided $\alpha\beta^{q-1} \leq n^\rho$, where we have applied Proposition 4.3 with n replaced by $n + j\alpha\beta^{q-1}$. Thus

$$|Q_{n,n^\rho}(g^qh)| \leq |Q_{n,n+\alpha\beta^q}(g^qh)| \leq n^\xi,$$

and again applying Proposition 4.3 for $\nu_1 = n$, $\nu_2 = n^\rho$ we get

$$|Q_{n,n^\rho}(g^{q+1}h)| \leq \frac{8\pi}{\kappa_1} n^{\rho\lambda}. \tag{4.14}$$

Finally, we observe that q is a constant depending only on λ and ρ and that $h(n) = \psi(n)^6$. Thus it is easily verified that

(4.15)
$$\lim_{n\to\infty} \frac{(g(n))^{q+1}h(n)}{(\psi(n))^3} = 0,$$

and so there exists $N_2 = N_2(\rho, \lambda) \geq \max\{N, N_1\}$ such that $(g(n))^{q+1}h(n) \leq \psi^3(n)$ for $n \geq N_2$. Hence, from (4.14) and the definition of Q, we get (4.13). ∎

Proposition 4.5. *There exists $N_0 = N_0(\lambda)$ such that whenever $n \geq N_0$ and $|c_{n+1}| = \max_{j>n} |c_j|$, we have $n \notin Q(\psi)$, where ψ is given in Proposition* 4.4.

Proof. We have $\lim_{\nu\to\infty} \|f - s_\nu\|_{L_1[-\pi,\pi]} = 0$ and so there exists an $M = M(n) > n$ such that

(4.16)
$$\|f - s_M\|_{L_1[-\pi,\pi]} \leq \psi^2(n)E_n.$$

Let Δ be one of the intervals that form U_n. Then from $\frac{1}{2} > |\Delta| \geq \kappa_1/(16n^{1+\lambda})$ and (4.1) we have

(4.17)
$$\|f - s_n\|_{L_1(\Delta)} \leq \int_{G_n} |f(t) - s_n(t)|\, dt \leq E_n \exp(-\kappa_2 n^{1-\lambda}) \leq \psi^2(n)E_n.$$

Set $\hat{D}_1 := \{l \in \mathbf{Z}: n + 1 \leq |l| \leq M \text{ and } |l| - 1 \notin Q(\psi^3)\}$ and

$$\hat{D}_2 = \{l \in \mathbf{Z}: n + 1 \leq |l| \leq M \text{ and } |l| - 1 \in Q(\psi^3)\}$$

and put

$$\hat{P}_1(t) := \sum_{l\in\hat{D}_1} c_l e^{ilt} \quad \text{and} \quad \hat{P}_2(t) := \sum_{l\in\hat{D}_2} c_l e^{ilt}.$$

Then $s_M - s_n = \hat{P}_1 + \hat{P}_2$ and therefore

(4.18)
$$\|\hat{P}_2\|_{L_1(\Delta)} \leq \|f - s_n\|_{L_1(\Delta)} + \|f - s_M\|_{L_1(\Delta)} + \|\hat{P}_1\|_{L_1(\Delta)}.$$

Let $N_3 = N_3(\lambda)$ be such that for $n \geq N_3$

$$\sum_{j=n}^{\infty} \psi^3(j) = \sum_{j=n}^{\infty} e^{-(1/2)\kappa_2 j^{1-\lambda}} \leq e^{-(1/3)\kappa_2 n^{1-\lambda}} = \psi^2(n)$$

holds true. Then from the definition of Q we have

$$\|\hat{P}_1\|_{L_1(\Delta)} \leq 2|\Delta| \sum_{j=n}^{M} \psi^3(j)E_j \leq E_n \sum_{j=n}^{\infty} \psi^3(j) \leq \psi^2(n)E_n.$$

Hence from (4.16), (4.17), and (4.18) we get

(4.19)
$$\|\hat{P}_2\|_{L_1(\Delta)} \leq 3\psi^2(n)E_n.$$

Next, write $\hat{D}_2 = \{n_1, n_2, \dots, n_p, -n_1, -n_2, \dots, -n_p\}$, where

$$n < n_1 < n_2 < \cdots < n_p \le M.$$

Then

$$\hat{P}_2(t) = \sum_{k=1}^{p} (c_{n_k} e^{in_k t} + \bar{c}_{n_k} e^{-in_k t}).$$

If $n + 1 \notin \hat{D}_2$, then $|c_{n+1}| \le \psi^3(n) E_n \le \psi(n) E_n$ and therefore $n \notin Q(\psi)$. So we consider the case $n + 1 \in \hat{D}_2$. Then $n_1 = n + 1$. Set $\rho := 1 + \sqrt{3}$. Since $\lambda < 2 - \sqrt{3}$ we have

$$\rho\lambda = (1 + \sqrt{3})\lambda = (2 + \sqrt{3})\lambda - \lambda < 1 - \lambda \tag{4.20}$$

and

$$2\rho\lambda + 1 + \lambda = (3 + 2\sqrt{3})\lambda + 1 = \frac{\sqrt{3}\lambda}{2 - \sqrt{3}} + 1 < \sqrt{3} + 1 = \rho. \tag{4.21}$$

There are two cases to be considered.

Let first $n^\rho \le n_p$. Then there exists $q < p$ such that $n_q < n^\rho \le n_{q+1}$. Then from (4.20) and Proposition 4.4, $q \le (8\pi/\kappa_1) n^{\rho\lambda}$ and therefore there exists $n_\nu \in \hat{D}_2$, $1 \le \nu \le q$, such that

$$n_{\nu+1} - n_\nu \ge \frac{n_{q+1} - n_1}{q} \ge \frac{n^\rho - n - 1}{(8\pi/\kappa_1) n^{\rho\lambda}} =: m.$$

Now, we apply Lemma 3.7 to $\hat{P}_2$ with

$$\mu = \left[\frac{16\pi}{\kappa_1} n^{\rho\lambda}\right], \qquad |\Delta| \ge \frac{\kappa_1}{16 n^{1+\lambda}},$$

and the above defined m. The necessary inequality

$$m \ge \frac{\kappa_5 \mu \ln(1/|\Delta|)}{|\Delta|}$$

holds true for $n \ge N_4$ provided (4.21) since

$$\frac{\kappa_5 \mu \ln(1/|\Delta|)}{m|\Delta|} \le \frac{2048 \kappa_5 \pi^2}{\kappa_1^2} n^{2\rho\lambda + 1 + \lambda - \rho} \ln\left(\frac{16}{\kappa_1} n^{1+\lambda}\right) \to 0$$

as $n \to \infty$. By Lemma 3.7 we get

$$\|\hat{P}_2\|_{L_1(\Delta)} \ge (\kappa_6 |\Delta|)^{4\mu - 1} |c_{n+1}| \ge \left(\frac{\kappa_6 \kappa_1}{16} n^{-1-\lambda}\right)^{(64\pi/\kappa_1) n^{\rho\lambda}} |c_{n+1}|, \tag{4.22}$$

for the case $n_p \ge n^\rho$.

Now let $n_p < n^\rho$. Then from Proposition 4.4, $p \le (8\pi/\kappa_1)n^{\rho\lambda}$. From Lemma 3.6 we get

$$\|\hat{P}_2\|_{L_1(\Delta)} \ge (\kappa_4|\Delta|)^{2p+1}|c_{n+1}| \ge \left(\frac{\kappa_4\kappa_1}{16} n^{-1-\lambda}\right)^{(16\pi/\kappa_1)n^{\rho\lambda}} |c_{n+1}|, \tag{4.23}$$

for $n_p < n^\rho$. Since $\rho\lambda < 1 - \lambda$, there exists $N_5 = N_5(\lambda)$ such that for $n \ge N_5$

$$3e^{-(1/3)\kappa_2 n^{1-\lambda}} \max\left\{\left(\frac{16n^{1+\lambda}}{\kappa_1\kappa_4}\right)^{(16\pi/\kappa_1)n^{\rho\lambda}}, \left(\frac{16}{\kappa_6\kappa_1} n^{1+\lambda}\right)^{(64\pi/\kappa_1)n^{\rho\lambda}}\right\} < e^{-(1/6)\kappa_2 n^{1-\lambda}} = \psi(n).$$

Finally, for $n \ge N_0 = \max\{N_2, N_3, N_4, N_5\}$, we obtain from (4.19) and (4.23)

$$|c_{n+1}| < \psi(n)E_n. \quad \blacksquare$$

We have thus proved that

$$|c_{n+1}| < \psi(n)E_n, \qquad \text{when} \quad |c_{n+1}| = \max_{l>n} |c_l|.$$

Now, consider the case

$$|c_{n+1}| \ne \max_{l>n} |c_l| = |c_j|, \qquad j > n+1.$$

Then $|c_j| < \psi(j)E_j$ and $|c_{n+1}| < |c_j| < \psi(j)E_j < \psi(n)E_n$. Therefore $n \notin Q(\psi)$. We have shown that for all $n \ge N_0$ we have $n \notin Q(\psi)$, i.e.,

$$|c_{n+1}| < e^{-(1/6)\kappa_2 n^{1-\lambda}} E_n \qquad \text{for every} \quad n \ge N_0.$$

Since this contradicts the statement of Lemma 3.8, Theorem 2.1 is proved. ■

Proof of Theorem 2.2. The proof of Theorem 2.2 is quite similar to the proof of Theorem 2.1. For this reason, we mention only the necessary changes. Lemma 3.1 where $\varepsilon = n^{-\lambda}$ is applied instead of Lemma 3.5 with $\varepsilon = n^{-(2/3)\lambda}$. Then G_n is a single interval and, therefore, we have the estimate

$$|\Delta| \ge \frac{\kappa_1}{16} \nu^{-\lambda}$$

in Proposition 4.1. This leads to the estimate $2\rho\lambda + \lambda < \rho$ instead of

$$2\rho\lambda + 1 + \lambda < \rho.$$

Therefore, we can choose $\rho = (1 + \sqrt{5})/2$ and then for $0 < \lambda < (3 - \sqrt{5})/2$ we have $\rho\lambda = (1 + \sqrt{5})\lambda/2 = (3 + \sqrt{5})\lambda/2 - \lambda < 1 - \lambda$ and

$$2\rho\lambda + \lambda = (2 + \sqrt{5})\lambda < (2 + \sqrt{5})(3 - \sqrt{5})/2 = \rho.$$

All other details are omitted. ■

5. Proof of Theorem 2.4

Lemma 5.1. *Let the Fourier coefficients $\{c_n\}$ of the function f be such that*

$$\limsup_{n\to\infty} |c_n|^{1/n} = q < 1.$$

Let ε be a positive number with $\varepsilon < (1-q)/q$ if $q \neq 0$. Then there exist infinitely many integers n such that

$$|c_n| \geq \varepsilon \sum_{v>n} |c_v|.$$

Proof. Suppose to the contrary that there exist N and $\varepsilon > 0$ with $\varepsilon < (1-q)/q$ if $q \neq 0$ such that

$$|c_n| < \varepsilon \sum_{v>n} |c_v| \qquad \text{for every} \quad n > N.$$

Set

$$r_n := \sum_{v>n} |c_v|.$$

Then $r_{n-1} - r_n < \varepsilon r_n$ and $r_{n-1} < (1+\varepsilon) r_n$. Consequently,

$$r_n(1+\varepsilon)^{-p} \leq r_{n+p},$$

and so $\limsup_{p\to\infty} r_p^{1/p} \geq \limsup_{p\to\infty} r_{n+p}^{1/p} \geq (1+\varepsilon)^{-1}$. But, for analytic functions,

$$\limsup_{p\to\infty} r_p^{1/p} = \limsup_{p\to\infty} |c_p|^{1/p} = q.$$

Therefore,

$$\limsup_{p\to\infty} r_p^{1/p} = q \geq \frac{1}{1+\varepsilon} > q,$$

which is a contradiction. ∎

Proof of Theorem 2.4. Suppose to the contrary, that $\limsup_{k\to\infty} (n_{k+1}/n_k) = 1$. Without loss of generality, we may assume that $2\delta < 1$ and that $|\Delta_k| = 2\delta$.

Let μ be an integer with $2\pi/\delta < \mu < 4\pi/\delta$ and divide the interval $[-\pi, \pi)$ into μ equal intervals $J_1, J_2, \ldots, J_\mu$, where

$$J_t := [-\pi + 2\pi(t-1)/\mu, \ -\pi + 2\pi t/\mu).$$

Let $I_t := [-\pi + 2\pi(t-\frac{3}{4})/\mu, \ -\pi + 2\pi(t-\frac{1}{4}))$. Since $|\Delta_k| \geq \delta$ for $k = 1, 2, \ldots$, then for each k there exists an integer $p(k)$, such that:

(i) $1 \leq p(k) \leq \mu$; and
(ii) $J_{p(k)} \subset \Delta_k$.

Choose an integer M, such that

$$M > \frac{8\kappa_5 \ln(4/\delta)\mu^2}{\delta}, \tag{5.1}$$

where κ_5 is the constant from Lemma 3.7. Furthermore, choose $\varepsilon > 0$ such that

$$(1+\varepsilon)^M \le 2,$$

$$(1+\varepsilon)^M - 1 \le \frac{\kappa_0 \delta}{16 \ln(4/(\kappa_4 \delta))}, \tag{5.2}$$

$$\varepsilon \le \frac{\kappa_0 \delta}{320\mu \ln(4/(\kappa_6 \delta))},$$

where κ_0, κ_4, κ_6 are the constants from Lemmas 3.1, 3.6, and 3.7, respectively.
Since $\limsup_{k\to\infty} n_{k+1}/n_k = 1$, there exists an N such that

$$\frac{n_{k+1}}{n_k} \le (1+\varepsilon)^{1/2} \qquad \text{for} \quad k > N. \tag{5.3}$$

It is possible to "throw away" some of the elements of the sequence $\{n_k\}$, in such a way that the condition

$$(1+\varepsilon)^{1/2} \le \frac{n_{k+1}}{n_k} \le 1 + \varepsilon \tag{5.4}$$

holds for every k. Indeed, suppose we have chosen $\{m_1, \ldots, m_p\} \subset \{n_k\}_{k=1}^{\infty}$ and

$$(1+\varepsilon)^{1/2} \le \frac{m_{i+1}}{m_i} \le 1 + \varepsilon \qquad \text{for} \quad i = 0, 1, \ldots, p-1.$$

Let n_s be the largest element of the sequence $\{n_k\}$, such that $n_s \le (1+\varepsilon)m_p$. Then $(1+\varepsilon)^{1/2} m_p \le n_s \le (1+\varepsilon)m_p$. (If $n_s < (1+\varepsilon)^{1/2} m_p$, since $n_{s+1} > (1+\varepsilon)m_p$, we obtain $n_{s+1}/n_s > (1+\varepsilon)^{1/2}$ and this is a contradiction with (5.3).) Choose $m_{p+1} = n_s$. Then, by induction we obtain that there exists a subsequence $\{m_k\} \subset \{n_k\}$ such that $(1+\varepsilon)^{1/2} \le m_{k+1}/m_k \le 1+\varepsilon$ for $k = 1, 2, \ldots$. Without loss of generality, we may assume that $\{m_k\} \equiv \{n_k\}$.

For every $v > n_1$ we define $k(v)$ to be the integer such that $n_{k(v)} < v \le n_{k(v)+1}$. Let $A := \{v > n_1 \mid \exists l = l(v) \text{ such that } k(v) < l \le k(v) + M, \text{ and } p(k(v)) = p(l)\}$. Let $B := \{v > n_1 \mid v \notin A\}$.

From Lemma 3.1 it follows that

$$\int_{I_{p(k(v))}} |f - s_{n_{k(v)}}| \le e^{-\kappa_0 \delta n_{k(v)}/4} E_{n_{k(v)}},$$

$$\int_I |f - s_{n_{l(v)}}| \le e^{-\kappa_0 \delta n_{l(v)}/4} E_{n_{l(v)}},$$

$$I = I_{p(k(v))} = I_{p(l(v))}.$$

Then

$$\int_I |s_{n_{k(v)}} - s_{n_{l(v)}}| \le 2e^{-\kappa_0 \delta n_{k(v)}/4} E_{n_{k(v)}}.$$

From Lemma 3.6 it follows that

$$\int_I |s_{n_{k(v)}} - s_{n_{l(v)}}| \ge |c_v| \left(\kappa_4 \frac{\delta}{4}\right)^{n_{l(v)} - n_{k(v)}}.$$

But,

$$n_{l(v)} - n_{k(v)} \le n_{k(v)}((1+\varepsilon)^{l(v)-k(v)} - 1) < v((1+\varepsilon)^M - 1).$$

Since

$$\left(\kappa_4 \frac{\delta}{4}\right)^{(1+\varepsilon)^M - 1} \ge e^{-\kappa_0 \delta/16}$$

(see (5.2)), we have

$$|c_v| \le 2e^{-\kappa_0 \delta n_{k(v)}/8} E_{n_{k(v)}} \qquad \left(n_{k(v)} \ge \frac{n_{k(v)+1}}{1+\varepsilon} \ge \frac{v}{2}\right). \tag{5.5}$$

The inequality (5.5) holds for all $v \in A$.

We can easily see that the set B possesses the following property: for every $r \ge 1$ among the numbers $n_r, n_{r+1}, \dots, n_{r+M-1}$ there are no more than μ numbers that belong to B. Now, we estimate $|c_v|$, when $v \in B$.

Let $v \in B$. Since $c_v \to 0$ as $v \to \infty$, there exists $m = m(v) \ge v$, such that $|c_m| = \max_{k \ge v} |c_k|$. Among $n_{k(m)}, n_{k(m)+1}, \dots, n_{k(m)+M-1}$ there are no more than μ that belong to B. Thus there exists an integer r, $1 \le r \le M$, such that

$$n_{k(m)+r}, n_{k(m)+r+1}, \dots, n_{k(m)+r+(M/\mu)-1}$$

are elements of A, and from the definition of A it follows that all integers in $[n_{k(m)+r}, n_{k(m)+r+(M/\mu)}]$ belong to A.

Since f is a continuous function,

$$\int_{[-\pi,\pi]} |f - s_N| \to 0 \qquad \text{as} \quad N \to \infty,$$

and so there exists an N such that

$$\int_{[-\pi,\pi]} |f - s_N| \le e^{-\kappa_0 \delta n_{k(m)}/4} E_{n_{k(m)}}.$$

But, from Lemma 3.1, it follows that

$$\int_I |f - s_{n_{k(m)}}| \le e^{-\kappa_0 \delta n_{k(m)}/4} E_{n_{k(m)}}, \qquad I = I_{n_{k(m)}}.$$

Therefore,

$$\int_I |s_N - s_{n_{k(m)}}| \le 2e^{-\kappa_0 \delta n_{k(m)}/4} E_{n_{k(m)}}.$$

Now write $s_N - s_{n_{k(m)}} = P_1 + P_2$, where

$$P_1(x) := \sum_{\substack{n_{k(m)} < |\nu| \le N \\ \nu \in B}} c_\nu e^{i\nu x} \quad \text{and} \quad P_2(x) := \sum_{\substack{n_{k(m)} < |\nu| \le N \\ \nu \in A}} c_\nu e^{i\nu x}.$$

Then

$$\int_I |P_1| \le 2e^{-\kappa_0 \delta/4} E_{n_{k(m)}} + \int_I |P_2|.$$

But

$$\int_I |P_2| \le 2 \sum_{\substack{n_{k(m)} < \nu < N \\ \nu \in A}} |c_\nu| \le 2 \sum_{n_{k(m)} < \nu} (2e^{-\kappa_0 \delta n_{k(\nu)}/8} E_{n_{k(m)}})$$

$$\le 4E_{n_{k(m)}} \sum_{n_{k(m)} < \nu} e^{-\kappa_0 \delta \nu/16} \le \kappa(\delta) e^{-\kappa_0 \delta n_{k(m)}/16} E_{n_{k(m)}},$$

where $\kappa(\delta) := 4/(1 - e^{-\kappa_0 \delta/16})$. We obtain

$$\int_I |P_1| \le (2 + \kappa(\delta)) e^{-\kappa_0 \delta n_{k(m)}/16} E_{n_{k(m)}}. \tag{5.6}$$

Now, we apply Lemma 3.7. There are no more than $\mu n_{k(m)+r}\varepsilon =: \mu^*$ elements of B between $n_{k(m)}$ and $n_{k(m)+r}$. The "gap" after $n_{k(m)+r}$ is bigger than

$$n_{k(m)+r}((1 + \varepsilon)^{M/(2\mu)} - 1) \ge n_{k(m)+r}\varepsilon(M/2\mu) =: m^*.$$

We can use Lemma 3.7 to estimate $\int_I |P_1|$ since

$$m^* > \frac{4\kappa_5 \ln(4/\delta)\mu^*}{\delta} \quad \text{and} \quad |I| \ge \frac{\delta}{4}$$

(see (5.1)). We obtain

$$\int_I |P_1| \ge \left(\kappa_6 \frac{\delta}{4}\right)^{4\mu^* + 1} |c_m|$$

$$\ge \left(\kappa_6 \frac{\delta}{4}\right)^{5\mu\varepsilon(1+\varepsilon)^r n_{k(m)}} |c_m| \ge \left(\kappa_6 \frac{\delta}{4}\right)^{10\mu\varepsilon n_{k(m)}} |c_m|$$

(see (5.2)). Since $(\kappa_6(\delta/4))^{10\mu\varepsilon} \ge e^{-\kappa_0 \delta/32}$ (see (5.2)), it follows that

$$\int_I |P_1| \ge |c_m| e^{-\kappa_0 \delta b_{k(m)}/32}. \tag{5.7}$$

Combining (5.6) and (5.7) we get

$$|c_m| \le (2 + \kappa(\delta))e^{-\kappa_0 \delta n_{k(m)}/32} E_{n_{k(m)}}.$$

Since $|c_\nu| \le |c_m|$ we obtain

$$|c_\nu| \le (2 + \kappa(\delta))e^{-\kappa_0 \delta n_{k(\nu)}/32} E_{n_{k(\nu)}}, \tag{5.8}$$

for all $\nu \in B$. Because (5.8) is weaker than (5.5), the former estimate is valid for all $\nu > n_1$.

Thus, we have proved that for every ν with $n_k < \nu \le n_{k+1}$ the following inequality holds:

$$\begin{aligned} |c_\nu| &\le (2 + \kappa(\delta))e^{-\kappa_0 \delta n_k/32} E_{n_k} \\ &\le (2 + \kappa(\delta))e^{-\kappa_0 \delta \nu/(32(1+\varepsilon))} E_{n_k} \quad \text{(see (5.4))}. \end{aligned}$$

Put $a := 2 + \kappa(\delta) > 0$, $b := \kappa_0 \delta/(32(1 + \varepsilon)) > 0$. Then

$$|c_\nu| \le a e^{-b\nu} E_{n_k}. \tag{5.9}$$

Note that the last inequality holds for every integer ν in $(n_k, n_{k+1}]$. Thus we get

$$\max_{n_k < \nu \le n_{k+1}} |c_\nu| \le a e^{-b n_k} E_{n_k}. \tag{5.10}$$

From the inequality (5.9) we obtain a contradiction if $\limsup_{n\to\infty} |c_n|^{1/n} = 1$, and so the theorem is proved in this case.

Now, consider the case $\limsup_{n\to\infty} |c_n|^{1/n} = q < 1$. Lemma 5.1 implies that there exist infinitely many integers n such that

$$|c_n| \ge \varepsilon_1 \sum_{\nu > n} |c_\nu|,$$

where $0 < \varepsilon_1$ and $\varepsilon_1 < (1 - q)/q$ if $q \ne 0$. Hence there exist infinitely many integers k such that

$$\sum_{n_k < \nu \le n_{k+1}} |c_\nu|^2 \ge \varepsilon_1^2 \sum_{\nu > n_{k+1}} |c_\nu|^2 > 0. \tag{5.11}$$

Let k be an integer such that (5.11) holds. Then

$$\begin{aligned} E_{n_k}^2 &\le \frac{1 + \varepsilon_1^2}{\varepsilon_1^2} \sum_{n_k < \nu \le n_{k+1}} |c_\nu|^2 \le (n_{k+1} - n_k) \frac{1 + \varepsilon_1^2}{\varepsilon_1^2} \max_{n_k < \nu \le n_{k+1}} |c_\nu|^2 \\ &\le \varepsilon n_k \frac{1 + \varepsilon_1^2}{\varepsilon_1^2} \max_{n_k < \nu \le n_{k+1}} |c_\nu|^2 \quad \text{(see (5.4))}. \end{aligned}$$

Therefore,

$$E_{n_k} \le c\sqrt{n_k} \max_{n_k < \nu \le n_{k+1}} |c_\nu|, \qquad \text{where} \quad c := \frac{\sqrt{(1 + \varepsilon_1^2)\varepsilon}}{\varepsilon_1}.$$

Combining this inequality with (5.10) we obtain

$$0 < \max_{n_k < \nu \le n_{k+1}} |c_\nu| \le ace^{-bn_k}\sqrt{n_k} \max_{n_k < \nu \le n_{k+1}} |c_\nu|.$$

Hence

$$1 \le ace^{-bn_k}\sqrt{n_k}. \tag{5.12}$$

Finally, observe that (5.12) holds for infinitely many k. But,

$$e^{-bn_k}\sqrt{n_k} \to 0 \quad \text{as} \quad n_k \to \infty,$$

which is a contradiction. This completes the proof. ∎

Remark. In fact, we have proved a slightly stronger result:

$$\limsup_{n\to\infty} \frac{n_{k+1}}{n_k} \ge 1 + \varepsilon(\delta),$$

where $\varepsilon(\delta) = \kappa_7 \delta^4 \ln^{-2}(\delta)$, κ_7 is an absolute constant.

Theorem 5.2. *Let $f \in C^*[-\pi, \pi]$ and assume that its Fourier coefficients satisfy $\limsup_{n\to\infty} |c_n|^{1/n} < \frac{1}{2}$. Then there exist subsequences $\{n_k\}_{k=1}^{\infty}$ and $\{\alpha_{n_k}\}_{k=1}^{\infty}$ such that $f - s_{n_k}$ has a zero in the interval $((l\pi)/(n_k + 1) + \alpha_{n_k}, ((l + 1)\pi)/(n_k + 1))$ for each $k = 1, 2, \ldots$ and each $l = 1, 2, \ldots, 2n_k - 1$.*

Proof. We will apply Lemma 5.1. Note that for $q < \frac{1}{2}$ we have $(1 - q)/q > 1$ and so there exist s and ε such that $1 < \varepsilon < (1 - q)/q$. Therefore, there exists a sequence $\{n_k\}_{k=1}^{\infty}$ such that

$$|c_{n_k}| > \sum_{l=n_k+1}^{\infty} |c_l| = r_{n_k}.$$

Now consider the sequence $\{n_k - 1\}_{k=1}^{\infty}$. Then

$$f(x) - s_{n_k-1}(x) = \sum_{p=n_k}^{\infty} |c_p| \cos(p - \alpha_p)x.$$

But $|c_{n_k}| > \sum_{l=n_k+1}^{\infty} |c_l|$, and so

$$\operatorname{sign}\left(f\left(\frac{l\pi}{n_k} + \alpha_{n_k}\right) - s_{n_k-1}\left(\frac{l\pi}{n_k} + \alpha_{n_k}\right)\right) = \operatorname{sign}(\cos(l\pi)) = (-1)^l.$$

Hence $f - s_{n_k-1}$ has a sign change in the interval $[l\pi/n_k + \alpha_{n_k}, ((l + 1)\pi)/n_k + \alpha_{n_k}]$ and this completes the proof. ∎

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