

REMEZ- AND NIKOLSKII-TYPE INEQUALITIES FOR LOGARITHMIC POTENTIALS*

TAMÁS ERDÉLYI†, XIN LI‡, AND E. B. SAFF§

Abstract. Remez- and Nikolskii-type inequalities on line segments, on circles, and on certain bounded domains of the complex plane are established for exponentials of logarithmic potentials with respect to probability measures on \mathbb{C} having compact support.

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1. Introduction and notation. Generalized nonnegative polynomials of the form

$$f(z) = |\omega| \prod_{j=1}^k |z - z_j|^{r_j} \quad (\omega \in \mathbb{C}, z_j \in \mathbb{C}, 0 < r_j \in \mathbb{R}, j = 1, 2, \dots, k)$$

were studied in a sequence of recent papers [1], [2], [3], [4], [6], [7]. Several important polynomial inequalities were extended to this class by utilizing the generalized degree

$$N := \sum_{j=1}^k r_j$$

of f in place of the ordinary one. Since

$$\log f(z) = \sum_{j=1}^k r_j \log |z - z_j| + \log |\omega|,$$

a generalized nonnegative polynomial can be considered as a constant times the exponential of a logarithmic potential with respect to a finite Borel measure on \mathbb{C} that is supported in finitely many points (the measure has mass $r_j > 0$ at each $z_j, j = 1, 2, \dots, k$). This suggests that some of the inequalities holding for generalized nonnegative polynomials may be true for exponentials of logarithmic potentials of the form

$$Q_{\mu,c}(z) = \exp \left(\int_{\mathbb{C}} \log |z - t| d\mu(t) + c \right),$$

where μ is a finite, nonnegative Borel measure on \mathbb{C} having compact support and $c \in \mathbb{R}$. The quantity $\mu(\mathbb{C})$ plays the role of the generalized degree N , defined by (1.2). In this paper we extend a number of classical polynomial inequalities for exponentials

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† Department of Mathematics, Ohio State University, 231 West Eighteenth Avenue, Columbus, Ohio 43210. This author's research was supported in part by National Science Foundation grant DMS-902-4901.

‡ Department of Mathematics, University of Central Florida, Orlando, Florida 32816.

§ Institute for Constructive Mathematics, Department of Mathematics, University of South Florida, Tampa, Florida 33620. This author's research was supported in part by National Science Foundation grants DMS-881-4026 and DMS-891-2423.

of logarithmic potentials. Typically such extensions are not straightforward; indeed our proofs are far from simple density arguments.

Denote by \mathcal{P}_n^r the set of all algebraic polynomials of degree at most n with real coefficients and let \mathcal{P}_n^c be the set of all algebraic polynomials of degree at most n with complex coefficients. Let \mathcal{M} denote the set of all probability measures on \mathbb{C} with compact support. For $\mu \in \mathcal{M}$ and $c \in \mathbb{R}$ we define

$$P_{\mu,c}(z) := \int_{\mathbb{C}} \log |z - t| d\mu(t) + c \quad (z \in \mathbb{C})$$

$$Q_{\mu,c}(z) := \exp(P_{\mu,c}(z)) \quad (z \in \mathbb{C}).$$

Associated with $\mu \in \mathcal{M}$ and $c \in \mathbb{R}$ we introduce the sets

$$\begin{aligned} E_{\mu,c} &:= \{x \in [-1, 1] \mid P_{\mu,c}(x) \leq 0\} \\ &= \{x \in [-1, 1] \mid Q_{\mu,c}(x) \leq 1\} \end{aligned}$$

We will denote by $m_1(A)$ and $m_2(B)$ the one-dimensional Lebesgue measure of a set $A \subset \mathbb{R}$, and the two-dimensional Lebesgue measure of a set $B \subset \mathbb{C}$, respectively.

The Remez inequality [12] asserts that

$$\max_{-1 \leq x \leq 1} |p(x)| \leq T_n \left(\frac{2+s}{2-s} \right)$$

for every $p \in \mathcal{P}_n^r$ such that

$$m_1(\{x \in [-1, 1] \mid |p(x)| \leq 1\}) \geq 2 - s \quad (0 < s < 2),$$

where T_n is the Chebyshev polynomial of degree n , defined by $T_n(x) := \cos n\theta$, $x = \cos \theta$. Proofs of this inequality appear in [9, pp. 119–121] and [5]. In Theorem 2.1 we establish a sharp upper bound for $\max_{-1 \leq x \leq 1} Q_{\mu,c}(x)$ when $m_1(E_{\mu,c}) \geq 2 - s$, and (assuming $Q_{\mu,c}(x)$ is continuous) we find all $\mu \in \mathcal{M}$ and $c \in \mathbb{R}$ with $m_1(E_{\mu,c}) \geq 2 - s$ for which this sharp upper bound is achieved.

In Theorem 2.2 we establish pointwise upper bounds for $Q_{\mu,c}(x)$ for fixed $x \in [-1, 1]$, if $\mu \in \mathcal{M}$, $c \in \mathbb{R}$, and $m_1(E_{\mu,c}) \geq 2 - s$. An obvious bound for $Q_{\mu,c}(x)$ follows immediately from Theorem 2.1, but it turns out that for any fixed $-1 < x < 1$ this can be substantially improved. Indeed, Theorem 2.2 establishes essentially sharp upper bounds, which extend the validity of a pointwise Remez-type inequality [4, Thm. 4] proved for generalized nonnegative polynomials.

In Corollary 2.3 we offer another, slightly weaker version of the Remez-type inequality of Theorem 2.1, and in Theorem 2.4 we establish an analogue of Corollary 2.3, where the interval $[-1, 1]$ is replaced by the closure of a bounded domain $\Omega \subset \mathbb{C}$ with C^2 boundary, and the one-dimensional Lebesgue measure m_1 is replaced by m_2 . Such two-dimensional Remez-type inequalities seem to be new even for ordinary polynomials and for special domains, such as the open unit disk. Therefore we formulate a two-dimensional Remez-type inequality in this special case first, which turns out to be essentially sharp by Theorem 2.6.

Concerning L_p -versions of Remez's inequality, we study the following question: How large can the ratio

$$\frac{\int_{-1}^1 (Q_{\mu,c}(x))^p dx}{\int_A (Q_{\mu,c}(x))^p dx}$$

be if $\mu \in \mathcal{M}$, $c \in \mathbb{R}$, $A \subset [-1, 1]$, $m_1(A) \geq 2 - s$, $0 < s < 2$, and $p > 0$? We give an essentially sharp answer in Theorem 2.7 for the case when $0 < s \leq \frac{1}{2}$. In Theorem 2.8 we establish an essentially sharp upper bound for the ratio

$$\frac{\int_{\Omega} (Q_{\mu,c}(z))^p dm_2(z)}{\int_A (Q_{\mu,c}(z))^p dm_2(z)}$$

when $\Omega \subset \mathbb{C}$ is a bounded domain with C^2 boundary, $\mu \in \mathcal{M}$, $c \in \mathbb{R}$, $A \subset \bar{\Omega}$, $m_2(A) \geq m_2(\Omega) - s$, $s > 0$ sufficiently small, and $p > 0$. In Theorems 2.9 and 2.10 we give essentially the best possible Remez-type inequalities for exponentials of logarithmic potentials on the unit circle. The Remez-type inequalities of Corollary 2.3 and Theorems 2.4 and 2.9 will play a central role in establishing the Nikolskii-type inequalities for exponentials of potentials on $[-1, 1]$, on the unit circle and on bounded domains of \mathbb{C} with C^2 boundary. These Nikolskii-type inequalities are formulated in Theorems 3.1, 3.2, and 3.3.

2. Remez-type inequalities: statement of results. In this section we state our main results concerning Remez-type inequalities for logarithmic potentials on $[-1, 1]$, on the unit circle and on bounded domains having smooth boundaries. The proofs of these results will be given in §§6, 7, 8, and 9.

THEOREM 2.1. *Let $\mu \in \mathcal{M}$, $c \in \mathbb{R}$, and $E_{\mu,c}$ be defined as in (1.7). Then*

$$m_1(E_{\mu,c}) \geq 2 - s \quad (0 < s < 2)$$

implies

$$\max_{-1 \leq x \leq 1} Q_{\mu,c}(x) \leq \frac{\sqrt{2} + \sqrt{s}}{\sqrt{2} - \sqrt{s}}$$

Furthermore, if $Q_{\mu,c}$ restricted to $[-1, 1]$ is continuous on $[-1, 1]$, then the equality holds in (2.2) if and only if

$$\mu = \mu_{[-1, 1-s]}^* \quad \text{or} \quad \mu = \mu_{[-1+s, 1]}^*$$

and

$$c = -\log \frac{2-s}{4}$$

where μ_K^ denotes the equilibrium measure (cf. [14, §III.2]) of a compact set $K \subset \mathbb{C}$.*

We remark that $Q_{\mu,c}$ is upper semicontinuous on \mathbb{C} , so the maximum on $[-1, 1]$ is attained.

Concerning pointwise upper bounds for $Q_{\mu,c}(x)$ we shall prove the following result that extends the validity of Theorem 4 of [4].

THEOREM 2.2. *There is an absolute constant k_1 such that*

$$Q_{\mu,c}(x) \leq \exp \left(\left\{ \frac{s}{\sqrt{1-x^2}}, \sqrt{s} \right\} \right)$$

for every $-1 \leq x \leq 1, \mu \in \mathcal{M}$ and $c \in \mathbb{R}$ satisfying

$$m_1(E_{\mu,c}) \geq 2 - s \quad (0 < s \leq 1)$$

Here we do not examine what happens when $1 < s < 2$; the case $0 < s \leq 1$ is more important in applications. The sharpness of Theorem 2.2 (in the corresponding polynomial case) is shown in [4, §12].

Observe that the first assertion of Theorem 2.1 is equivalent to the following.

THEOREM 2.1*. *For every $\mu \in \mathcal{M}, c \in \mathbb{R}$ and $0 < t < 1$,*

$$m_1 \left(\left\{ x \in [-1, 1] : Q_{\mu,c}(x) > \frac{1 - \sqrt{t}}{1 + \sqrt{t}} \max_{-1 \leq y \leq 1} Q_{\mu,c}(y) \right\} \right) \geq 2t$$

Consequently, we obtain the following.

COROLLARY 2.3. *There is an absolute constant $k_2 > 0$ such that*

$$(2.6) \quad m_1 \left(\left\{ x \in [-1, 1] : Q_{\mu,c}(x) > \exp(-\sqrt{s}) \max_{-1 \leq y \leq 1} Q_{\mu,c}(y) \right\} \right) \geq k_2 s$$

for every $\mu \in \mathcal{M}, c \in \mathbb{R}$, and $0 < s < 2$.

In our next theorem we establish the analogue of Corollary 2.3 for the case when $[-1, 1]$ is replaced by the closure of a bounded domain $\Omega \subset \mathbb{C}$ with C^2 boundary.

THEOREM 2.4. *Let $\Omega \subset \mathbb{C}$ be a bounded domain with C^2 boundary. Then, there is a constant $k_3 = k_3(\Omega) > 0$ depending only on Ω such that*

$$m_2 \left(\left\{ z \in \bar{\Omega} : Q_{\mu,c}(z) > \exp(-\sqrt{s}) \max_{w \in \bar{\Omega}} Q_{\mu,c}(w) \right\} \right) \geq k_3 s$$

for every $\mu \in \mathcal{M}, c \in \mathbb{R}$, and $0 < s < m_2(\Omega)$.

Actually, in the above theorem it suffices to have a somewhat weaker geometric assumption for the boundary of Ω , namely, the following: there is an $\tau > 0$ depending only on Ω such that for each $z \in \partial\Omega$ there is an open disk D_z with radius τ such that $D_z \subset \Omega$ and $\bar{D}_z \cap \partial\Omega = \{z\}$. It is well known that if $\partial\Omega$ is a C^2 curve, then this property holds.

To prove Theorem 2.4 we will need the following result for polynomials. To formulate this, we introduce the notation

$$(2.8) \quad D := \{z \in \mathbb{C} : |z| < 1\}$$

and

$$(2.9) \quad \mathcal{P}_n^c(\bar{D}, s) := \{p \in \mathcal{P}_n^c : m_2(\{z \in \bar{D} : |p(z)| \leq 1\}) \geq \pi - s\} \quad (0 < s < \pi),$$

along with the analogous definition for $\mathcal{P}_n^r(\bar{D}, s)$.

THEOREM 2.5. *There is an absolute constant $k_4 > 0$ such that*

$$(2.10) \quad \max_{u \in \bar{D}} |p(u)| \leq \exp(k_4 n \sqrt{s})$$

for every $p \in \mathcal{P}_n^c(\bar{D}, s)$ and $0 < s \leq \frac{1}{4}$.

Our next theorem shows that the result of Theorem 2.5 is essentially sharp.

THEOREM 2.6. *There is an absolute constant $k_5 > 0$ such that*

$$(2.11) \quad \sup(\{|p(1)| : p \in \mathcal{P}_{3n}^r(\bar{D}, s)\}) \geq \exp(k_5 n \sqrt{s})$$

for every $0 < s \leq \frac{1}{2}$.

Using Theorems 2.1 and 2.4, we establish Remez-type inequalities in L_p ($0 < p < \infty$) for exponentials of potentials on both $[-1, 1]$ and bounded domains $\Omega \subset \mathbb{C}$ and C^2 boundary.

THEOREM 2.7. *There is an absolute constant $k_6 > 0$ such that*

$$(2.12) \quad \int_{-1}^1 (Q_{\mu,c}(x))^p dx \leq \left(1 + \left(\frac{1 + \sqrt{s}}{1 - \sqrt{s}}\right)^p\right) \int_A (Q_{\mu,c}(x))^p dx \\ \leq (1 + \exp(k_6 p \sqrt{s})) \int_A (Q_{\mu,c}(x))^p dx$$

for every $\mu \in \mathcal{M}$, $c \in \mathbb{R}$, $p > 0$, $0 < s \leq \frac{1}{2}$, and $A \subset [-1, 1]$ with $m_1(A) \geq 2 - s$. If $0 < s \leq \frac{1}{4}$, then $k_6 = 4$ is a suitable choice.

THEOREM 2.8. *Let $\Omega \subset \mathbb{C}$ be a bounded domain with C^2 boundary. Then there are constants $0 < k_7 = k_7(\Omega)$ and $0 < k_8 = k_8(\Omega)$ depending only on Ω such that*

$$(2.13) \quad \int_{\Omega} (Q_{\mu,c}(z))^p dm_2(z) \leq (1 + \exp(k_7 p \sqrt{s})) \int_A (Q_{\mu,c}(z))^p dm_2(z)$$

for every $\mu \in \mathcal{M}$, $c \in \mathbb{R}$, $p > 0$, $0 < s \leq k_8$, and $A \subset \bar{\Omega}$ with $m_2(A) \geq m_2(\Omega) - s$.

The following theorem establishes a Remez-type inequality for exponentials of logarithmic potentials on the unit circle, extending a Remez-type inequality for trigonometric polynomials [4, Thm. 3].

THEOREM 2.9. *There is an absolute constant $k_9 > 0$ such that*

$$\max_{-\pi \leq t \leq \pi} Q_{\mu,c}(e^{it}) \leq \exp(k_9 s)$$

for every $\mu \in \mathcal{M}$, $c \in \mathbb{R}$ and $0 < s \leq \pi/2$ whenever

$$(2.15) \quad m_1(\{t \in [-\pi, \pi] : Q_{\mu,c}(e^{it}) \leq 1\}) \geq 2\pi - s.$$

From this we will easily obtain the following.

THEOREM 2.10. *We have*

$$\int_{-\pi}^{\pi} (Q_{\mu,c}(e^{it}))^p dt \leq (1 + \exp(2k_9 ps)) \int_A (Q_{\mu,c}(e^{it}))^p dt$$

for every $\mu \in \mathcal{M}$, $c \in \mathbb{R}$, $p > 0$, $0 < s \leq \pi/4$, and $A \subset [-\pi, \pi]$ with $m_1(A) \geq 2\pi - s$. Here k_9 is the same as in Theorem 2.9.

We have formulated each of our results for probability measures on \mathbb{C} with compact support. This was done only for the sake of brevity. As an example, we rewrite the result of Theorem 2.1 for all finite Borel measures on \mathbb{C} with compact support.

COROLLARY 2.11. *Let μ be a finite Borel measure on \mathbb{C} with compact support, and let $c \in \mathbb{R}$ and $E_{\mu,c}$ be defined as in (1.7). Then*

$$(2.16) \quad m_1(E_{\mu,c}) \geq 2 - s \quad (0 < s < 2)$$

implies

$$\max_{-1 \leq x \leq 1} Q_{\mu,c}(x) \leq \left(\frac{\sqrt{2} + \sqrt{s}}{\sqrt{2} - \sqrt{s}} \right)^{\mu(\mathbb{C})}$$

Furthermore, if $Q_{\mu,c}$ restricted to $[-1, 1]$ is continuous, then equality holds in (2.17) if and only if

$$\mu = \mu(\mathbb{C})\mu_{[-1, 1-s]}^* \quad \text{or} \quad \mu = \mu(\mathbb{C})\mu_{[-1+s, 1]}^*$$

and

$$c = -\mu(\mathbb{C}) \log \frac{2-s}{4},$$

where μ_K^* denotes the equilibrium measure of a compact set $K \subset \mathbb{C}$.

3. Nikolskii-type inequalities: statement of results. Using Corollary 2.3 and Theorem 2.4 we will prove the following Nikolskii-type inequalities. The proofs will be given in §10.

THEOREM 3.1. *There is an absolute constant $k_{10} > 0$ such that*

$$\|Q_{\mu,c}\|_{L_p(-1,1)} \leq (k_{10}(1+q^2))^{1/q-1/p} \|Q_{\mu,c}\|_{L_q(-1,1)}$$

for every $\mu \in \mathcal{M}$, $c \in \mathbb{R}$, and $0 < q < p \leq \infty$.

THEOREM 3.2. *Let $\Omega \subset \mathbb{C}$ be a bounded domain with C^2 boundary. There exists a constant $k_{11} = k_{11}(\Omega) > 0$ depending only on Ω such that*

$$(3.2) \quad \|Q_{\mu,c}\|_{L_p(\Omega)} \leq (k_{11}(1+q^2))^{1/q-1/p} \|Q_{\mu,c}\|_{L_q(\Omega)}$$

for every $\mu \in \mathcal{M}$, $c \in \mathbb{R}$, and $0 < q < p \leq \infty$.

We remark that Theorem 3.1 is an extension of [7, Thm. 6], where the same inequality was proved when the support of μ is a finite set.

THEOREM 3.3. *There is an absolute constant $k_{12} > 0$ such that*

$$(3.3) \quad \|Q_{\mu,c}(e^{it})\|_{L_p(-\pi,\pi)} \leq (k_{12}(1+q))^{1/q-1/p} \|Q_{\mu,c}(e^{it})\|_{L_q(-\pi,\pi)}$$

for every $\mu \in \mathcal{M}$, $c \in \mathbb{R}$, and $0 < q < p \leq \infty$.

For general finite measures μ , Theorem 3.1 yields the following.

COROLLARY 3.4. *There is an absolute constant $k_{10} > 0$ such that*

$$(3.4) \quad \|Q_{\mu,c}\|_{L_p(-1,1)} \leq (k_{10}(1+(q\mu(\mathbb{C}))^2))^{1/q-1/p} \|Q_{\mu,c}\|_{L_q(-1,1)}$$

for every finite Borel measure μ on \mathbb{C} with compact support, $c \in \mathbb{R}$ and $0 < q < p \leq \infty$.

Theorems 3.2 and 3.3 have similar straightforward extensions.

4. Lemmas for Theorem 2.1. To prove Theorem 2.1 we need a series of lemmas, which we state in this section and prove in §5.

For a compact set $K \subset \mathbb{C}$ containing infinitely many points, let $T_{n,K} \in \mathcal{P}_n^c$ be the n th degree monic Chebyshev polynomial with respect to K , i.e.,

$$(4.1) \quad \|T_{n,K}\|_K = \inf_{p \in \mathcal{P}_{n-1}^c} \|z^n - p(z)\|_K$$

where $\|\cdot\|_K$ denotes the uniform norm on K . We also define the normalized Chebyshev polynomials

$$\widehat{T}_{n,K} := \frac{T_{n,K}}{\|T_{n,K}\|_K}.$$

LEMMA 4.1. Let $0 \leq \delta < 2, \delta < s < 2$, and $z \in \mathbb{C}$ with $\operatorname{Re} z \geq 1 - \delta$ fixed. Then

$$(4.3) \quad \sup |p(z)| = |\widehat{T}_{n,[-1,1-\delta]}(z)|,$$

where the supremum in (4.3) is taken over all $p \in \mathcal{P}_n^r$ satisfying

$$m_1(\{x \in [-1, 1 - \delta] : |p(x)| \leq 1\}) \geq 2 - s.$$

If $K \subset \mathbb{C}$ is a compact set we denote by $D_\infty(K)$ the unbounded component of the complement $\mathbb{C} \setminus K$. This domain is referred to as the *outer domain* of K and its boundary $\partial D_\infty(K)$ is called the *outer boundary* of K . If K has positive logarithmic capacity [14, p. 55], we denote by $g_{D_\infty(K)}(z, \infty)$ the Green function with pole at ∞ for $D_\infty(K)$. We remark that $g_{D_\infty(K)}(z, \infty)$ is the smallest positive harmonic function in $D_\infty(K) \setminus \{\infty\}$ that behaves like $\log |z| + \text{const.}$ near ∞ (cf. [11, p. 333]).

LEMMA 4.2. Let $K \subset [-1, 1]$ be compact with $m_1(K) \geq 2 - s$ ($0 < s < 2$). Then the inequality

$$(4.5) \quad g_{D_\infty(K)}(z, \infty) \leq g_{D_\infty([-1,1-s])}(z, \infty)$$

holds for all z such that $\operatorname{Re} z \geq \sup(K)$.

To prove Lemma 4.2 we need the following result of Myrberg and Lega [11, Thm. 11.1, p. 333].

LEMMA 4.3. Let $K \subset \mathbb{C}$ be compact with $\operatorname{cap}(K) > 0$, where “cap” denotes the logarithmic capacity. Then

$$(4.6) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{|F_{n,K}(z)|}{\|F_{n,K}\|_K} = g_{D_\infty(K)}(z, \infty)$$

for every $z \in D_\infty(K)$, where $F_{n,K}$ denotes an n th degree monic Fekete polynomial for K . The convergence in (4.6) is locally uniform in $D_\infty(K)$.

LEMMA 4.4. Let $\mu \in \mathcal{M}$ and $c \in \mathbb{R}$. If

$$(4.7) \quad m_1(E_{\mu,c}) \geq 2 - s \quad (0 < s < 2),$$

then the inequality

$$(4.8) \quad P_{\mu,c}(z) \leq g_{D_\infty([-1,1-s])}(z, \infty)$$

holds for all z such that $\operatorname{Re} z \geq \sup(E_{\mu,c})$.

To formulate our last lemma, for $0 < s < 2$ we introduce the notation

$$\mathcal{M}(s) := \{(\mu, c) \in \mathcal{M} \times \mathbb{R} : m_1(E_{\mu,c}) \geq 2 - s\}.$$

LEMMA 4.5. Let $0 < s < 2$ be fixed. Then

$$(4.10) \quad \sup \left(\max_{-1 \leq x \leq 1} P_{\mu,c}(x) \right) = \sup(P_{\mu,c}(1)),$$

where the supremum on each side of (4.10) is taken over all $(\mu, c) \in \mathcal{M}(s)$.

5. Proofs of the lemmas for Theorem 2.1.

Proof of Lemma 4.1. We prove the lemma only when $\delta = 0$, since the case when $0 < \delta < 2$ can be handled similarly. For the sake of brevity we introduce the classes

$$(5.1) \quad \mathcal{P}_n^r([-1, 1], s) := \{p \in \mathcal{P}_n^r : m_1(\{x \in [-1, 1] : |p(x)| \leq 1\}) \geq 2 - s\} \\ (n = 0, 1, 2, \dots; \quad 0 < s < 2).$$

It is easy to see that $\mathcal{P}_n^r([-1, 1], s)$ is a closed and bounded subset of \mathcal{P}_n^r in the uniform norm on $[-1, 1]$; hence it is compact. If $z \in \mathbb{C}$ is fixed, then the map $p \rightarrow |p(z)|$ is continuous; therefore, there exists a $p^* \in \mathcal{P}_n^r([-1, 1], s)$ such that

$$|p^*(z)| = \sup_{p \in \mathcal{P}_n^r([-1, 1], s)} |p(z)|$$

Now we show that $\operatorname{Re} z \geq 1$ implies

$$p^* = \pm \widehat{T}_{n, [-1, 1-s]}$$

To see this, we analyze the properties of p^* .

PROPOSITION 5.1. *p^* has only real zeros.*

Proof. Assume to the contrary that p^* has a nonreal zero w . Then

$$p(x) = (1 + \eta)p^*(x) \left(1 - \frac{\varepsilon(x-z)(x-\bar{z})}{(x-w)(x-\bar{w})} \right) \in \mathcal{P}_n^r([-1, 1], s)$$

with sufficiently small $\eta > 0$ and $\varepsilon > 0$ contradicts the maximality of p^* . This proves the proposition. \square

PROPOSITION 5.2. *All zeros of p^* are in $[-1, 1]$.*

Proof. Assume to the contrary that p^* has a nonreal zero w outside $[-1, 1]$. Then $\operatorname{Re} w$ is real by Proposition 5.1. We now distinguish three cases.

Case 1. $w > \operatorname{Re} z$. Let $w^* \in \mathbb{R}$ be the symmetric image of w with respect to $\operatorname{Re} z$, i.e., $w^* := 2\operatorname{Re} z - w$. Then

$$p(x) := (1 + \eta)p^*(x) \frac{x - w^*}{x - w} \in \mathcal{P}_n^r([-1, 1], s)$$

with a sufficiently small $\eta > 0$ contradicts the maximality of p^* .

Case 2. $1 < w \leq \operatorname{Re} z$. Now

$$p(x) := (1 + \eta)p^*(x) \frac{x - 1}{x - w} \in \mathcal{P}_n^r([-1, 1], s)$$

with a sufficiently small $\eta > 0$ contradicts the maximality of p^* .

Case 3. $w < -1$. Observe that $(x+1)(x-w)^{-1}$ is strictly increasing in $[-1, \infty)$, and so

$$(5.7) \quad \left| \frac{\operatorname{Re} z + 1}{\operatorname{Re} z - w} \right| \leq \left| \frac{z + 1}{z - w} \right|$$

$$(5.8) \quad p(x) := (1 + \eta)p^*(x) \frac{(x+1)(\operatorname{Re} z - w)}{(x-w)(\operatorname{Re} z + 1)} \in \mathcal{P}_n^r([-1, 1], s)$$

with a sufficiently small $\eta > 0$ contradicts the maximality of p^* .

By considering Cases 1, 2, and 3, Proposition 5.2 is completely proved \square
 Now we introduce the notation

$$I := \{x \in [-1, 1] : |p^*(x)| \leq 1\}.$$

Obviously I is the union of pairwise disjoint subintervals of $[-1, 1]$ that will be called the components of I . Every component of I contains at least one zero of p^* ; otherwise a routine application of Rolle's Theorem, together with Propositions 5.1 and 5.2 would imply that p'^* has at least as many zeros as p^* , a contradiction. Using this observation we prove the following.

PROPOSITION 5.3. *The set I is a single interval; in fact, $I = [-1, 1 - s]$.*

Proof. To see that I is an interval, assume to the contrary that I has at least two components, and let I_1 be the component closest to 1. Let η and η' be the left-hand endpoint of I_1 and the right-hand endpoint of the component closest to I_1 , respectively. If $w_j (j = 1, 2, \dots, m)$ are the zeros of p^* lying in I_1 , then it is easy to check that

$$p(x) := p^*(x) \frac{\prod_{j=1}^m (x - w_j + h)}{\prod_{j=1}^m (x - w_j)} \in \mathcal{P}_n^r([-1, 1], s)$$

with $0 < h \leq \eta - \eta'$ contradicts the maximality of p^* . Therefore I is an interval with $m_1(I) \geq 2 - s$. Now if $I \neq [-1, 1 - s]$, then

$$p(x) := p^*(x + \varepsilon) \in \mathcal{P}_n^r([-1, 1], s)$$

with sufficiently small $\varepsilon > 0$ contradicts the maximality of p^* , which proves the proposition. \square

Now Proposition 5.3 together with a result of Erdős [13, p. 64] yield that $p^* \equiv \widehat{T}_{n, [-1, 1-s]}$ and Lemma 4.1 is proved. \square

Proof of Lemma 4.2. Let $F_{n,K}$ denote an n th degree monic Fekete polynomial for K and set

$$\widehat{F}_{n,K}(x) := \frac{F_{n,K}(x)}{\|F_{n,K}\|_K} \quad \text{and} \quad \delta := 1 - \sup(K)$$

$$K \subset \{x \in [-1, 1 - \delta] : |\widehat{F}_{n,K}(x)| \leq 1\}.$$

Therefore,

$$m_1(\{x \in [-1, 1 - \delta] : |\widehat{F}_{n,K}(x)| \leq 1\}) \geq 2 - s,$$

and Lemma 4.1 implies that

$$|\widehat{F}_{n,K}(z)| \leq |\widehat{T}_{n, [-1, 1-s]}(z)|$$

holds for every $z \in \mathbb{C}$ such that $\text{Re } z \geq \sup(K)$. Since $\text{cap}(K) \geq m_1(K)/4 > 0$ [14, Cor. 4, p. 84], by Lemma 4.3 we have

$$\lim_{n \rightarrow \infty} |\widehat{F}_{n,K}(z)|^{1/n} = \exp(g_{D_\infty(K)}(z, \infty)), \quad z \in D_\infty(K),$$

and, as is well known,

$$(5.14) \quad \lim_{n \rightarrow \infty} |\widehat{T}_{n,[-1,1-s]}(z)|^{1/n} = \exp\{g_{D_\infty([-1,1-s])}(z, \infty)\}$$

for every $z \in \mathbb{C}$ such that $z \notin [-1, 1-s]$. Now (5.12)–(5.14) yield the lemma except for the point $z_0 = \sup(K) \geq 1-s$. That (4.5) also holds at z_0 can be seen from the limiting argument given in the next proof. \square

Proof of Lemma 4.4. For a fixed $0 < \varepsilon < 2-s$ we choose a compact set $K \subset E_{\mu,c}$ such that $\sup(E_{\mu,c}) > \sup(K)$ and

$$m_1(E_{\mu,c} \setminus K) \leq \varepsilon.$$

The last inequality, together with (4.7), yields $m_1(K) \geq 2-s-\varepsilon$. Note that the function

$$g_{D_\infty(K)}(z, \infty) - P_{\mu,c}(z)$$

is superharmonic on $\overline{\mathbb{C}} \setminus K$ and, since $K \subset E_{\mu,c}$,

$$\liminf_{\substack{z \rightarrow K \\ z \in D_\infty(K)}} (g_{D_\infty(K)}(z, \infty) - P_{\mu,c}(z)) \geq 0.$$

Therefore the minimum principle for superharmonic functions gives

$$(5.15) \quad g_{D_\infty(K)}(z, \infty) - P_{\mu,c}(z) \geq 0$$

for all $z \in D_\infty(K)$, and in particular, for all $z \in \mathbb{C}$ with $\operatorname{Re} z \geq \sup(E_{\mu,c}) > \sup(K)$. On the other hand, by the preceding proof,

$$g_{D_\infty(K)}(z, \infty) \leq g_{D_\infty([-1,1-s-\varepsilon])}(z, \infty)$$

for all $z \in \mathbb{C}$ with $\operatorname{Re} z > \sup(K)$. This, together with (5.15) yields

$$(5.16) \quad P_{\mu,c}(z) \leq g_{D_\infty([-1,1-s-\varepsilon])}(z, \infty)$$

for all $z \in \mathbb{C}$ with $\operatorname{Re} z \geq \sup(E_{\mu,c})$. Taking the limit in (5.16) as $\varepsilon \rightarrow 0+$, we obtain the desired result. \square

Proof of Lemma 4.5. Note that $P_{\mu,c}$ is upper semicontinuous; hence there exists $y \in [-1, 1]$ such that

$$P_{\mu,c}(y) = \max_{-1 \leq x \leq 1} P_{\mu,c}(x)$$

If $(\mu, c) \in \mathcal{M}(s)$, then either

$$(5.17) \quad m_1(\{x \in [-1, y] : P_{\mu,c}(x) \leq 0\}) \geq \frac{1}{2}(1+y)(2-s)$$

or

$$(5.18) \quad m_1(\{x \in [y, 1] : P_{\mu,c}(x) \leq 0\}) \geq \frac{1}{2}(1-y)(2-s).$$

We may assume that $y > -1$ and that (5.17) holds; otherwise, we study $(\mu(-t), c) \in \mathcal{M}(s)$. Now (5.17) implies that $(\nu, \hat{c}) \in \mathcal{M}(s)$, where

$$\nu(t) := \mu \left(\frac{y+1}{2}t + \frac{y-1}{2} \right) \quad \hat{c} := c - \log \left(\frac{2}{y+1} \right)$$

and

$$\max_{-1 \leq x \leq 1} P_{\mu,c}(x) = P_{\mu,c}(y) = P_{\nu, \hat{c}}(1),$$

which proves the lemma. \square

6. Proofs of Theorems 2.1 and 2.2.

Proof of Theorem 2.1. From Lemmas 4.4 and 4.5 we deduce that

$$\max_{-1 \leq x \leq 1} Q_{\mu,c}(x) \leq \exp(g_{D_\infty([-1,1-s])}(1, \infty))$$

whenever $m_1(E_{\mu,c}) \geq 2 - s$. Note that

$$\exp\{g_{D_\infty([-1,1-s])}(1, \infty)\} = \frac{\sqrt{2} + \sqrt{s}}{\sqrt{2} - \sqrt{s}}$$

which, together with (6.1), yields the first part of the theorem.

Now we prove the unicity part of the theorem. Assume that $(\mu, c) \in \mathcal{M}(s)$, $Q_{\mu,c}$ restricted to $[-1, 1]$ is continuous and

$$Q_{\mu,c}(1) = \frac{\sqrt{2} + \sqrt{s}}{\sqrt{2} - \sqrt{s}}$$

Then, by continuity, $\sup(E_{\mu,c}) < 1$. First we show that

$$P_{\mu,c}(z) = g_{D_\infty([-1,1-s])}(z, \infty)$$

for all z in the half plane

$$\mathcal{H} := \{z \in \mathbb{C} : \operatorname{Re} z > \sup(E_{\mu,c})\}$$

Indeed, (6.3) can be written as $h(1) = 0$, where

$$h(z) := g_{D_\infty([-1,1-s])}(z, \infty) - P_{\mu,c}(z).$$

Since h is superharmonic in the domain and $1 \in \mathcal{H}$, Lemma 4.4 and the minimum principle for superharmonic functions imply that $h(z) \equiv 0$ in \mathcal{H} . Thus (6.4) holds in \mathcal{H} .

Next we show that

$$\operatorname{supp}(\mu) \subset \mathbb{R}.$$

Assume that $\operatorname{supp}(\mu) \setminus \mathbb{R} \neq \emptyset$. If $w \in \operatorname{supp}(\mu) \setminus \mathbb{R}$, then the disk

$$D(w, \varepsilon) := \{z \in \mathbb{C} : |z - w| < \varepsilon\}$$

has positive μ -measure for every $\varepsilon > 0$. Now let

$$(6.5) \quad A := \operatorname{supp}(\mu) \cap D(w, \varepsilon) \quad \text{with } \varepsilon = \frac{1}{3} |\operatorname{Im} w|$$

We define the linear transformation $\varphi : \mathbb{C} \rightarrow \mathbb{C}$ by

$$(6.6) \quad \varphi(z) := 1 + (z - 1) \exp(i(\pi - \arg(w - 1)))$$

Obviously,

$$(6.7) \quad |1 - \varphi(t)| = |1 - t| \quad \text{for all } t \in \mathbb{C},$$

and there is a $0 < \delta < 1$ depending only on w and $\sup(E_{\mu,c})$ such that

$$(6.8) \quad |x - \varphi(t)| < \delta|x - t| \quad \text{for all } t \in A \quad \text{and} \quad -1 \leq x \leq \sup(E_{\mu,c})$$

We denote the restriction of a measure ν on a measurable set B by $\nu|_B$, and define the measure $\sigma(t) := \mu(\varphi^{-1}(t))$. Then (6.7), (6.8), and $\mu \in \mathcal{M}$ imply

$$(6.9) \quad \int_{\mathbb{C}} \log |1-t| d\mu|_A(t) = \int_A \log |1-\varphi(t)| d\mu(t) \\ = \int_{\varphi(A)} \log |1-t| d\sigma(t)$$

and

(6.10)

$$\int_{\mathbb{C}} \log |x-t| d\sigma|_{\varphi(A)} = \int_A \log |x-\varphi(t)| d\mu(t) \\ < \int_A \log |x-t| d\mu(t) + \mu(A) \log \delta \quad \text{for all } -1 \leq x \leq \sup(E_{\mu,c})$$

Now let

$$\hat{\mu}(t) := \mu|_{\mathbb{C} \setminus A}(t) + \sigma|_{\varphi(A)}(t)$$

We have $\hat{\mu} \in \mathcal{M}$, since $\mu \in \mathcal{M}$ and

$$\int_{\mathbb{C}} d\hat{\mu} = \int_{\mathbb{C} \setminus A} d\mu + \int_{\varphi(A)} d\sigma \\ = \int_{\mathbb{C} \setminus A} d\mu + \int_A d\mu = \int_{\mathbb{C}} d\mu = 1$$

From (6.9)–(6.11) we obtain

$$(6.12) \quad \int_{\mathbb{C}} \log |1-t| d\hat{\mu}(t) = \int_{\mathbb{C}} \log |1-t| d\mu(t)$$

and, for $-1 \leq x \leq \sup(E_{\mu,c})$,

$$\int_{\mathbb{C}} \log |x-t| d\hat{\mu}(t) < \int_{\mathbb{C}} \log |x-t| d\mu(t) + \mu(A) \log \delta.$$

Now (6.13) and $(\mu, c) \in \mathcal{M}(s)$ imply

$$(\hat{\mu}, c - \mu(A) \log \delta) \in \mathcal{M}(s),$$

while (6.12) and $0 < \delta < 1$ yield

$$P_{\hat{\mu}, c - \mu(A) \log \delta}(1) = P_{\mu, c}(1) - \mu(A) \log \delta > P_{\mu, c}(1),$$

which contradicts the extremal property of $P_{\mu, c}$. Therefore $\text{supp}(\mu) \subset \mathbb{R}$.

Let $[\alpha, \beta]$ be the smallest interval containing $\text{supp}(\mu) \cup [-1, 1-s]$. Since the function $g_{D_\infty([-1, 1-s])} - P_{\mu, c}$ is harmonic on $\overline{\mathbb{C}} \setminus [\alpha, \beta]$ and vanishes in the half plane \mathcal{H} , we have

$$(6.14) \quad g_{D_\infty([-1, 1-s])}(z, \infty) \equiv P_{\mu, c}(z)$$

for all $z \in \mathbb{C} \setminus [\alpha, \beta]$. In particular, letting $z \rightarrow \infty$ in (6.14),

$$c = -\log \frac{2-s}{4}.$$

Since (6.14) can be written as

$$\int_{\mathbb{C}} \log |z-t| d\mu(t) = \int_{\mathbb{C}} \log |z-t| d\mu_{[-1,1-s]}^*(t)$$

for all $z \in \mathbb{C} \setminus [\alpha, \beta]$, the result of [10, Thm. 1.12', p. 76] yields $\mu = \mu_{[-1,1-s]}^*$.

Finally, if $(\mu, c) \in \mathcal{M}(s)$ and

$$Q_{\mu,c}(y) = \frac{\sqrt{2} + \sqrt{s}}{\sqrt{2} - \sqrt{s}}, \quad y \in [-1, 1],$$

then, as in the proof of Lemma 4.5, we have either

$$(\hat{\nu}, \hat{c}) \in \mathcal{M}(s) \quad \text{or} \quad (\tilde{\nu}, \tilde{c}) \in \mathcal{M}(s),$$

where

$$\hat{\nu}(t) := \mu((y+1)t/2 + (y-1)/2), \quad \hat{c} := c - \log \left(\frac{2}{y+1} \right)$$

and

$$\tilde{\nu}(t) := \mu((1-y)t/2 + (1+y)/2), \quad \tilde{c} := c - \log \left(\frac{2}{1-y} \right)$$

If $(\hat{\nu}, \hat{c}) \in \mathcal{M}(s)$, then

$$Q_{\hat{\nu}, \hat{c}}(1) = Q_{\mu,c}(y) = \frac{\sqrt{2} + \sqrt{s}}{\sqrt{2} - \sqrt{s}},$$

and so, by the first part of the proof,

$$\hat{\nu} = \mu_{[-1,1-s]}^* \quad \text{and} \quad \hat{c} = -\log \frac{2-s}{4}.$$

Since $Q_{\hat{\nu}, \hat{c}}(1) \geq Q_{\mu,c}(1) = Q_{\mu,c}((3-y)/(1+y))$, it follows that $y = 1$. Hence

$$\mu = \hat{\nu} = \mu_{[-1,1-s]}^* \quad \text{and} \quad c = \hat{c} = -\log \frac{2-s}{4}.$$

If $(\tilde{\nu}, \tilde{c}) \in \mathcal{M}(s)$, then in exactly the same way we obtain

$$\mu = \tilde{\nu} = \mu_{[-1+s,1]}^* \quad \text{and} \quad c = \tilde{c} = -\log \frac{2-s}{4}$$

which completes the proof. \square

Proof of Theorem 2.2. We assume that $0 < s < 1$, since the case $s = 1$ can be obtained by Theorem 2.1. Let $\mu \in \mathcal{M}$ and $c \in \mathbb{R}$ be such that (2.4) holds. For a fixed $0 < \varepsilon < 1 - s$ we choose a compact set $K \subset E_{\mu,c}$ such that

$$m_1(E_{\mu,c} \setminus K) \leq \varepsilon.$$

Then, as in the proof of Lemma 4.4, we deduce that

$$(6.15) \quad g_{D_\infty(K)}(z, \infty) - P_{\mu,c}(z) \geq 0$$

for all $z \in D_\infty(K)$. Note that assumption (2.4) and the choice of K imply $m_1(K) \geq 2 - s + \varepsilon$. Applying [4, Thm. 4] to the Fekete polynomials $F_{n,K} \in \mathcal{P}_n^r$, we have

$$(6.16) \quad \frac{1}{n} \log \frac{|F_{n,K}(x)|}{\|F_{n,K}\|_K} \leq k_1 \min \left\{ \frac{s + \varepsilon}{\sqrt{1 - x^2}}, \sqrt{s + \varepsilon} \right\}$$

for every $-1 \leq x \leq 1$, where $k_1 > 0$ is an absolute constant. By Lemma 4.3 and $m_1(K) \geq 2 - s - \varepsilon > 0$, the limit of the left-hand side of (6.16), as $n \rightarrow \infty$, exists for every $x \in [-1, 1] \setminus K$, and equals $g_{D_\infty(K)}(x, \infty)$. Therefore (6.16) and Lemma 4.3 imply

$$g_{D_\infty(K)}(x, \infty) \leq k_1 \min \left\{ \frac{s + \varepsilon}{\sqrt{1 - x^2}}, \sqrt{s + \varepsilon} \right\}$$

for every $x \in [-1, 1] \setminus K$, and together with (6.15) this yields

$$P_{\mu,c}(x) \leq k_1 \min \left\{ \frac{s + \varepsilon}{\sqrt{1 - x^2}}, \sqrt{s + \varepsilon} \right\}$$

for every $x \in [-1, 1] \setminus K$. Since $P_{\mu,c}(x) \leq 0$ for every $x \in K \subset E_{\mu,c}$, (6.17) holds for every $x \in [-1, 1]$. Taking the limit in (6.17) as $\varepsilon \rightarrow 0+$, we get the desired result. \square

7. Proofs of Theorems 2.4, 2.5, and 2.6. Denote by \mathcal{T}_n the set of all real trigonometric polynomials of degree at most n . Note that $p \in \mathcal{P}_n^c$ implies that $q_r(t) := |p(re^{it})|^2 \in \mathcal{T}_n$ for every $r > 0$. This follows immediately from the identity

$$\begin{aligned} |z - z_j|^2 &= |re^{it} - r_j e^{it_j}|^2 = (re^{it} - r_j e^{it_j})(re^{-it} - r_j e^{-it_j}) \\ &= r^2 + r_j^2 - 2rr_j \cos(t - t_j) \\ &\quad (z = re^{it}, z_j = r_j e^{it_j}, t, t_j \in \mathbb{R}, r > 0, r_j > 0). \end{aligned}$$

In the proof of Theorem 2.5 a Remez-type inequality on the size of trigonometric polynomials will play a central role. To formulate this we introduce the notation

$$\mathcal{T}_n(s) := \{q \in \mathcal{T}_n : m_1(\{t \in [-\pi, \pi] : |q(t)| \leq 1\}) \geq 2\pi - s\} \quad (0 < s < 2\pi).$$

LEMMA 7.1 *There is an absolute constant $k_{13} > 0$ such that*

$$\max_{-\pi \leq t \leq \pi} |q(t)| \leq \exp(k_{13}ns) \quad (0 < s \leq \pi/2)$$

for every $q \in \mathcal{T}_n(s)$.

Lemma 7.1 is proved in [4, Thm. 3]. Our next lemma is a well-known, simple consequence of the maximum principle for analytic functions.

LEMMA 7.2. *Let $\bar{D} := \{z \in \mathbb{C} : |z| \leq 1\}$. We have*

$$\max_{u \in \bar{D}} |p(u)| \leq (1 - r)^{-n} \max_{|u| \leq 1-r} |p(u)|$$

for every $p \in \mathcal{P}_n^c$ and $0 < r < 1$.

Theorem 2.5 will be used in the proof of Theorem 2.4, so we prove Theorem 2.5 first. The proof of Theorem 2.4 will be given at the end of this section.

Proof of Theorem 2.5. Let $p \in \mathcal{P}_n^c(\overline{D}, s)$ ($0 < s \leq \frac{1}{4}$). Observe that if $q_r(t) := |p(re^{it})|^2 \notin \mathcal{T}_n(2\sqrt{s})$ for every $1 - \sqrt{s} \leq r \leq 1$, then

$$m_2(\{z \in \overline{D} : |p(z)|^2 > 1\}) > \int_{1-\sqrt{s}}^1 2\sqrt{s}rdr \geq \sqrt{s}2\sqrt{s}(1 - \sqrt{s}) \geq s$$

($0 < s \leq \frac{1}{4}$ was used in the last inequality), which contradicts the fact that $p \in \mathcal{P}_n^c(\overline{D}, s)$. Thus there exists an r_0 , ($1 - \sqrt{s} \leq r_0 \leq 1$) such that

$$(7.4) \quad q_{r_0}(t) = |p(r_0e^{it})|^2 \in \mathcal{T}_n(2\sqrt{s}).$$

Then, by Lemma 7.1, we obtain

$$\max_{-\pi \leq t \leq \pi} |p(r_0e^{it})|^2 = \max_{-\pi \leq t \leq \pi} |q_{r_0}(t)| \leq \exp(2k_{13}n\sqrt{s}).$$

Furthermore, Lemma 7.2, together with $1 - \sqrt{s} \leq r_0 \leq 1$ and $0 < s \leq \frac{1}{4}$, yields

$$\begin{aligned} \max_{u \in \overline{D}} |p(u)|^2 &\leq (1 - \sqrt{s})^{-2n} \max_{|u| \leq r_0} |p(u)|^2 \\ &\leq \exp(4n\sqrt{s}) \max_{|u| \leq r_0} |p(u)|^2 = \exp(4n\sqrt{s}) \max_{-\pi \leq t \leq \pi} |p(r_0e^{it})|^2. \end{aligned}$$

Now (7.5) and (7.6) give the theorem with $k_4 := k_{13} + 2$. \square

Proof of Theorem 2.6. Let $T_n(x) = \cos(n \arccos x)$ ($-1 \leq x \leq 1$) be the Chebyshev polynomial of degree n . For $0 < s \leq 1$, define the polynomials

$$T_{n,s}(z) := T_n\left(\frac{z}{\cos \sqrt{s}}\right)$$

and

$$Q_{3n,s}(z) := z^{2n}T_{n,s}\left(\frac{z + z^{-1}}{2}\right)$$

Obviously,

$$\max_{|u| \leq 1} |Q_{3n,s}(u)| = |Q_{3n,s}(1)| = T_n\left(\frac{1}{\cos \sqrt{s}}\right) \geq T_n\left(1 + \frac{s}{2}\right)$$

Let

$$\begin{aligned} D_{s,c} := \{z \in \mathbb{C} : |z| \leq 1, \arg z \in [\sqrt{s}, \pi - \sqrt{s}] \cup [\pi + \sqrt{s}, 2\pi - \sqrt{s}]\} \\ \cup \{z \in \mathbb{C} : |z| \leq 1 - c\sqrt{s}\}. \end{aligned}$$

where $0 < c \leq 1$ will be chosen later. We examine the maximum of $|Q_{3n,s}|$ on $D_{s,c}$. By the maximum principle, it is sufficient to examine the maximum of $|Q_{3n,s}|$ on the boundary of $D_{s,c}$. For z satisfying $|z| = 1$, $\arg z \in [\sqrt{s}, \pi - \sqrt{s}] \cup [\pi + \sqrt{s}, 2\pi - \sqrt{s}]$, we have

$$(7.11) \quad |Q_{3n,s}(z)| \leq \max_{|x| \leq \cos \sqrt{s}} \left| T_n\left(\frac{x}{\cos \sqrt{s}}\right) \right| \leq 1$$

Furthermore, by the maximum principle, we have for $|z| = 1 - c\sqrt{s}$,

$$\begin{aligned} |Q_{3n,s}(z)| &\leq (1 - c\sqrt{s})^n \max_{|u| \leq 1 - c\sqrt{s}} \left| u^n T_{n,s} \left(\frac{u + u^{-1}}{2} \right) \right| \\ &\leq \exp(-cn\sqrt{s}) \max_{|u|=1} \left| u^n T_{n,s} \left(\frac{u + u^{-1}}{2} \right) \right| = \exp(-cn\sqrt{s}) T_n \left(\frac{1}{\cos \sqrt{s}} \right) \end{aligned}$$

Now let

$$z = r(\cos \sqrt{s} + i \sin \sqrt{s}) \quad \text{with } 1 - c\sqrt{s} \leq r \leq 1.$$

If $c = \frac{1}{8}$ and $0 < s \leq 1$ in (7.13), then

$$\begin{aligned} (7.14) \quad \left| \frac{z + z^{-1}}{2} - \cos \sqrt{s} \right| &= \left| \left(\frac{r + r^{-1}}{2} - 1 \right) \cos \sqrt{s} + i \frac{r - r^{-1}}{2} \sin \sqrt{s} \right| \\ &\leq \frac{1}{2} \frac{c^2 s}{1 - c\sqrt{s}} \cos \sqrt{s} + \frac{c\sqrt{s}}{1 - c\sqrt{s}} \sin \sqrt{s} \leq \frac{s}{4} \cos \sqrt{s}. \end{aligned}$$

Therefore, using the fact that the zeros of $T_{n,s}$ are in $(-\cos \sqrt{s}, \cos \sqrt{s})$, we easily conclude

$$(7.15) \quad |Q_{3n,s}(z)| \leq \left| T_{n,s} \left(\frac{z + z^{-1}}{2} \right) \right| \leq T_{n,s} \left(\left(1 + \frac{s}{4} \right) \cos \sqrt{s} \right) = T_n \left(1 + \frac{s}{4} \right).$$

By the reason of symmetry (7.15) holds when $z = r(\pm \cos \sqrt{s} \pm i \sin \sqrt{s})$, $1 - \sqrt{s}/8 \leq r \leq 1$, and $0 < s \leq 1$. We define

$$(7.16) \quad K(n, s) := \max \left\{ T_n \left(\frac{1}{\cos \sqrt{s}} \right) \exp \left(-\frac{1}{8} n \sqrt{s} \right), T_n \left(1 + \frac{s}{4} \right) \right\}$$

and

$$P_{3n,s}(z) := \frac{Q_{3n,s}(z)}{K(n, s)}$$

By (7.11), (7.12), (7.15)–(7.17) and the maximum principle we can easily deduce that

$$|P_{3n,s}(z)| \leq 1 \quad \text{for } z \in D_{s,1/8}.$$

Hence

$$P_{3n,s} \in \mathcal{P}_{3n}^r(\overline{D}, s) \quad (0 < s \leq 1).$$

Finally, by (7.9), (7.16), and (7.17) we obtain

$$\begin{aligned} P_{3n,s}(1) &= \frac{T_n(1/\cos \sqrt{s})}{K(n, s)} \\ &\geq \min \left\{ \exp \left(\frac{1}{8} n \sqrt{s} \right) \frac{T_n(1 + s/2)}{T_n(1 + s/4)} \right\} = \exp \left(\frac{1}{8} n \sqrt{s} \right) \end{aligned}$$

which completes the proof. \square

Proof of Theorem 2.4. Denote the boundary of Ω by Γ . Since Γ is a C^2 curve, there is an $r > 0$ depending only on Ω such that for each $z \in \Gamma$ there is an open disk D_z with radius r such that $D_z \subset \Omega$ and $\overline{D_z} \cap \Gamma = \{z\}$. By the maximum principle for analytic functions, for every $p \in \mathcal{P}_n^c$ there is a $z_0 \in \Gamma$ such that $|p(z_0)| = \max_{u \in \overline{\Omega}} |p(u)|$. It follows from Theorem 2.5, by a linear transformation, that there are constants $k_{14} := k_4 \sqrt{1/r^2} = k_4/r > 0$ and $k_{15} := r^2/4 > 0$ depending only on Ω (k_4 is the same as in Theorem 2.5) such that

$$(7.21) \quad \max_{u \in \overline{\Omega}} |p(u)| = \max_{u \in \overline{D_{z_0}}} |p(u)| \leq \exp(k_4 n \sqrt{s/r^2}) = \exp(k_{14} n \sqrt{s}) \quad (0 < s \leq k_{15})$$

for every $p \in \mathcal{P}_n^c$ satisfying

$$(7.22) \quad m_2(\{z \in \overline{\Omega} : |p(z)| \leq 1\}) \geq m_2(\overline{\Omega}) - s.$$

Now let $\mu \in \mathcal{M}, c \in \mathbb{R}$,

$$(7.23) \quad E_{\Omega, \mu, c} := \{z \in \overline{\Omega} : Q_{\mu, c}(z) \leq 1\},$$

and assume that

$$(7.24) \quad m_2(E_{\Omega, \mu, c}) > m_2(\overline{\Omega}) - s.$$

For a fixed $0 < \varepsilon < m_2(\overline{\Omega}) - s$ we choose a compact set $K \subset E_{\Omega, \mu, c}$ such that

$$(7.25) \quad m_2(E_{\Omega, \mu, c} \setminus K) \leq \varepsilon.$$

This, together with (7.24), gives

$$(7.26) \quad m_2(K) \geq m_2(\overline{\Omega}) - (s + \varepsilon).$$

As in the proof of Lemma 4.4, we obtain that

$$(7.27) \quad g_{D_\infty(K)}(z, \infty) - P_{\mu, c}(z) \geq 0$$

for all $z \in D_\infty(K)$. Applying (7.21) to the normalized Fekete polynomials

$$(7.28) \quad \widehat{F}_{n, K} := \frac{F_{n, K}}{\|F_{n, K}\|_K}$$

we obtain

$$(7.29) \quad \frac{1}{n} \log |\widehat{F}_{n, K}(z)| \leq K_{14} \sqrt{s + \varepsilon}$$

for every $z \in \overline{\Omega}$ and $0 < s + \varepsilon \leq k_{15}$. By Lemma 4.3, the limit of the left-hand side of (7.29), as $n \rightarrow \infty$, exists for every $z \in D_\infty(K)$ and equals $g_{D_\infty(K)}(z, \infty)$. Therefore, (7.29) and Lemma 4.3 imply

$$(7.30) \quad g_{D_\infty(K)}(z, \infty) \leq k_{14} \sqrt{s + \varepsilon}$$

for every $z \in D_\infty(K) \cap \overline{\Omega}$ and $0 < s + \varepsilon \leq k_{15}$, and together with (7.27) this yields

$$(7.31) \quad P_{\mu, c}(z) \leq k_{14} \sqrt{s + \varepsilon}$$

for every $z \in D_\infty(K) \cap \overline{\Omega}$ and $0 < s + \varepsilon \leq k_{15}$. Since $K \subset E_{\Omega, \mu, c}$ and $P_{\mu, c}(z)$ is subharmonic in \mathbb{C} , it follows from the maximum principle that (7.31) holds for all $z \in \overline{\Omega}$. Taking the limit as $\varepsilon \rightarrow 0+$, we get

$$(7.32) \quad Q_{\mu, c}(z) \leq \exp(k_{14} \sqrt{s})$$

for every $z \in \overline{\Omega}$ and $0 < s \leq k_{15}$, which completes the proof. \square

8. Proofs of Theorems 2.7 and 2.8.

Proof of Theorem 2.7. From Theorem 2.1* we can easily deduce that

$$m_1(E_{\mu,c,s,A}) \geq 2s - s = s$$

for every $\mu \in \mathcal{M}, c \in \mathbb{R}, 0 < s < 1$, and $A \subset [-1, 1]$ with $m_1(A) \geq 2 - s$, where

$$E_{\mu,c,s,A} = \left\{ x \in A : Q_{\mu,c}(x) > \frac{1 - \sqrt{s}}{1 + \sqrt{s}} \max_{-1 \leq y \leq 1} Q_{\mu,c}(y) \right\}$$

Hence

$$\begin{aligned} \int_{[-1,1] \setminus A} (Q_{\mu,c}(x))^p dx &\leq s \max_{-1 \leq y \leq 1} (Q_{\mu,c}(y))^p \\ &< \left(\frac{1 + \sqrt{s}}{1 - \sqrt{s}} \right)^p \int_{E_{\mu,c,s,A}} (Q_{\mu,c}(x))^p dx \\ &< \left(\frac{1 + \sqrt{s}}{1 - \sqrt{s}} \right)^p \int_A (Q_{\mu,c}(x))^p dx \\ &\leq \exp(k_6 p \sqrt{s}) \int_A (Q_{\mu,c}(x))^p dx \end{aligned}$$

for every $\mu \in \mathcal{M}, c \in \mathbb{R}, 0 < s \leq \frac{1}{2}$ and $A \subset [-1, 1]$ with $m_1(A) \geq 2 - s$, where $k_6 = 4$ is a suitable choice. From (8.3) we immediately get (2.12). \square

Theorem 2.8 follows from Theorem 2.4 by straightforward modifications of the proof of Theorem 2.7.

9. Proofs of Theorems 2.9 and 2.10.

Proof of Theorem 2.9. Assume that $p \in \mathcal{P}_n^c$ and

$$m_1(\{t \in [-\pi, \pi) : |p(e^{it})| \leq 1\}) \geq 2\pi - s \quad (0 < s \leq \pi/2).$$

Applying Lemma 7.1 to $q(t) := |p(e^{it})|^2 \in \mathcal{T}_n(s)$, we obtain

$$|p(e^{it})| \leq \exp(k_{13}ns) \quad (t \in \mathbb{R})$$

The above polynomial inequality can be extended to exponentials of logarithmic potentials with compact support by the technique used in the proof of Theorems 2.2 and 2.4; so we omit the details. \square

Theorem 2.10 follows immediately from Theorem 2.9 in exactly the same way as Theorem 2.7 was obtained from Theorem 2.1* ; so we omit the details.

10. Proofs of Theorems 3.1, 3.2, and 3.3.

Proof of Theorem 3.1. For the sake of brevity we denote the norm $\|\cdot\|_{L_p(-1,1)}$ by $\|\cdot\|_p$. It is sufficient to prove the theorem when $p = \infty$, and then a simple argument gives the desired result for arbitrary $0 < q < p < \infty$. To see this, assume that

$$\|f\|_\infty \leq M^{1/q} \|f\|_q$$

for an $f \in L_\infty$ and $0 < q < \infty$, with some factor M . Then

$$\begin{aligned} \|f\|_p^p &= \int |f|^{p-q+q} < \|f\|_\infty^{p-q} \|f\|_q^q \\ &\leq M^{p/q-1} \|f\|_q^{p-q} \|f\|_q^q \end{aligned}$$

and therefore

$$\|f\|_p \leq M^{1/q-1/p} \|f\|_q$$

for every $0 < q < p < \infty$. Thus, in the sequel let $0 < q < p = \infty$. Applying Corollary 2.3 with

$$(10.1) \quad s = \min\{1, q^{-2}\},$$

we obtain

$$m_1 \left(\left\{ x \in [-1, 1] : (Q_{\mu,c}(x))^q \geq e^{-1} \max_{-1 \leq y \leq 1} (Q_{\mu,c}(y))^q \right\} \right) \geq k_2 s$$

for every $\mu \in \mathcal{M}$, $c \in \mathbb{R}$, and $q > 0$. Now, integrating only on the subset E of $[-1, 1]$, where

$$(10.3) \quad (Q_{\mu,c}(x))^q \geq e^{-1} \max_{-1 \leq y \leq 1} (Q_{\mu,c}(y))^q,$$

and using (10.1) and (10.2), we conclude that

$$(10.4) \quad \|Q_{\mu,c}\|_\infty^q \leq \frac{e}{m_1(E)} \int_E (Q_{\mu,c}(x))^q dx \leq \frac{e}{k_2} (1+q^2) \|Q_{\mu,c}\|_q^q$$

for every $\mu \in \mathcal{M}$, $c \in \mathbb{R}$, and $q > 0$, and the theorem follows by taking the q th root. \square

Theorems 3.2 and 3.3 follow from Theorems 2.4 and 2.9, respectively, by straightforward modifications of the proof of Theorem 3.1.

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