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Local behaviour of the error in the Bergman kernel method for numerical conformal mapping *

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Abstract

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Let Ω be a simply-connected domain in the complex plane, let $\zeta \in \Omega$ and let $K(z, \zeta)$ denote the Bergman kernel function of Ω with respect to ζ. Also, let $K_n(z, \zeta)$ denote the nth-degree polynomial approximation to $K(z, \zeta)$, given by the classical Bergman kernel method, and let π_n denote the corresponding nth-degree Bieberbach polynomial approximation to the conformal map f of Ω onto a disc. Finally, let B be any subdomain of Ω. In this paper we investigate the two local errors $\| K(\cdot, \zeta) - K_n(\cdot, \zeta) \|_{L^2(B)}$, $\| f_\zeta' - \pi_n' \|_{L^2(B)}$, and compare their rates of convergence with those of the corresponding global errors with respect to $L^2(\Omega)$. Our results show that if ∂B contains a subarc of $\partial\Omega$, then the rates of convergence of the local errors are not substantially different from those of the global errors.

Keywords: Bergman kernel function; Bergman kernel method; Bieberbach polynomials; conformal mapping; local and global errors.

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1. Introduction

Let Ω be a simply-connected domain of the complex plane $\mathbb{C}$, whose boundary Γ is a closed Jordan curve, and let $\zeta \in \Omega$. Then, by the Riemann mapping theorem, there exists a unique conformal mapping $w = f_\zeta(z)$ of Ω onto a disc $\{w: |w| < r_\zeta\}$, such that

$$f_\zeta(\zeta) = 0, \quad f'_\zeta(\zeta) = 1.$$

The radius r_ζ of this disc is called the *conformal radius* of Ω with respect ot ζ. (To avoid the study of uninteresting cases we shall assume throughout this paper that $f_\zeta(z)$ is not a polynomial function.)

For the inner product

$$(g, h) := \iint_\Omega g(z)\overline{h(z)}\, \mathrm{d}m,$$

where $\mathrm{d}m$ is the two-dimensional Lebesgue measure, we consider the Hilbert space

$$L^2(\Omega) := \left\{g: g \text{ analytic in } \Omega,\ \|g\|^2_{L^2(\Omega)} = (g, g) < \infty\right\}.$$

Let $K(z, \zeta)$ denote the *Bergman kernel function* of Ω which has the reproducing property

$$g(\zeta) = (g, K(\cdot, \zeta)), \quad \forall g \in L^2(\Omega), \tag{1.1}$$

(cf. [1–3,6]). Then it is known (cf. [3, p.34]) that $r_\zeta = (\pi K(\zeta, \zeta))^{-1/2}$ and that for $z \in \Omega$,

$$f'_\zeta(z) = \frac{K(z, \zeta)}{K(\zeta, \zeta)}, \quad f_\zeta(z) = \frac{1}{K(\zeta, \zeta)} \int_{t=\zeta}^{z} K(t, \zeta)\, \mathrm{d}t. \tag{1.2}$$

Next let $Q_n(z) = \gamma_n z^n + \cdots$, $\gamma_n > 0$, be the sequence of orthonormal polynomials for the inner product $(\cdot, \cdot)$, i.e.,

$$\iint_\Omega Q_k(z)\overline{Q_l(z)}\, \mathrm{d}m = \delta_{k,l}.$$

Since Ω is a Jordan region, it is known (cf. [3, p.17]) that $\{Q_n\}_0^\infty$ forms a complete orthonormal system for $L^2(\Omega)$ and, from the reproducing property (1.1), it follows that (with respect to this system) the Fourier coefficients of $K(\cdot, \zeta)$ are given by $\overline{Q_n(\zeta)}$, $n = 0, 1, \ldots$. Thus, for the partial sums

$$K_n(z, \zeta) := \sum_{j=0}^{n} \overline{Q_j(\zeta)} Q_j(z) \tag{1.3}$$

we have

$$E_n(K, \Omega) := \| K(\cdot, \zeta) - K_n(\cdot, \zeta) \|_{L^2(\Omega)} \to 0, \quad \text{as } n \to \infty. \tag{1.4}$$

From the least-squares property of the Fourier sections, we also have that

$$E_n(K, \Omega) \leqslant \| K(\cdot, \zeta) - p_n \|_{L^2(\Omega)}, \quad \forall p_n \in \Pi_n, \tag{1.5}$$

where Π_n denotes the collection of all polynomials having degree at most n.

In the classical Bergman kernel method (BKM) for numerically computing the conformal mapping f_ζ, we replace K by K_{n-1} in (1.2) and obtain the polynomial approximations

$$\pi_n'(z) = \frac{K_{n-1}(z, \zeta)}{K_{n-1}(\zeta, \zeta)} \quad \text{and} \quad \pi_n(z) := \frac{1}{K_{n-1}(\zeta, \zeta)} \int_{t=\zeta}^{z} K_{n-1}(t, \zeta)\, \mathrm{d}t, \tag{1.6}$$

to f_ζ' and f_ζ, respectively. The polynomials π_n are the *Bieberbach polynomials* for Ω, and it is easily seen from (1.4) that they satisfy

$$\epsilon_n'(f'\zeta, \Omega) := \| f_\zeta' - \pi_n' \|_{L^2(\Omega)} \to 0, \quad \text{as } n \to \infty, \tag{1.7}$$

and

$$\epsilon_n(f_\zeta, \Omega) := \| f_\zeta - \pi_n \|_{L^2(\Omega)} \to 0, \quad \text{as } n \to \infty. \tag{1.8}$$

Roughly speaking, the rates of convergence in (1.4), (1.7) and (1.8) are governed by the smoothness properties of the boundary Γ or, equivalently, by the nature and location of the singularities of f_ζ in $\mathbb{C}\setminus\Omega$. For example, if Γ is an analytic Jordan curve, then these rates are geometric, i.e.,

$$\limsup_{n\to\infty} \left[E_n(K, \Omega) \right]^{1/n} < 1$$

(and similarly for $\epsilon_n'(f_\zeta', \Omega)$ and $\epsilon_n(f_\zeta, \Omega)$), while for piecewise analytic boundaries these rates are typically of the form $1/n^\gamma$, for some constant $\gamma > 0$ (cf. [3,4,9,10]).

The purpose of this paper is to investigate *local* rates of convergence in the BKM. To be more precise, let B be any (arbitrarily small) Jordan subdomain of Ω and consider the norm

$$\| g \|_{L^2(B)} := \left[\int\int_B | g |^2 \, \mathrm{d}m \right]^{1/2}. \tag{1.9}$$

Then our goal is to investigate the rates of convergence of the following two errors:

$$E_n(K, B) := \| K(\cdot, \zeta) - K_n(\cdot, \zeta) \|_{L^2(B)}, \tag{1.10}$$

$$\epsilon_n'(f_\zeta', B) := \| f_\zeta' - \pi_n' \|_{L^2(B)}. \tag{1.11}$$

If the closure $\overline{B}$ is contained in Ω, then it is indeed possible for the local errors (1.10), (1.11) to tend to zero geometrically faster than the corresponding global errors with respect to $L^2(\Omega)$ (see Example 3.1). If, however, the boundary ∂B of B contains a subarc of Γ (and Γ satisfies certain smoothness conditions), then we shall show that the rates of convergence of the local errors are not "substantially" different from those of the corresponding global errors. This fact is somewhat surprising, because it implies that the BKM errors in small subregions of Ω that are near to the singularities of f_ζ are "essentially" the same as those in small subregions that are far from these singularities. This behaviour is, however, consistent with the second author's *principle of contamination* in best approximation (cf. [8]).

2. Statements of results

Our results will be established by assuming that the boundary curve Γ satisfies certain smoothness conditions. In particular, we shall assume that Γ belongs to a class $C(p, \alpha)$. This class is defined as follows (cf. [10, p.5]).

Definition 2.1. A rectifiable Jordan curve γ is said to *belong to the class $C(p, \alpha)$*, where p is a positive integer and $0 < \alpha \leqslant 1$, if γ has a parametrization $z = z(s)$, where s is arc length, and the function $z(s)$ is p times continuously differentiable with $z^{(p)}(s) \in \mathrm{Lip}\ \alpha$.

Our principal result is as follows.

Theorem 2.2. *With the notations of Section* 1, *suppose that $\Gamma \in C(p+1, \alpha)$, $p \geqslant 0$, $0 < \alpha \leqslant 1$. If $B \subset \Omega$ is any Jordan domain such that its boundary ∂B contains a subarc of Γ, then*

$$\sum_{n=0}^{\infty} \left[\frac{\| K(\cdot, \zeta) - K_n(\cdot, \zeta) \|_{L^2(B)}}{\| K(\cdot, \zeta) - K_n(\cdot, \zeta) \|_{L^2(\Omega)}} \right]^2 = \infty. \tag{2.1}$$

An immediate consequence of Theorem 2.2 is the following.

Corollary 2.3. *Let Γ and $B \subset \Omega$ be as in Theorem* 2.2. *Then, given $\epsilon > 0$, there exists a subsequence $\Lambda \subset N$ such that*

$$\| K(\cdot, \zeta) - K_n(\cdot, \zeta) \|_{L^2(B)} \geqslant \frac{c}{n^{1/2+\epsilon}} \| K(\cdot, \zeta) - K_n(\cdot, \zeta) \|_{L^2(\Omega)}, \quad n \in \Lambda, \tag{2.2}$$

where c is a positive constant.

The following two results are also relatively simple consequences of Theorem 2.2 and its proof.

Theorem 2.4. *Suppose that $\Gamma \in C\ (p+1, \alpha)$ with $p + \alpha > \frac{1}{2}$ and let $B \subset \Omega$ be as in Theorem* 2.2. *Then, given $\epsilon > 0$, there exists a subsequence $\Lambda \subset N$ such that*

$$\| f_\zeta' - \pi_n' \|_{L^2(B)} \geqslant \frac{c}{n^{1/2+\epsilon}} \| f_\zeta' - \pi_n' \|_{L^2(\Omega)}, \quad n \in \Lambda, \tag{2.3}$$

where c is a positive constant.

Theorem 2.5. *Suppose that Γ is an analytic Jordan curve and let $B \subset \Omega$ be as in Theorem* 2.2. *Then there exists a subsequence $\Lambda \subset N$ and a positive constant c such that*

$$\| K(\cdot, \zeta) - K_n(\cdot, \zeta) \|_{L^2(B)} \geqslant c \| K(\cdot, \zeta) - K_n(\cdot, \zeta) \|_{L^2(\Omega)}, \quad n \in \Lambda, \tag{2.4}$$

and

$$\| f_\zeta' - \pi_n' \|_{L^2(B)} \geqslant c \| f_\zeta' - \pi_n' \|_{L^2(\Omega)}, \quad n \in \Lambda. \tag{2.5}$$

We expect that the corresponding errors for the mapping function f_ζ satisfy similar results but, so far, we have not been able to prove this.

3. Examples

Example 3.1. Consider the case where $\Omega = \{z: |z| < 1\}$ and $\zeta \in \Omega$ is different than zero. Then

$$Q_n(z) = \sqrt{\frac{n+1}{\pi}}\, z^n, \quad n = 0, 1, \ldots,$$

and hence

$$K(z, \zeta) = \frac{1}{\pi} \sum_{j=0}^{\infty} (j+1)(\bar{\zeta} z)^j = \frac{1}{\pi} \frac{1}{(1-\bar{\zeta} z)^2}, \quad \zeta, z \in \Omega, \tag{3.1}$$

$$K_{n-1}(z, \zeta) = \frac{1}{\pi} \sum_{j=0}^{n-1} (j+1)(\bar{\zeta} z)^j = \frac{1}{\pi} \frac{n(\bar{\zeta} z)^{n+1} - (n+1)(\bar{\zeta} z)^n + 1}{(1-\bar{\zeta} z)^2}. \tag{3.2}$$

Also,

$$f_\zeta(z) = (1 - |\zeta|^2)\left(\frac{z-\zeta}{1-\bar{\zeta} z}\right),$$

so that the mapping function f_ζ has a simple pole at the point $z = 1/\bar{\zeta}$ but is otherwise analytic in the extended plane.

Equations (3.1) and (3.2) imply that

$$K(z, \zeta) - K_{n-1}(z, \zeta) = \frac{1}{\pi} \frac{(\bar{\zeta} z)^n}{(1-\bar{\zeta} z)^2} \{-n\bar{\zeta} z + n + 1\},$$

and from this it follows that

$$\limsup_{n \to \infty} \| K(\cdot, \zeta) - K_{n-1}(\cdot, \zeta) \|_{L^2(\Omega)}^{1/n} = |\zeta|.$$

It also follows that if $B := \{z: |z| < r < 1\}$, then

$$\limsup_{n \to \infty} \| K(\cdot, \zeta) - K_{n-1}(\cdot, \zeta) \|_{L^2(B)}^{1/n} = r|\zeta|.$$

This illustrates the fact that if $\bar{B} \subset \Omega$, then the local error (1.10) can tend to zero geometrically faster than the global error with respect to $L^2(\Omega)$.

Let z_1, z_2 denote, respectively, the two boundary points nearest and furthest away from the singularity of f_ζ at $z = 1/\bar{\zeta}$, i.e., $z_1 := e^{i\alpha}$ and $z_2 := -e^{i\alpha}$, where $\alpha := \arg \zeta$. Then,

$$K(z_1, \zeta) - K_{n-1}(z_1, \zeta) = \frac{1}{\pi} \frac{|\zeta|^n}{(1-|\zeta|)^2} \{-n|\zeta| + n + 1\} =: e_n(z_1),$$

$$K(z_2, \zeta) - K_{n-1}(z_2, \zeta) = \frac{1}{\pi} \frac{(-1)^n |\zeta|^n}{(1+|\zeta|)^2} \{n|\zeta| + n + 1\} =: e_n(z_2),$$

and hence

$$\lim_{n \to \infty} \frac{|e_n(z_1)|}{|e_n(z_2)|} = \frac{1+|\zeta|}{1-|\zeta|}.$$

This supports (in a pointwise sense) the remark made at the end of Section 1 concerning the BKM errors in small subregions close to and far away from the singularities of f_ζ.

Example 3.2. Let Ω be bounded by $\Gamma = \Gamma_1 \cup \Gamma_2$, where Γ_1 and Γ_2 are respectively the half circle

$$\Gamma_1 := \{z = x + \mathrm{i}y:\ |z| = 1,\ x \leqslant 0\}$$

and the half ellipse

$$\Gamma_2 := \left\{z = x + \mathrm{i}y:\ \tfrac{1}{4}x^2 + y^2 = 1,\ x > 0\right\},$$

and set $\zeta = 0$.

In this case $\Gamma \in C(1, 1)$ and the mapping function f_0 has a branch point singularity at each of the points $z_1 = \mathrm{i}$ and $z_2 = -\mathrm{i}$ where the two curves Γ_1 and Γ_2 meet, in the sense that

$$f_0(z) - f_0(z_j) \sim (z - z_j)^2 \log(z - z_j), \quad \text{as } z \to z_j,\ j = 1, 2,$$

(cf. [7, p.651]).

Let B_1 and B_2 denote the two subdomains of Ω whose boundaries ∂B_1 and ∂B_2 are as follows: (i) ∂B_1 consists of the subarc

$$z = \mathrm{e}^{\mathrm{i}\theta}, \quad \tfrac{1}{2}\pi \leqslant \theta \leqslant \tfrac{7}{12}\pi,$$

of Γ_1 and the two straight lines that join the point 0.5i respectively to the boundary points i and $\mathrm{e}^{\mathrm{i}(7\pi/12)}$; (ii) ∂B_2 consists of the subarc

$$z = \mathrm{e}^{\mathrm{i}\theta}, \quad \pi \leqslant \theta \leqslant \tfrac{13}{12}\pi,$$

of Γ_1 and the two straight lines that join the point -0.5 respectively to the boundary points -1 and $\mathrm{e}^{\mathrm{i}(13\pi/12)}$. (Observe that ∂B_1 contains the point $z_1 = \mathrm{i}$, where f_0 has a branch point singularity, while ∂B_2 does not involve any singular points of f_0.)

In Table 1 we have listed (for various values of n) estimates of the errors

$$\epsilon_n'(f_0', \Omega) := \| f_0' - \pi_n' \|_{L^2(\Omega)}, \qquad \epsilon_n'(f_0', B_j) := \| f_0' - \pi_n' \|_{L^2(B_j)}, \quad j = 1, 2,$$

and also of the ratios

$$r_n^{\{j\}} := \frac{\epsilon_n'(f_0', B_j)}{\epsilon_n'(f_0', \Omega)}, \quad j = 1, 2, \qquad r_n^{\{1,2\}} := \frac{\epsilon_n'(f_0', B_1)}{\epsilon_n'(f_0', B_2)}.$$

Table 1

n	$\epsilon_n'(f_0', \Omega)$	$\epsilon_n'(f_0', B_1)$	$\epsilon_n'(f_0', B_2)$	$r_n^{\{1\}}$	$r_n^{\{2\}}$	$r_n^{\{1,2\}}$
5	$7.7\cdot10^{-2}$	$1.6\cdot10^{-2}$	$7.2\cdot10^{-4}$	0.209	0.009	22.2
6	$5.7\cdot10^{-2}$	$1.6\cdot10^{-2}$	$8.8\cdot10^{-3}$	0.287	0.154	1.86
7	$3.6\cdot10^{-2}$	$9.1\cdot10^{-3}$	$6.7\cdot10^{-4}$	0.252	0.019	13.5
8	$2.7\cdot10^{-2}$	$8.6\cdot10^{-3}$	$4.1\cdot10^{-3}$	0.324	0.153	2.12
9	$2.0\cdot10^{-2}$	$5.7\cdot10^{-3}$	$7.3\cdot10^{-4}$	0.292	0.037	7.84
10	$1.3\cdot10^{-2}$	$4.8\cdot10^{-3}$	$2.0\cdot10^{-3}$	0.361	0.150	2.41
11	$1.2\cdot10^{-2}$	$3.9\cdot10^{-3}$	$8.2\cdot10^{-4}$	0.333	0.070	4.74
12	$7.5\cdot10^{-3}$	$2.9\cdot10^{-3}$	$8.9\cdot10^{-4}$	0.382	0.120	3.20
13	$7.4\cdot10^{-3}$	$2.8\cdot10^{-3}$	$7.3\cdot10^{-4}$	0.374	0.099	3.78
14	$5.0\cdot10^{-3}$	$1.9\cdot10^{-3}$	$3.0\cdot10^{-4}$	0.384	0.060	6.41
15	$4.8\cdot10^{-3}$	$1.9\cdot10^{-3}$	$5.6\cdot10^{-4}$	0.409	0.116	3.51

Table 2

n	$\epsilon_n(f_0, \Omega)$	$\epsilon_n(f_0, B_1)$	$\epsilon_n(f_0, B_2)$	$r_n^{\{1\}}$	$r_n^{\{2\}}$	$r_n^{\{1,2\}}$
5	$1.5\cdot10^{-2}$	$2.8\cdot10^{-3}$	$1.2\cdot10^{-4}$	0.182	0.008	24.1
6	$9.7\cdot10^{-2}$	$2.6\cdot10^{-3}$	$1.5\cdot10^{-3}$	0.265	0.151	1.76
7	$5.5\cdot10^{-3}$	$1.2\cdot10^{-3}$	$1.6\cdot10^{-4}$	0.217	0.029	7.41
8	$3.5\cdot10^{-3}$	$1.0\cdot10^{-3}$	$4.8\cdot10^{-4}$	0.294	0.138	2.13
9	$2.4\cdot10^{-3}$	$6.0\cdot10^{-4}$	$8.8\cdot10^{-5}$	0.248	0.036	6.80
10	$1.4\cdot10^{-3}$	$4.4\cdot10^{-4}$	$2.0\cdot10^{-4}$	0.324	0.146	2.22
11	$1.2\cdot10^{-3}$	$3.4\cdot10^{-4}$	$8.3\cdot10^{-5}$	0.283	0.069	4.11
12	$6.4\cdot10^{-4}$	$2.1\cdot10^{-4}$	$7.3\cdot10^{-5}$	0.335	0.114	2.93
13	$6.4\cdot10^{-4}$	$2.1\cdot10^{-4}$	$6.0\cdot10^{-5}$	0.322	0.093	3.47
14	$3.9\cdot10^{-4}$	$1.3\cdot10^{-4}$	$2.1\cdot10^{-5}$	0.324	0.053	6.09
15	$3.5\cdot10^{-4}$	$1.3\cdot10^{-4}$	$4.0\cdot10^{-5}$	0.360	0.113	3.20

Table 2 contains the corresponding estimates for the mapping function f_0, i.e.,

$$\epsilon_n(f_0, \Omega) := \| f_0 - \pi_n \|_{L^2(\Omega)}, \qquad \epsilon_n(f_0, B_j) := \| f_0 - \pi_n \|_{L^2(B_j)}, \quad j = 1, 2,$$

$$r_n^{\{j\}} := \frac{\epsilon_n(f_0, B_j)}{\epsilon_n(f_0, \Omega)}, \quad j = 1, 2, \qquad r_n^{\{1,2\}} := \frac{\epsilon_n(f_0, B_1)}{\epsilon_n(f_0, B_2)}.$$

All these estimates were computed by using the FORTRAN conformal mapping package BKMPACK [11].

As might be expected, the results of the two tables show that the local errors $\epsilon_n'(f_0', B_2)$ and $\epsilon_n(f_0, B_2)$, for the subregion B_2, are smaller than the errors $\epsilon_n'(f_0', B_1)$ and $\epsilon_n(f_0, B_1)$ for the subregion B_1 whose boundary contains the singular point $z_1 = \mathrm{i}$. However, the numerical results also show that the rates of decrease of $\epsilon_n'(f_0', B_1)$ and $\epsilon_n(f_0, B_1)$ are not substantially different than those of $\epsilon_n'(f_0', B_2)$ and $\epsilon_n(f_0, B_2)$. The numerics for $\epsilon_n'(f_0', B_2)$ are therefore consistent with the result of Theorem 2.4, while those for $\epsilon_n(f_0, B_2)$ support our statement at the end of Section 2.

4. Proofs

To establish Theorems 2.2, 2.4 and 2.5 we shall make use of several lemmas. The first two of these are due to Suetin [10].

Lemma 4.1 (Suetin [10, p.20]). *Suppose $\Gamma \in C(p+1, \alpha)$, $p \geqslant 0$, $0 < \alpha \leqslant 1$. Then the orthonormal polynomials Q_n of Section 1 satisfy*

$$Q_n(z) = \sqrt{\frac{n+1}{\pi}}\, \phi'(z)[\phi(z)]^n \left[1 + \mathrm{O}\left(\frac{\log n}{n^{p+\alpha}}\right)\right], \quad z \in \Gamma, \tag{4.1}$$

where $w = \phi(z)$ is the conformal mapping of $\overline{C}/\overline{\Omega}$ onto $\{w\colon |w| > 1\}$ normalized by $\phi(\infty) = \infty$ and $\phi'(\infty) > 0$.

Lemma 4.2 (Suetin [10, p.38]). *Let A be a simply-connected domain bounded by a Jordan curve $\gamma \in C(1, \alpha)$, $0 < \alpha \leqslant 1$. Then for every polynomial $P_n \in \Pi_n$ we have*

$$\int_\gamma |P_n(z)|^2 |\mathrm{d}z| \leqslant c(n+1) \int\int_A |P_n(z)|^2 \,\mathrm{d}m, \tag{4.2}$$

where the constant c depends only on γ.

Lemma 4.3. *Suppose $\Gamma \in C(p+1, \alpha)$, $p \geqslant 0$, $0 < \alpha \leqslant 1$, and let $B \subset \Omega$ be as in Theorem* 2.2. *Then there exists a positive constant $\tau = \tau(B)$ such that*

$$\| Q_n \|_{L^2(B)} \geqslant \tau > 0, \quad n = 0, 1, \ldots, \tag{4.3}$$

where the Q_n are the orthonormal polynomials of Section 1.

Proof. Since ∂B contains a subarc of Γ, it is always possible to choose a subarc $\gamma_0 \subset \Gamma \cap \partial B$ and construct a Jordan domain $A \subset B$ such that $\gamma := \partial A \in C(1, \alpha)$ and $\gamma_0 \subset \gamma$. Then from (4.1) it follows that

$$\int_\gamma |Q_n(z)|^2 |\mathrm{d}z| \geqslant \int_{\gamma_0} |Q_n(z)|^2 |\mathrm{d}z| \geqslant c_0(n+1), \quad n = 0, 1, \ldots, \tag{4.4}$$

(recall that $|\phi(z)| = 1$ on $\gamma_0 \subset \Gamma$). On the other hand, from (4.2) we get

$$\int_\gamma |Q_n(z)|^2 |\mathrm{d}z| \leqslant c(n+1) \int\int_A |Q_n(z)|^2 \,\mathrm{d}m \leqslant c(n+1) \int\int_B |Q_n(z)|^2 \,\mathrm{d}m. \tag{4.5}$$

Thus combining (4.4) and (4.5), we obtain

$$0 < \frac{c_0}{c} \leqslant \int\int_B |Q_n(z)|^2 \,\mathrm{d}m,$$

and this gives the desired inequality (4.3). □

Lemma 4.4. *With the notations and assumptions of Section* 1, *there exist positive constants c_1 and c_2 such that*

$$c_1 \| K_{n-1}(\cdot, \zeta) - K(\cdot, \zeta) \|_{L^2(\Omega)} \leqslant \| f_\zeta' - \pi_n' \|_{L^2(\Omega)} \leqslant c_2 \| K_{n-1}(\cdot, \zeta) - K(\cdot, \zeta) \|_{L^2(\Omega)}, \tag{4.6}$$

for $n = 0, 1, \ldots$.

Thus the $L^2(\Omega)$ norms of $f_\zeta' - \pi_n'$ and $K_{n-1}(\cdot, \zeta) - K(\cdot, \zeta)$ are equivalent.

Proof. Recall that $K(\zeta, \zeta) f_\zeta'(z) = K(z, \zeta)$ and that $K_{n-1}(\zeta, \zeta)\pi_n'(z) = K_{n-1}(z, \zeta)$. Thus

$$\begin{aligned} &\| f_\zeta' - \pi_n' \|_{L^2(\Omega)} \\ &\quad = \left\| \frac{K(\cdot, \zeta)}{K(\zeta, \zeta)} - \frac{K_{n-1}(\cdot, \zeta)}{K_{n-1}(\zeta, \zeta)} \right\|_{L^2(\Omega)} \\ &\quad = \frac{\| K(\cdot, \zeta)[K_{n-1}(\zeta, \zeta) - K(\zeta, \zeta)] + K(\zeta, \zeta)[K(\cdot, \zeta) - K_{n-1}(\cdot, \zeta)] \|_{L^2(\Omega)}}{K(\zeta, \zeta) K_{n-1}(\zeta, \zeta)}. \end{aligned} \tag{4.7}$$

Since $K(\zeta, \zeta)$ is finite and positive and $K_{n-1}(\zeta, \zeta) \to K(\zeta, \zeta)$ as $n \to \infty$, the estimates (4.6) will follow from (4.7) and the triangle inequality, provided we show that

$$|K_{n-1}(\zeta, \zeta) - K(\zeta, \zeta)| = o(\| K_{n-1}(\cdot, \zeta) - K(\cdot, \zeta) \|_{L^2(\Omega)}), \quad \text{as } n \to \infty. \tag{4.8}$$

But

$$|K_{n-1}(\zeta, \zeta) - K(\zeta, \zeta)| = \sum_{k=n}^{\infty} |Q_k(\zeta)|^2$$

and

$$\| K_{n-1}(\cdot, \zeta) - K(\cdot, \zeta) \|_{L^2(\Omega)} = \left(\sum_{k=n}^{\infty} |Q_k(\zeta)|^2 \right)^{1/2}.$$

Thus

$$\frac{|K_{n-1}(\zeta, \zeta) - K(\zeta, \zeta)|}{\| K_{n-1}(\cdot, \zeta) - K(\cdot, \zeta) \|_{L^2(\Omega)}} = \left(\sum_{k=n}^{\infty} |Q_k(\zeta)|^2 \right)^{1/2}, \tag{4.9}$$

and this yields (4.8), since the right-hand side of (4.9) clearly tends to zero as $n \to \infty$. □

Proof of Theorem 2.2. We shall follow closely the argument of [5]. Let

$$E_n := \| K(\cdot, \zeta) - K_n(\cdot, \zeta) \|_{L^2(\Omega)} = \left(\sum_{k=n+1}^{\infty} |Q_k(\zeta)|^2 \right)^{1/2}$$

and

$$r_n := \frac{\| K(\cdot, \zeta) - K_n(\cdot, \zeta) \|_{L^2(B)}}{E_n}.$$

Then, from the triangle inequality, it follows that

$$\begin{aligned} \| Q_n(\zeta) Q_n(\cdot) \|_{L^2(B)} &= \| K_n(\cdot, \zeta) - K_{n-1}(\cdot, \zeta) \|_{L^2(B)} \\ &\leqslant \| K_n(\cdot, \zeta) - K(\cdot, \zeta) \|_{L^2(B)} + \| K_{n-1}(\cdot, \zeta) - K(\cdot, \zeta) \|_{L^2(B)} \\ &= r_n E_n + r_{n-1} E_{n-1}, \end{aligned}$$

that is,

$$\| Q_n(\zeta) Q_n(\cdot) \|_{L^2(B)} \leqslant (E_n + E_{n-1}) \max(r_n, r_{n-1}), \quad n = 1, 2, \ldots . \tag{4.10}$$

On the other hand, by Lemma 4.3, we have that

$$\| Q_n(\zeta) Q_n(\cdot) \|_{L^2(B)} \geqslant |Q_n(\zeta)| \tau = (E_{n-1}^2 - E_n^2)^{1/2} \tau, \quad n = 1, 2, \ldots, \tag{4.11}$$

for some $\tau > 0$. Thus, from (4.10) and (4.11), we get

$$(E_{n-1}^2 - E_n^2)^{1/2} \tau \leqslant (E_{n-1} + E_n) \max(r_n, r_{n-1}),$$

and this implies that

$$\tau^2 \left(\frac{E_{n-1} - E_n}{E_{n-1} + E_n} \right) \leqslant \max(r_n^2, r_{n-1}^2), \quad n = 1, 2, \ldots . \tag{4.12}$$

Next we note that E_n decreases to zero as $n \to \infty$ ($E_n \downarrow 0$, as $n \to \infty$). Hence it follows from elementary properties of series that

$$\sum_{n=1}^{\infty} \left(\frac{E_{n-1} - E_n}{E_{n-1} + E_n} \right) = \infty.$$

(This divergence can be seen from the comparison

$$\frac{E_{n-1}-E_n}{E_{n-1}+E_n} \geqslant \frac{1}{2}\left(1-\frac{E_n}{E_{n-1}}\right)$$

and the observation that $\prod_{n=1}^{\infty} E_n/E_{n-1}$ diverges to zero.) Therefore, from (4.12), we get that $\sum_{n=1}^{\infty} \max(r_n^2, r_{n-1}^2)=\infty$ and this implies the desired result $\sum_{n=0}^{\infty} r_n^2=\infty$. □

Proof of Theorem 2.4. From (4.7) (with Ω replaced by B) and (4.9) we get

$$\begin{aligned}\| f_\zeta' - \pi_n' \|_{L^2(B)} \geqslant{}& \frac{1}{K_{n-1}(\zeta,\zeta)} \| K(\cdot,\zeta) - K_{n-1}(\cdot,\zeta)\|_{L^2(B)} \\ &- \frac{\| K(\cdot,\zeta)\|_{L^2(B)}}{K(\zeta,\zeta)K_{n-1}(\zeta,\zeta)} | K_{n-1}(\zeta,\zeta) - K(\zeta,\zeta)| \\ \geqslant{}& c_1 \| K(\cdot,\zeta) - K_{n-1}(\cdot,\zeta)\|_{L_2(B)} \\ &- c_2\left(\sum_{k=n}^{\infty} |Q_k(\zeta)|^2\right)^{1/2} \| K(\cdot,\zeta) - K_{n-1}(\cdot,\zeta)\|_{L^2(\Omega)},\end{aligned}$$

where c_1, c_2 are positive constants. Thus, by Corollary 2.3, there exists a subsequence $\Lambda \subset \mathbb{N}$ such that

$$\| f_\zeta' - \pi_n' \|_{L^2(B)} \geqslant \left[\frac{c_3}{n^{1/2+\epsilon}} - c_2\left(\sum_{k=n}^{\infty} |Q_k(\zeta)|^2\right)^{1/2}\right] \| K(\cdot,\zeta) - K_{n-1}(\cdot,\zeta)\|_{L^2(\Omega)},$$
$$n \in \Lambda. \tag{4.13}$$

Also, from [10, p.35], we have

$$\left(\sum_{k=n}^{\infty} |Q_k(\zeta)|^2\right)^{1/2} \leqslant \frac{c_4}{n^{p+\alpha}}, \quad n=1,2,\ldots. \tag{4.14}$$

We now assume, without loss of generality, that $0<\epsilon<p+\alpha-\frac{1}{2}$. Then, from (4.13) and (4.14), we have that

$$\| f_\zeta' - \pi_n' \|_{L^2(B)} \geqslant \frac{c_5}{n^{1/2+\epsilon}} \| K(\cdot,\zeta) - K_{n-1}(\cdot,\zeta)\|_{L^2(\Omega)}, \quad n \in \Lambda. \tag{4.15}$$

Finally, by using Lemma 4.4, we get from this last inequality that (2.3) holds for all $n \in \Lambda$. □

Proof of Theorem 2.5. Let E_n and r_n have the same meanings as in the proof of Theorem 2.2. Since Γ is an analytic curve, it is well known that

$$\limsup_{n\to\infty} E_n^{1/n} = \frac{1}{\rho},$$

for some $\rho>1$ (cf. [3, p.35]). Furthermore, since

$$\liminf_{n\to\infty} \frac{E_n}{E_{n-1}} \leqslant \limsup_{n\to\infty} E_n^{1/n},$$

there exists a subsequence $\Lambda_0 \subset \mathbb{N}$ such that

$$\lim_{\substack{n\to\infty \\ n\in\Lambda_0}} \frac{E_n}{E_{n-1}} = \frac{1}{\rho_0} \leqslant \frac{1}{\rho} < 1.$$

For this subsequence we have

$$\lim_{\substack{n\to\infty \\ n\in\Lambda_0}} \frac{E_{n-1}-E_n}{E_{n-1}+E_n} = \frac{1-\rho_0^{-1}}{1+\rho_0^{-1}},$$

and so from (4.12) we get

$$0 < \tau^2\left(\frac{\rho_0 - 1}{\rho_0 + 1}\right) \leqslant \liminf_{\substack{n\to\infty \\ n\in\Lambda_0}} \max\left(r_n^2, r_{n-1}^2\right).$$

It follows that there exists a subsequence $\Lambda \subset \mathbb{N}$ such that

$$r_n \geqslant c > 0, \quad n \in \Lambda,$$

and this yields the desired result (2.4). The second result (2.5) follows by modifying in an obvious manner the proof of Theorem 2.4. □

References

[1] S. Bergman, *The Kernel Function and Conformal Mapping*, Math. Surveys **5** (Amer. Mathematical Soc., Providence, RI, 1970).

[2] D. Gaier, *Konstruktive Methoden der Konformen Abbildung* (Springer, Berlin, 1964).

[3] D. Gaier, *Lectures on Complex Approximation* (Birkhäuser, Boston, 1987).

[4] D. Gaier, On the convergence of the Bieberbach polynomials in regions with corners, *Constr. Approx.* **4** (1988) 289–305.

[5] X. Li, E.B. Saff and Z. Sha, Behavior of best L_p polynomial approximants on the unit interval and on the unit circle, *J. Approx. Theory* **63** (1990) 170–190.

[6] Z. Nehari, *Conformal Mapping* (McGraw-Hill, New York, 1952).

[7] N. Papamichael and M.K. Warby, Stability and convergence properties of Bergman kernel methods in numerical conformal mapping, *Numer. Math.* **48** (1986) 639–669.

[8] E.B. Saff, A principle of contamination in best polynomial approximation, in: Gomez, Guerra, Jimeniz and Lopez, Eds., *Approximation and Optimization*, Lecture Notes in Math. **1354** (Springer, Berlin, 1988) 79–97.

[9] I.B. Simonenko, On the convergence of Bieberbach polynomials in the case of a Lipschitz domain, *Math. USSR-Izv.* **13** (1979) 166–174.

[10] P.K. Suetin, *Polynomials Orthogonal over a Region and Bieberbach Polynomials,* Proc. Steklov Inst. Math. **100** (1971) (Amer. Mathematical Soc., Providence, RI, 1974).

[11] M.K. Warby, BKMPACK, user's guide, Technical Report, Dept. Math. Statist., Brunel Univ. 1992.