Reprinted from JOURNAL OF APPROXIMATION THEORY Vol. 9, No. 2, October 1973
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On the Degree of Best Rational Approximation to the Exponential Function

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Communicated by Joseph L. Walsh

Received May 2, 1971

In two recent papers [4, 5] the author investigated the convergence of rational functions of best approximation to the exponential function. The purpose of the present note is to obtain some asymptotic results on the degree of best uniform approximation to the exponential function by rational functions of prescribed types.

A rational function $r_{m,n}(z)$ is said to be *of type* (m, n) if it is of the form $r_{m,n}(z) = p_m(z)/q_n(z)$, where $p_m(z)$ and $q_n(z)$ are polynomials of degree at most m and n respectively. Let $V_{m,n}$ denote the set of all rational functions of type (m, n) and set

$$E_{m,n}(e^{az}, \rho) \equiv \inf_{r_{m,n} \in V_{m,n}} [\max | e^{az} - r_{m,n}(z)|; | z | \leqslant \rho].$$

Our main result (compare [2, p. 168]) is the following.

THEOREM 1. *For each fixed n we have*

$$E_{m,n}(e^{az}, \rho) = \frac{m!n!(| a | \rho)^{m+n+1}}{(m+n)!(m+n+1)!}(1 + o(1)) \qquad \text{as} \quad m \to \infty. \quad (1)$$

For the special case $n = 0$, Eq. (1) gives the known result on polynomial approximation [1].

To prove Theorem 1 we first investigate the degree of convergence of the Padé approximants [3, Sect. 73] for e^z. The Padé approximant of type (m, n) for the function e^z is the unique rational function $R_{m,n}(z)$ of type (m, n) with the property that $e^z - R_{m,n}(z)$ has a zero of order $m + n + 1$ at $z = 0$. We shall write

$$R_{m,n}(z) = P_{m,n}(z)/Q_{m,n}(z),$$

* The research of the author was supported, in part, by NSF Grant GF-19275.

where $P_{m,n}(z)$ and $Q_{m,n}(z)$ are polynomials of respective degrees m and n, and $Q_{m,n}(0) = 1$.

LEMMA. *Let* $\epsilon_{m,n} \equiv m!n!/(m+n)!(m+n+1)!$. *Then for each fixed* n *there holds*

$$\lim_{m\to\infty} \frac{e^z - R_{m,n}(z)}{(-1)^n\, \epsilon_{m,n} z^{m+n+1}} = 1, \tag{2}$$

uniformly for z *on each bounded subset of the plane.*

Proof. Set $F_m(z) \equiv (e^z Q_{m,n}(z) - P_{m,n}(z))/(-1)^n\, \epsilon_{m,n} z^{m+n+1}$. From the known identity [3, p. 436]

$$e^z Q_{m,n}(z) - P_{m,n}(z) = \frac{(-1)^n\, z^{m+n+1}}{(m+n)!} \int_0^1 e^{tz} t^n (1-t)^m\, dt$$

it follows that

$$F_m(z) - 1 = \sum_{k=1}^{\infty} \frac{z^k}{k!} \prod_{\nu=1}^{k} \frac{(n+\nu)}{(m+n+\nu+1)}.$$

Let ϵ and τ be two positive numbers and choose m_0 so that

$$(n+1)/(m_0+n+1) < \epsilon/\tau(1+\epsilon).$$

Then for $|z| \leqslant \tau$ and $m \geqslant m_0$ we have

$$|F_m(z) - 1| \leqslant \sum_{k=1}^{\infty} \frac{\tau^k}{k!} \prod_{\nu=1}^{k} \frac{\nu(n+1)}{(m+n+\nu+1)} \leqslant \sum_{k=1}^{\infty} \frac{\epsilon^k}{(1+\epsilon)^k} = \epsilon,$$

and hence $\lim_{m\to\infty} F_m(z) = 1$ uniformly on each bounded subset of the plane. Since $\lim_{m\to\infty} Q_{m,n}(z) = 1$ for all z [3, p. 436], the lemma follows.

Proof of Theorem 1. It suffices to prove Theorem 1 for the case $a = 1$. From (2) it is clear that $\limsup_{m\to\infty} E_{m,n}(e^z, \rho)/\epsilon_{m,n}\rho^{m+n+1} \leqslant 1$. Hence it remains to show that $\lambda \equiv \liminf_{m\to\infty} E_{m,n}(e^z, \rho)/\epsilon_{m,n}\rho^{m+n+1} \geqslant 1$.

Let $W_{m,n}(z) \in V_{m,n}$ satisfy

$$E_{m,n}(e^z, \rho) = [\max |e^z - W_{m,n}(z)|;\ |z| \leqslant \rho],$$

and write $W_{m,n}(z) = p_{m,n}(z)/q_{m,n}(z)$, where $p_{m,n}(z)$ and $q_{m,n}(z)$ are polynomials of degree at most m and n, respectively, and $q_{m,n}(0) = 1$. It is proved in [4] that $\lim_{m\to\infty} q_{m,n}(z) = 1$ for all z. From the extremal property of the

$W_{m,n}(z)$ and the fact that the sequence $\{q_{m,n}(z)\,Q_{m,n}(z)\}_{m=0}^{\infty}$ is uniformly bounded on $|z| = \rho$, there follow the inequalities

$$\begin{aligned} |R_{m,n}(z) - W_{m,n}(z)| &\leqslant A_1 \epsilon_{m,n} \rho^{m+n+1}, \qquad |z| \leqslant \rho, \\ |P_{m,n}(z)\,q_{m,n}(z) - p_{m,n}(z)\,Q_{m,n}(z)| &\leqslant A_2 \epsilon_{m,n} \rho^{m+n+1}, \qquad |z| \leqslant \rho, \end{aligned} \tag{3}$$

where the constants A_1, A_2 are independent of m. Since the functions

$$g_m(z) = (P_{m,n}(z)\,q_{m,n}(z) - p_{m,n}(z)\,Q_{m,n}(z))/\epsilon_{m,n} z^{m+n+1}$$

are each analytic for $|z| \geqslant \rho$, even at $z = \infty$, we have from (3) and the Maximum Principle that $|g_m(z)| \leqslant A_2$ for $|z| \geqslant \rho$. Hence the sequence $g_m(z)$ forms a normal family in $|z| > \rho$.

Now choose an increasing sequence of positive integers k such that

$$\lim_{k\to\infty} E_{k,n}(e^z, \rho)/\epsilon_{k,n}\rho^{k+n+1} = \lambda,$$

and such that

$$\lim_{k\to\infty} g_k(z) = g(z)$$

uniformly on each closed subset of $|z| > \rho$. Note that $g_k(\infty) = 0$ for each k and so $g(\infty) = 0$. Dividing $g_k(z)$ by $q_{k,n}(z)\,Q_{k,n}(z)(\to 1)$ there follows

$$\lim_{k\to\infty} \frac{R_{k,n}(z) - W_{k,n}(z)}{\epsilon_{k,n} z^{k+n+1}} = g(z), \qquad |z| > \rho,$$

and hence from (2) we have

$$\lim_{k\to\infty} \frac{e^z - W_{k,n}(z)}{(-1)^n\,\epsilon_{k,n} z^{k+n+1}} = 1 + (-1)^n g(z),$$

uniformly on each compact subset of $|z| > \rho$. Let M be a uniform bound for the functions $h_k(z) \equiv (e^z - W_{k,n}(z))/(-1)^n\,\epsilon_{k,n} z^{k+n+1}$ on the circle $|z| = 2\rho$. Then by Hadamard's 3-Circles Theorem we have for $\rho < r < 2\rho$

$$\max_{|z|=r} |h_k(z)| \leqslant \left[\frac{E_{k,n}(e^z, \rho)}{\epsilon_{k,n}\rho^{k+n+1}}\right]^{\log(2\rho/r)/\log 2} M^{\log(r/\rho)/\log 2}.$$

Taking the limit as $k \to \infty$ in the last inequality there follows

$$\max_{|z|=r} |1 + (-1)^n g(z)| \leqslant \lambda^{\log(2\rho/r)/\log 2} M^{\log(r/\rho)/\log 2}. \tag{4}$$

But

$$1 = | 1 + (-1)^n g(\infty) | \leqslant \max_{|z|=r} | 1 + (-1)^n g(z) |,$$

and so on letting $r \to \rho$ in (4) we deduce that $1 \leqslant \lambda$. This proves Theorem 1.

Let us consider another consequence of inequality (4). Since $\lambda = 1$ we have that

$$\sup_{|z|>\rho} | 1 + (-1)^n g(z) | \leqslant \lim_{r\to\rho} \max_{|z|=r} | 1 + (-1)^n g(z) | \leqslant 1.$$

Hence the function $1 + (-1)^n g(z)$ attains its maximum modulus at $z = \infty$ which implies that $1 + (-1)^n g(z) \equiv 1$. Thus $h_k(z) \to 1$ uniformly on each compact set in $| z | > \rho$.

By the same reasoning one can show that every subsequence of $\{(e^z - W_{m,n}(z))/(-1)^n \epsilon_{m,n} z^{m+n+1}\}$ possesses a subsequence which converges to unity for $| z | > \rho$. We have therefore established the following.

THEOREM 2 *For each fixed n the rational functions $W_{m,n}(z)$ of type (m, n) of best uniform approximation to e^z on $| z | \leqslant \rho$ satisfy*

$$\lim_{m\to\infty} \frac{e^z - W_{m,n}(z)}{(-1)^n \epsilon_{m,n} z^{m+n+1}} = 1, \tag{5}$$

uniformly for z on each compact set in $| z | > \rho$.

Theorem 2 appears to be new even for the case $n = 0$. It is of interest for two reasons.

First, (5) implies that the rational functions $W_{m,n}(z)$ of best uniform approximation to e^z on $| z | \leqslant \rho$ yield asymptotically (as $m \to \infty$) the best degree of uniform approximation to e^z on each larger disk $| z | \leqslant \tau$, $\rho < \tau$.

Second, by applying the Argument Principle to (5) one can deduce the following result concerning the points of interpolation of best approximating rational functions.

COROLLARY. *Let $\delta > 0$. Then for $m > m_\delta$ each of the best approximating rational functions $W_{m,n}(z)$ interpolates to e^z in precisely $m + n + 1$ points (counting multiplicity) inside $| z | = \rho + \delta$.*

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Printed by the St Catherine Press Ltd., Tempelhof 37, Bruges, Belgium.