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Szegö Polynomials and Frequency Analysis

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1. INTRODUCTION

Let $x_N = \{x_N(m)\}$ be an N-truncated causal signal of the form

$$x_N(m) = \begin{cases} \sum_{j=-I}^{I} \alpha_j e^{i\omega_j m}, & m = 0, 1, \ldots, N-1 \ (x_N(0) \neq 0) \\ 0, & \text{otherwise}, \end{cases}$$

$$\alpha_0 \geq 0,\ \alpha_{-j} = \bar{\alpha}_j,\ \omega_{-j} = -\omega_j,\ 0 = \omega_0 < \omega_1 < \cdots < \omega_I < \pi.$$

We consider the problem of determining the frequencies ω_j from the $x_N(m)$. Efficient numerical methods for finding the ω_j (especially in the presence of noise) have important applications to science and engineering. Various methods have been considered in [1], [2], [8]. We consider a method due to Levinson and Wiener [9] formulated here in terms of Szegö polynomials. Starting with autocorrelation coefficients $\mu_k^{(N)} := \sum_{m=0}^{N-1} x_N(m)x_N(m+k)$,

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one forms recursively the Szegö polynomials using:

$$(1.2a)\qquad \rho_0(\psi_N;z) := \rho_0^*(\psi_N;z) := 1,$$

$$(1.2b)\qquad \rho_n(\psi_N;z) := z\rho_{n-1}(\psi_N;z) + \delta_n^{(N)}\rho_{n-1}^*(\psi_N;z),\ n \geq 1$$

$$(1.2c)\qquad \rho_n^*(\psi_N;z) := \overline{\delta_n^{(N)}}\, z\rho_{n-1}(\psi_N;z) + \rho_{n-1}^*(\psi_N;z),\ n \geq 1,$$

$$(1.3)\qquad \delta_n^{(N)} := -\frac{\sum_{j=0}^{n-1} q_j^{(n-1,N)}\mu_{-j-1}^{(N)}}{\sum_{j=0}^{n-1} q_j^{(n-1,N)}\mu_{j+1-n}^{(N)}},\ \rho_{n-1}(\psi_N;z) =: \sum_{j=0}^{n-1} q_j^{(n-1,N)} z^j.$$

$\{\rho_n(\psi_N;z)\}$ is the sequence of monic orthogonal polynomials on the unit circle (Szegö polynomials) with respect to the distribution function $\psi_N(\theta)$ defined by

$$(1.4)\qquad \psi_N'(\theta) := \frac{1}{2\pi}|X_N(e^{i\theta})|^2,\ X_N(z) := \sum_{m=0}^{N-1} x_N(m)z^{-m},$$

$\rho_n^*(\psi_N;z) = z^n\rho_n(\psi_N;z^{-1})$, $\rho_n(\psi_N;z) = 0 \Rightarrow |z| < 1$, $|\delta_n^{(N)}| < 1$. The zeros of $\rho_n(\psi_N;z)$ are used to approximate the points $e^{i\omega_j}$. Success of this method depends on the following Conjecture [4]: *As $k \to \infty$ and $N \to \infty$ the $2I+1$ zeros of $\rho_k(\psi_N;z)$ of largest modulus approach the points $e^{i\omega_j}$, $-I \leq j \leq I$.* It is tacitly assumed that $\alpha_j \neq 0$ for $j \neq 0$. If $\alpha_0 = 0$ in (1.1), then there are only $2I$ critical points, since $e^{i0} = 1$ should not be counted. Hence the statement of the conjecture must be adjusted accordingly. Several theorems and numerical experiments that support the conjecture were given in [4]. Another related result is given here in Theorem 1. Our proof of it uses PPC-fractions

$$(1.5)\qquad \delta_0^{(N)} - \frac{2\delta_0^{(N)}}{1} + \frac{1}{\delta_1^{(N)}z} + \frac{(1-|\delta_1^{(N)}|)z}{\delta_2^{(N)}} + \frac{1}{\overline{\delta_2^{(N)}}z} + \cdots$$

$\delta_0^{(N)} := \sum_{m=0}^{N-1}(x_N(m))^2$, introduced in [5], [6] to study the trigonometric moment problem. If P_n and Q_n denote the nth numerator and denominator, respectively, of (1.5), then P_{2n}, Q_{2n}, P_{2n+1}, Q_{2n+1} are polynomials of degree at most n and $Q_{2n}(\psi_N;z) = \rho_n^*(\psi_N;z)$, $Q_{2n+1}(\psi_N;z) = \rho_n(\psi_N;z)$.

2. CONVERGENCE THEOREM

We consider the rational function

$$(2.1)\qquad F(z) := \sum_{j=-I}^{I} |\alpha_j|^2\,\frac{e^{i\omega_j}+z}{e^{i\omega_j}-z} = \int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\psi(\theta),$$

whose poles are the critical points $e^{i\omega_j}$, $-I \le j \le I$. Here $\psi(\theta)$ is a non-decreasing step function with a jump of $|\alpha_j|^2$ at each point ω_j, $j = -I, -I+1, \ldots, I$.

Theorem 1. *There exist constants A and B such that for all $k \ge 1$, $N \ge 1$ and $|z| \le R < 1$,*

$$\left|\frac{1}{N}\,\frac{P_{2k}(\psi_N; z)}{Q_{2k}(\psi_N; z)} - F(z)\right| \le \frac{AR^{k+1}}{1-R^2} + \frac{B}{\sqrt{N}(1-R)^4}.$$

The significance of this result for frequency analysis lies in the fact that the poles of P_{2k}/Q_{2k} are given by $1/\overline{z(j,k,N)}$ where $z(j,k,N)$ (for $1 \le j \le k$) are the zeros of $\rho_k(\psi_N; z)$.

Our proof of Theorem 1 uses several lemmas, the first involving

$$f_N(z) := \int_{-\pi}^{\pi} g(\theta, z)\, d\psi_N(\theta), \quad g(\theta, z) := \frac{e^{i\theta}+z}{e^{i\theta}-z}.$$

Lemma 1. *There exists a constant A such that, for $k \ge 1$, $N \ge 1$, and $|z| \le R < 1$,*

$$\left|\frac{P_{2k}(\psi_N; z)}{Q_{2k}(\psi_N; z)} - f_N(z)\right| \le \frac{ANR^{k+1}}{1-R^2}.$$

Proof. A proof of the lemma is given by [7, Theorem 3.1], [5, Theorem 3.2] and the fact that there exists a constant A such that

$$\delta_0^{(N)} := \sum_{m=0}^{N-1} (x_N(m))^2 \le (A/4)N$$

for $N \ge 1$. □

For the next lemma we consider complex valued functions $f(\theta, z)$ defined on $-\pi \le \theta \le \pi$, $|z| < 1$ such that, for each $0 < R < 1$, there exist constants $M_p(R, f)$ satisfying

$$\left|\frac{\partial^p f(\theta, z)}{\partial \theta^p}\right| \le M_p(R, f) \quad \text{for} \quad |\theta| \le \pi, \quad |z| \le R < 1, \quad p = 0, 1, 2, 3.$$

Moreover, it is assumed that, for fixed z, $f(\theta, z)$ is a periodic function of θ with period 2π and, for fixed θ, $f(\theta, z)$ is an analytic function of the complex variable z for $|z| < 1$. For $|\theta| \le \pi$, $|z| < 1$ and $N \ge 1$, we define

$$\sigma_N(\theta, z, f) := \frac{1}{2\pi N} \int_{-\pi}^{\pi} f(t, z) |h_N(\theta - t)|^2 dt, \tag{2.6a}$$

where

$$h_N(u) := \frac{1 - e^{iNu}}{1 - e^{iu}} = e^{i(N-1)\frac{u}{2}} \frac{\sin \frac{Nu}{2}}{\sin \frac{u}{2}}. \tag{2.6b}$$

Lemma 2. *For functions $f(\theta, z)$ described above and for $|\theta| \leq \pi$, $|z| \leq R < 1$ and $N \geq 1$,*

$$|f(\theta, z) - \sigma_N(\theta, z, f)| \leq \frac{2M_0(R, f) + 4M_3(R, f)}{N}$$

We consider the Fourier series partial sums

$$S_k(\theta, z, f) := \frac{a_0(z)}{2} + \sum_{j=1}^{k} [a_j(z) \cos j\theta + b_j(z) \sin j\theta], \tag{2.8a}$$

$$a_j(z) := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t, z) \cos jt \, dt, \; b_j(z) := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t, z) \sin jt \, dt.$$

Integration by parts (three times) in (2.8b) yields for $|z| \leq R < 1$,

$$|a_j(z)| \leq \frac{2M_p(R, f)}{j^p}, \quad |b_j(z)| \leq \frac{2M_p(R, f)}{j^p} \quad j \geq 1, \; 0 \leq p \leq 3.$$

Since $f(\theta, z) = \lim_{k \to \infty} S_k(\theta, z, f)$, we obtain from (2.8) and (2.9)

$$|f(\theta, z) - S_k(\theta, z, f)| \leq 4M_3(R, f) \sum_{j=k+1}^{\infty} \frac{1}{j^3} \leq \frac{2M_3(R, f)}{k^2}, \; k \geq 1. \tag{2.10}$$

In [3, p. 33] it is shown that

$$\sigma_N(\theta, z, f) = \frac{1}{N} \sum_{k=0}^{N-1} S_k(\theta, z, f).$$

Since $|f(\theta, z) - S_0(\theta, z, f)| \leq |f(\theta, z)| + \dfrac{|a_0(z)|}{2} \leq 2M_0(R, f)$, it follows from (2.10) and (2.11) that

$$|f(\theta, z) - \sigma_N(\theta, z, f)| \leq \frac{1}{N} \left[2M_0(R, f) + 2M_3(R, f) \sum_{k=1}^{N-1} \frac{1}{k^2} \right] \tag{2.12}$$

from which (2.7) follows. □

Lemma 3. *There exists a constant B such that for $|z| \leq R < 1$, and $N \geq 1$,*

$$\left| \frac{1}{N} f_N(z) - F(z) \right| \leq \frac{B}{\sqrt{N}(1 - R)^4} \tag{2.13}$$

From (1.1), (1.4) and (2.3) one has

$$\frac{1}{N} f_N(z) = \sum_{j=-I}^{I} \sum_{m=-I}^{I} \alpha_j \bar{\alpha}_m \int_{-\pi}^{\pi} G_{j,m}(t,z,N)\,dt \tag{2.14a}$$

where

$$G_{j,m}(t,z,N) := \frac{1}{2\pi N} g(t,z) h_N(\omega_j - t)\overline{h_N(\omega_m - t)}. \tag{2.14b}$$

With this notation we see from (2.6) that

$$\int_{-\pi}^{\pi} G_{j,j}(t,z,N)\,dt = \sigma_N(\omega_j, z, g), \qquad -I \le j \le I. \tag{2.15}$$

Therefore considering the terms in (2.14a) with $j = m$, one obtains by Lemma 2

$$\begin{aligned}\left|\sum_{j=-I}^{I} |\alpha_j|^2 \int_{-\pi}^{\pi} G_{j,j}(t,z,N)\,dt - F(z)\right| &= \left|\sum_{j=-I}^{I} |\alpha_j|^2 (\sigma_N(\omega_j,z,g) - g(\omega_j,z))\right| \\ &\le \frac{2M_0(R,g) + 4M_3(R,g)}{N} \sum_{j=-I}^{I} |\alpha_j|^2.\end{aligned} \tag{2.16}$$

To estimate the magnitude of $\int_{-\pi}^{\pi} G_{j,m}(t,z,N)\,dt$ when $j < m$, $\omega_j < \omega_m$, we write the integral in (2.14a) as a sum of five integrals

$$\int_{-\pi}^{\pi} = \int_{-\pi}^{\omega_j - \varepsilon_j} + \int_{\omega_j - \varepsilon_j}^{\omega_j + \varepsilon_j} + \int_{\omega_j + \varepsilon_j}^{\omega_m - \varepsilon_m} + \int_{\omega_m - \varepsilon_m}^{\omega_m + \varepsilon_m} + \int_{\omega_m + \varepsilon_m}^{\pi}, \tag{2.17}$$

where positive numbers ε_j are chosen such that $[\omega_j - \varepsilon_j, \omega_j + \varepsilon_j] \cap [\omega_m - \varepsilon_m, \omega_m + \varepsilon_m] = \phi$ if $j \ne m$ and $\omega_k \pm \varepsilon_k \in [-\pi, \pi]$ for all $-I \le k \le I$. Let

$$L := \max_{-I \le j \le I} \left[\left| \sin^2\left(\frac{\omega_j - t}{2}\right)\right|^{-1} : t \notin [\omega_j - \varepsilon_j, \omega_j + \varepsilon_j], |t| \le \pi \right].$$

For the first integral in (2.17) we obtain, for $|z| \le R$,

$$\left|\int_{-\pi}^{\omega_j - \varepsilon_j} G_{j,m}(t,z,N)\,dt\right| \le \frac{M_0(R,g)L}{N}. \tag{2.18}$$

The same upper bound applies to the third and fifth integrals on the right-hand side of (2.17). Applying the Schwarz inequality to the second integral yields

(2.19a)

$$\left|\int_{\omega_j - \varepsilon_j}^{\omega_j + \varepsilon_j} G_{j,m}(t,z,N)\,dt\right| \le \frac{M_0(R,g)}{2\pi N} \left[\int_{\omega_j - \varepsilon_j}^{\omega_j + \varepsilon_j} |h_N(\omega_j - t)|^2 dt\right]^{\frac{1}{2}} \cdot \left[\int_{\omega_j - \varepsilon_j}^{\omega_j + \varepsilon_j} |h_N(\omega_m - t)|^2 dt\right]^{\frac{1}{2}},$$

where

$$\text{(2.19b)} \qquad \int_{\omega_j-\epsilon_j}^{\omega_j+\epsilon_j} |h_N(\omega_m - t)|^2 dt = \int_{\omega_j-\epsilon_j}^{\omega_j+\epsilon_j} \frac{\sin^2 \frac{N(\omega_m - t)}{2}}{\sin^2(\frac{\omega_m - t}{2})} dt \leq 2\pi L$$

$$\text{(2.19c)} \qquad \int_{\omega_j-\epsilon_j}^{\omega_j+\epsilon_j} |h_N(\omega_j - t)|^2 dt \leq \int_{-\pi}^{\pi} \frac{\sin^2 \frac{Nu}{2}}{\sin^2 \frac{u}{2}} du = 2\pi N \sigma_N(\theta, z, 1) = 2\pi N.$$

For (2.19c) we have applied (2.6) to the constant function $f(\theta, t) \equiv 1$, so that $\sigma_N(\theta, z, 1) \equiv 1$. It follows from (2.19) that

$$\left| \int_{\omega_j-\epsilon_j}^{\omega_j+\epsilon_j} G_{j,m}(t, z, N) dt \right| \leq \frac{M_0(R, g)}{2\pi N} [2\pi L]^{\frac{1}{2}} [2\pi N]^{\frac{1}{2}} = \frac{M_0(R, g)\sqrt{L}}{\sqrt{N}}$$

A similar bound holds for the fourth integral on the right-hand side of (2.17). Combining the above results yields

$$\left| \int_{-\pi}^{\pi} G_{j,m}(t, z, N) dt \right| \leq \frac{3LM_0(R, g)}{N} + \frac{2\sqrt{L}M_0(R, g)}{\sqrt{N}}.$$

Let $\alpha := \max[|\alpha_j| : -I \leq j \leq I]$. Since there are $2I(2I + 1)$ integrals of the form (2.20) with $j \neq m$ in (2.14a), it follows from (2.14), (2.16) and (2.20) that for $|z| \leq R < 1$ and

$$\text{(2.21)} \qquad \begin{aligned} \left| \frac{1}{N} f_N(t) - F(z) \right| \leq & \frac{(2I + 1)\alpha^2 (2M_0(R, g) + 4M_3(R, g))}{N} \\ & + 2I(2I + 1)\alpha^2 \left[\frac{3LM_0(R, g)}{N} + \frac{2\sqrt{L}M_0(R, g)}{\sqrt{N}} \right] \end{aligned}$$

From the definition of $g(\theta, z)$ in (2.3) one can show that

$$M_0(R, g) = \frac{1 + R}{1 - R} \quad \text{and} \quad M_3(R, g) = \frac{12R}{(1 - R)^4}$$

satisfy (2.5). Lemma 3 follows from (2.21) and (2.22). □

Theorem 2.1 is an immediate consequence of Lemmas 1 and 3.

3. COMPUTATIONAL RESULTS

For a numerical experiment we chose $I = 4, \omega_1 = \pi/2, \omega_2 = \pi/3, \omega_3 = \pi/6, \omega_4 = 3\pi/4$, $\alpha_0 = 0$, $\alpha_1 = \alpha_2 = \alpha_3 = 1/2$ and $\alpha_4 = 5$ so that (1.1) becomes $x(m) = \cos(m\pi/2) + \cos(m\pi/3) + \cos(m\pi/6) + 10\cos(3m\pi/4)$. To these values we added a small component of white noise with mean zero and variance 0.02. We computed the Szegö polynomials $\rho_k(\psi_N; z)$ and their zeros $z(j,k,N)$ for $1 \le j \le k$, $1 \le k \le 50$ and $100 \le N \le 3000$ (in steps of 100 up to 2000). For each k and N we chose the 4 zeros in $\operatorname{Im} z > 0$ with largest modulus and associated with each such $z(j,k,N)$ the nearest point $e^{i\omega_j}$. Graphs of $|z(j,k,N) - e^{i\omega_j}|$ vs N in log-log scale (base 10) for $k = 8, 10, 20, 30, 40, 50$ are shown in Figures 1–4 for $\omega_j = \pi/6$, $\pi/3$, $\pi/2$, $3\pi/4$, respectively. Each graph appears to be roughly a straight line with negative slope $\lambda(j,k)$. Approximating $\lambda(j,k)$ by using the line determined by the two endpoints of each graph, we obtain the values shown in Table 1. It can be seen that for the three values of ω_j with $|\alpha_j| = 1$, the numbers $\lambda(j,k)$ decrease as k increases. These experimental results not only support the conjecture made in Section 1, but also suggest a regular convergence rate of the form $|z(j,k,N) - e^{i\omega_j}| = O(N^{\lambda(j,k)})$.

Table 1. Values of $\lambda(j,k) := [\operatorname{Log} f(j,k,3000) - \operatorname{Log} f(j,k,100)]/[\operatorname{Log} 3000 - \operatorname{Log} 100]$, where $f(j,k,N) := |z(j,k,N) - e^{i\omega_j}|$. $z(j,k,N)$ denotes the zero of $\rho_k(\psi_N; z)$ nearest to $e^{i\omega_j}$. $\rho_k(\psi_N; z)$ denotes the Szegö polynomial of degree k. N denotes the sample size of input data.

$2\lvert\alpha_j\rvert$	1	1	1	10
$k \backslash \omega_j$	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$
10				
20	-0.84	-0.84	-0.86	not computed
30	-0.91	-0.97	-0.96	not computed
40	-1.03	-1.14	-1.01	not computed
	-1.15	-1.19	-1.07	not computed

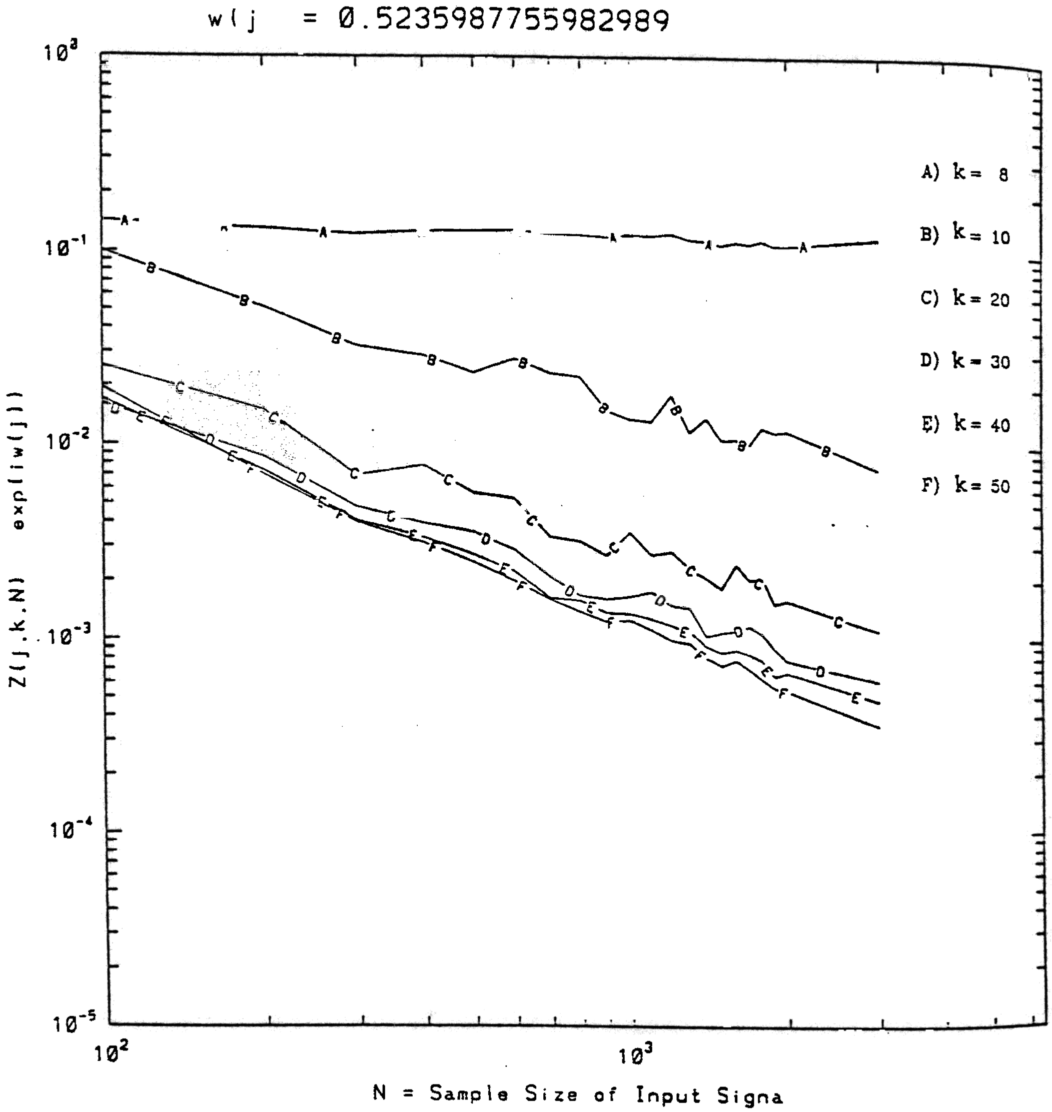

Figure 1. Graphs of $|z(j,k,N) - e^{i\omega_j}|$ versus N in a log-log scale suggest the convergence as $k \to \infty$ and $N \to \infty$ of zeros $z(j,k,N)$ of Szegö polynomials $\rho_k(\psi_N; z)$ to critical points $e^{i\omega_j}$. For this case $\omega_j = \pi/6$ and $2|\alpha_j| = 1$.

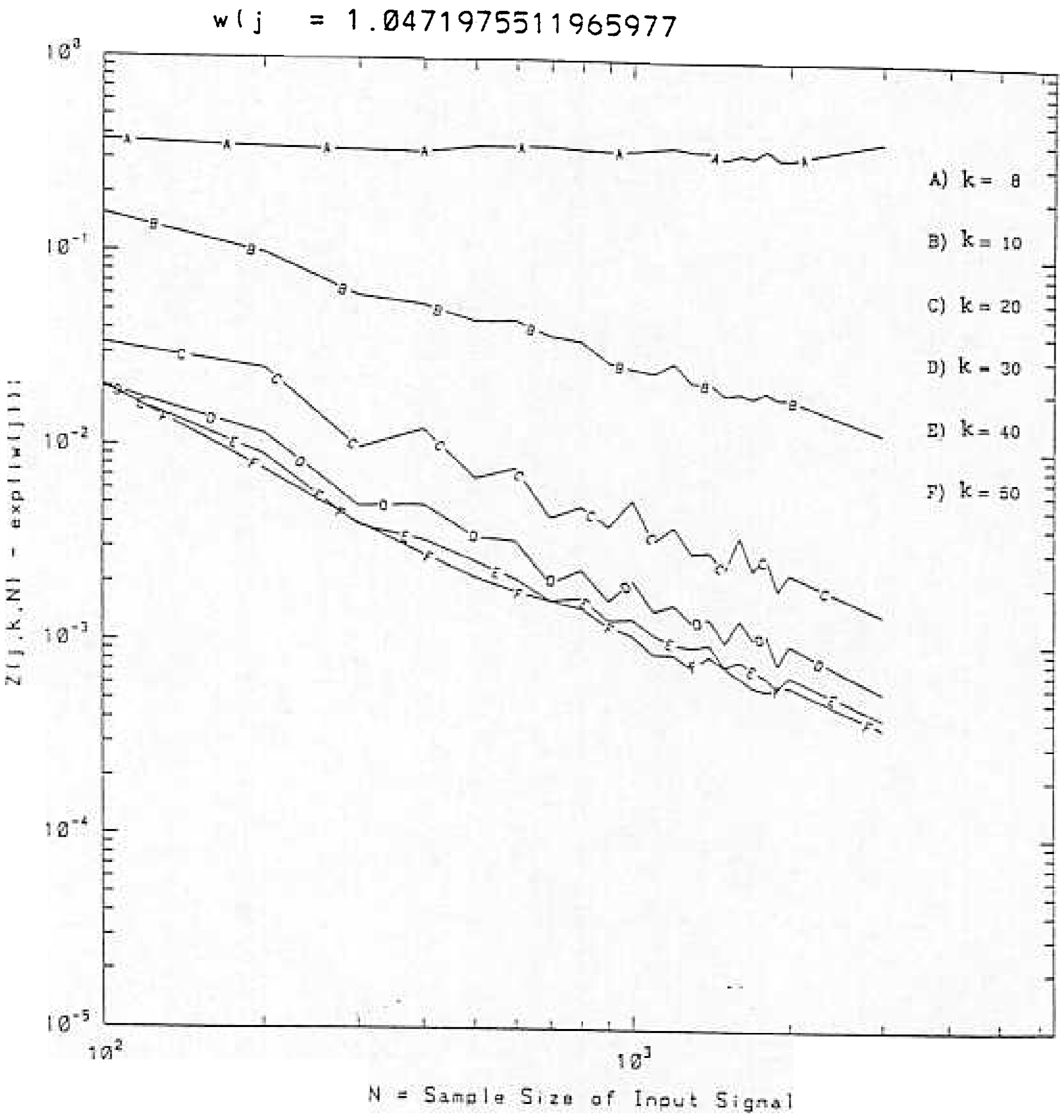

Figure 2. Graphs of $|z(j,k,N) - e^{i\omega_j}|$ versus N in a log-log scale suggest the convergence as $k \to \infty$ and $N \to \infty$ of zeros $z(j,k,N)$ of Szegö polynomials $\rho_k(\psi_N; z)$ to critical points $e^{i\omega_j}$. For this case $\omega_j = \pi/3$ and $2|\alpha_j| = 1$.

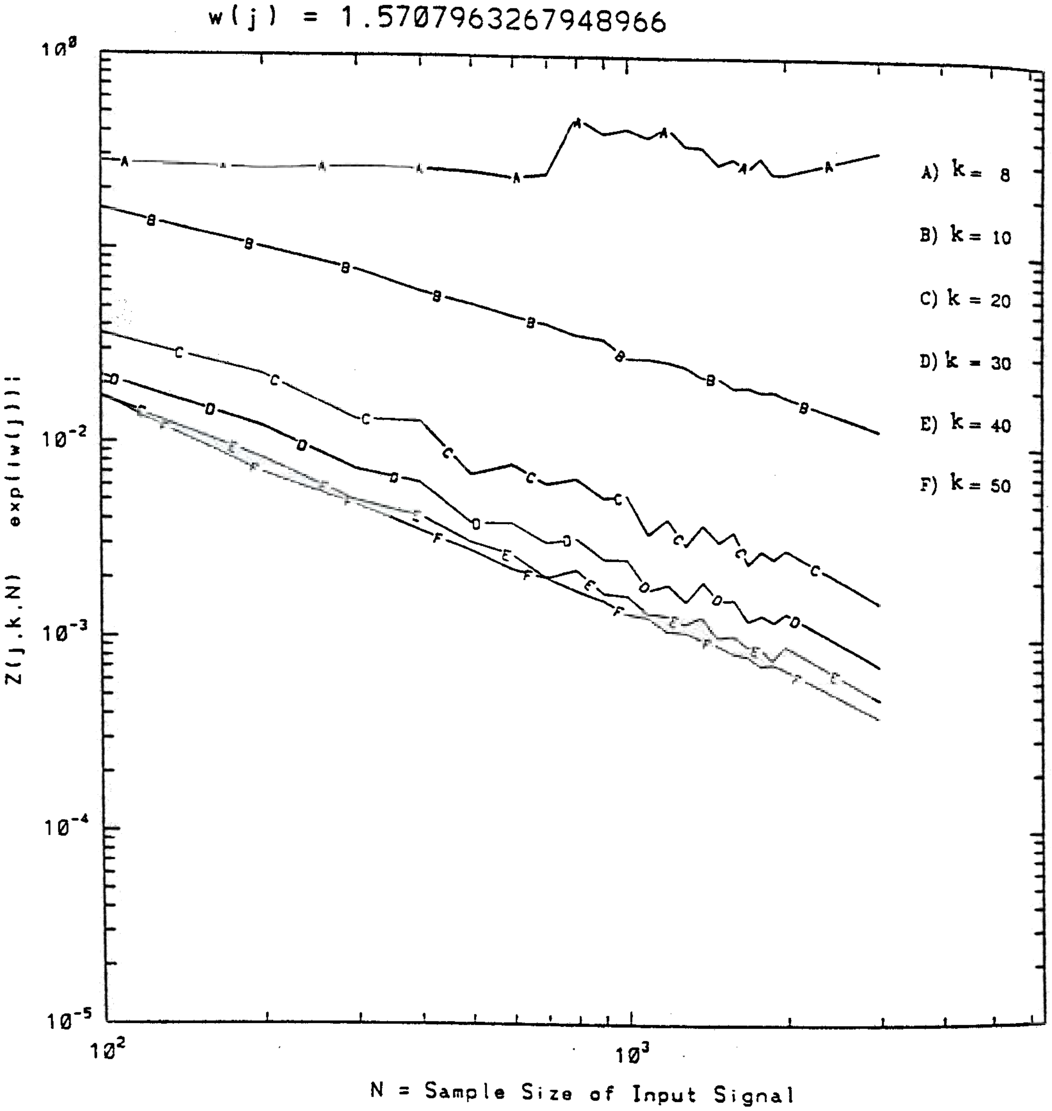

Figure 3. Graphs of $|z(j,k,N) - e^{i\omega_j}|$ versus N in a log-log scale suggest the convergence as $k \to \infty$ and $N \to \infty$ of zeros $z(j,k,N)$ of Szegő polynomials $\rho_k(\psi_N; z)$ to critical points $e^{i\omega_j}$. For this case $\omega_j = \pi/2$ and $2|\alpha_j| = 1$.

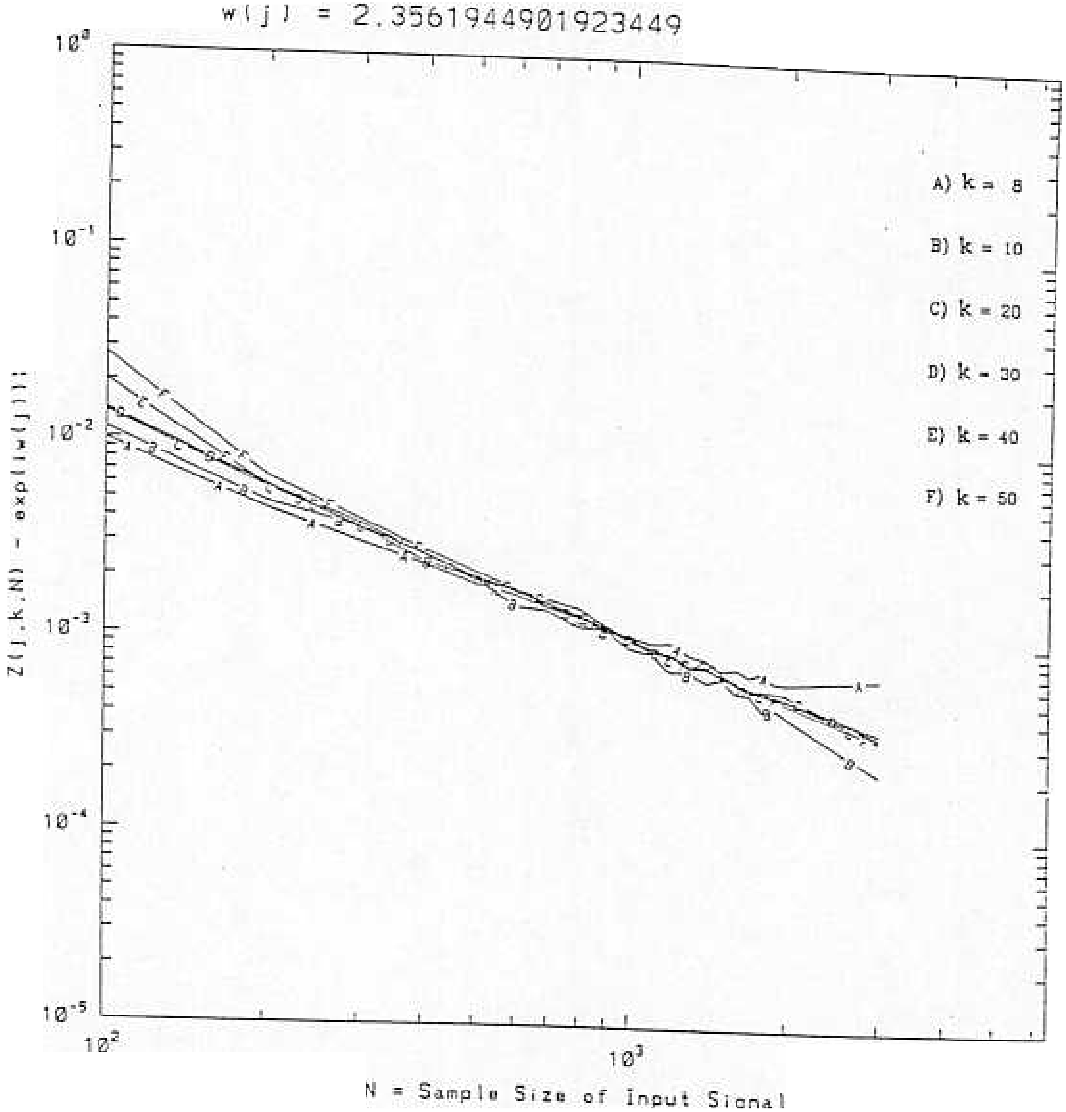

Figure 4. Graphs of $|z(j,k,N) - e^{i\omega_j}|$ versus N in a log-log scale suggest the convergence as $k \to \infty$ and $N \to \infty$ of zeros $z(j,k,N)$ of Szegö polynomials $\rho_k(\psi_N; z)$ to critical points $e^{i\omega_j}$. For this case $\omega_j = 3\pi/4$ and $2|\alpha_j| = 10$.

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REFERENCES

1. Hildebrand, F. B., *Introduction to Numerical Analysis*, McGraw–Hill Book Company, Inc., New York (1956).

2. Kumaresam, R., L. L. Scharf, A. K. Shaw, An algorithm for pole–zero modeling and spectral analysis, *IEEE Trans. ASSP* 34, No. 3 (June 1986), 637–640.

3. Jackson, D., *Fourier Series and Orthogonal Polynomials*, Math. Assoc. Amer. Carus Math. Mono. (1941).

4. Jones, William B., Olav Njåstad, E. B. Saff, Szegö Polynomials Associated with Wiener–Levinson Filters, *Journal of Computational and Applied Math.* 32 (1990), 387–406.

5. Jones, William B., Olav Njåstad and W. J. Thron, Continued fractions associated with the trigonometric and other strong moment problems, *Constructive Approximation* 2 (1986), 197–211.

6. Jones, William B., Olav Njåstad and W. J. Thron, Moment theory, orthogonal polynomials, quadrature and continued fractions associated with the unit circle, *Bull. London Math. Soc.* 21 (1989), 113–152.

7. Jones, William B. and W. J. Thron, A constructive proof of convergence of the even approximants of positive PC-fractions, *Rocky Mtn. J. Math.* 19, No. 1 (1989), 199–210.

8. Paul, A. K., Anharmonic frequency analysis, *Math. of Comp.* 26, No. 118 (April 1972), 437–447.

9. Wiener, Norbert, *Extrapolation, Interpolation and Smoothing of Stationary Time Series*, Published jointly by The Technology Press of the Massachusetts Institute of Technology, and John Wiley & Sons, Inc., New York (1949).