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A MINIMAL SOLUTION APPROACH TO POLYNOMIAL ASYMPTOTICS

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A general comparison method, based on the construction of a minimal solution to a given three term recurrence $Y_{n+1} - (z - a_n)Y_n + b_n^2 Y_{n-1} = 0$, $b_n^2 \neq 0$, and Pincherle's theorem, is applied to the case where $\sum^{\infty}(|a_n| + |b_n^2 - \frac{1}{4}|) < \infty$. A strong form of Nevai's theorem on the asymptotics of the polynomial solutions to the recurrence is obtained which extends to the case of complex coefficients.

1. INTRODUCTION

A monic orthogonal polynomial set $\{Q_n(z)\}_{n=0}^{\infty}$, (orthogonal with respect to a quasi-definite linear functional), is characterized by a three term recurrence (a_n, $b_n^2 \in \mathbf{C}$)

$$Q_{n+1}(z) - (z - a_n)Q_n(z) + b_n^2 Q_{n-1}(z) = 0, \quad b_{n+1}^2 \neq 0, \ n \geq 0$$

with initial conditions $Q_{-1} = 0$, $Q_0 = 1$ (Favard's theorem [1]). In the case of positive definite orthogonality one has $a_n \in \mathbf{R}$, $b_{n+1}^2 > 0$, $n \geq 0$ and a positive measure $d\alpha$ with infinite support such that

$$\int_{\mathbf{R}} Q_n(x)Q_m(x)d\alpha(x) = b_1^2 \cdots b_n^2 \delta_{nm}.$$

The connection between the polynomial asymptotics and those of the coefficients a_n, b_n^2 has been investigated extensively by Nevai and others in the positive definite case (see [6], [7] and the recent survey by Van Assche [8]).

We consider here a general approach to this problem based on the minimal solution to a given three term recurrence relation and Pincherle's theorem [2], [4]. Our strategy is to first construct the minimal solution and then deduce properties of the polynomial solutions.

This viewpoint is capable of dealing with bounded or unbounded coefficients which are real or complex. However to illustrate the approach we consider here the complex Nevai class $M(a,b)$ where $a_n \to a$ and $b_n^2 \to b^2/4 \neq 0$. Without loss of generality we take $a = 0$ and $b = 1$.

Since our construction of a minimal solution is based on a perturbation method we also, in all that follows, make the natural additional assumption

$$(1.3) \qquad \sum^{\infty}(|a_n| + |b_n^2 - \frac{1}{4}|) < \infty.$$

This allows a comparison with Chebyshev polynomials and an extension of positive definite results to the quasi-definite case. In particular we give a new derivation of a result of Nevai [6, p. 143] which extends to the case of complex coefficients. Note that if (1.3) does not hold, then a different comparison is required. For example, the case $\sum^{\infty}(|a_n - \frac{c}{n}| + |b_n^2 - \frac{1}{4} - \frac{d}{n}|) < \infty$, $c^2 + d^2 \neq 0$ necessitates a comparison with Pollaczek polynomials.

In §2 we construct minimal and dominant solutions and in §3 we consider some properties of the polynomial solutions of the first and second kind.

2. MINIMAL AND DOMINANT SOLUTIONS

The Chebyshev recurrence relation

$$(2.1) \qquad X_{n+1} - zX_n + \frac{1}{4}X_{n-1} = 0$$

has linearly independent solutions

$$(2.2) \qquad X_n^{(\pm)}(z) = (\lambda_{\pm}(z))^n, \ \lambda_{\pm}(z) = (z \pm \sqrt{z^2 - 1})/2, \quad z \neq -1, 1.$$

If $z \in \mathbf{C}\backslash[-1,1]$ we choose the square root branch so that $|\lambda_-(z)/\lambda_+(z)| < 1$. Then (2.1) has a minimal (or subdominant) solution

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$$X_n^{(s)}(z) = X_n^{(-)}(z) = (\lambda_-(z))^n, \quad z \in \mathbb{C}\backslash[-1,1]$$

and a dominant solution

$$X_n^{(d)}(z) = X_n^{(+)}(z) = (\lambda_+(z))^n, \quad z \in \mathbb{C}\backslash[-1,1]$$

(i.e. $X_n^{(s)}(z)/X_n^{(d)}(z) = (\lambda_-(z)/\lambda_+(z))^n \to 0$ as $n \to \infty$).
Note the boundary values of the minimal solution on the cut

$$X_n^{(s)}(x \pm i0) := \lim_{\epsilon \to 0+} X_n^{(s)}(x \pm i\epsilon) = (\lambda_\mp(x))^n,$$
$$\lambda_\mp(x) = (x \mp i\sqrt{1-x^2})/2, \; x \in (-1,1)$$

yields a ratio of solutions which oscillates as $n \to \infty$. Thus there is no minimal solution for $z = x \in (-1,1)$.

To construct the corresponding solutions for the three term recurrence
(2.5)
$Y_{n+1}(z) - (z-a_n)Y_n(z) + b_n^2 Y_{n-1}(z) = 0, \; b_{n+1}^2 \neq 0, \; n \geq 0$

$$\sum^{\infty}(|a_n| + |b_n^2 - \frac{1}{4}|) < \infty$$

we use the basic equation [3], [5]

$$Y_n^{(s)}(z) = X_n^{(s)}(z) - \sum_{m=n+1}^{\infty} G_{nm}(z)[a_m Y_m^{(s)}(z) + (b_m^2 - \frac{1}{4})Y_{m-1}^{(s)}(z)]$$

$$G_{nm}(z) := \frac{X_n^{(s)}(z)X_m^{(d)}(z) - X_n^{(d)}(z)X_m^{(s)}(z)}{W(X_m^{(s)}, X_m^{(d)})}$$

$$W(X_m^{(s)}, X_m^{(d)}) := X_m^{(s)}(z)X_{m+1}^{(d)}(z) - X_m^{(d)}(z)X_{m+1}^{(s)}(z)$$

Note that in the above summation $G_{nm}(z) = 2^{m-n+1}U_{m-n-1}(z)$ where $U_n(z)$ is the Chebyshev polynomial of the second kind.

The Volterra type equation (2.6) can be solved by iteration:

$$Y_n^{(s)}(z) = \sum_{r=0}^{\infty} Y_{n,r}$$

$$Y_{n,0}(z) = X_n^{(s)}(z)$$

(2.7)
$$Y_{n,r+1}(z) = -\sum_{m=n+1}^{\infty} G_{nm}(z)[a_m Y_{m,r}(z) + (b_m^2 - \frac{1}{4})Y_{m-1,r}(z)], \quad r \geq 0.$$

To establish the absolute and uniform convergence of the infinite summations above and justify (2.7) as a solution of (2.6) and the claim of minimality we use the following Gronwall type lemma.

Lemma 1. *Let $\{y_n\}$, $\{c_n\}$, $\{d_n\}$ be bounded positive sequences with $\sum^{\infty} c_n, \sum^{\infty} d_n < \infty$. If $y_n \leq 1 + \sum_{m=n+1}^{\infty}(c_m y_m + d_m y_{m-1})$, $n \geq n_0$, then there exists $N_0 \geq n_0$ such that*

(2.8)
$$y_n \leq \prod_{i=n+1}^{\infty} \frac{(1+c_m)}{(1-d_m)}, \quad n \geq N_0.$$

Proof. Let $y_n \leq M$, $\forall n$. Choose $N_0 < N$ such that $d_{n+1} < 1$, $n \geq N_0$ and $\sum_{k=N+1}^{\infty}(c_k + d_k) < \delta$. Then induction starting from $n = N$ down to $n = N_0$ establishes

$$y_n \leq \frac{(1 + c_{N+1}y_{N+1} + M\delta)}{(1-d_{N+1})} \prod_{k=n+1}^{N} \frac{(1+c_k)}{(1-d_k)},$$

$N_0 \leq n \leq N$. Now let $N \to \infty$, $\delta \to 0$. □

Theorem 2a. *(Asymptotics off the cut). Let $z \in \mathbb{C}\backslash[-1,1]$. Then (2.5) has minimal solution given by (2.7) with*

(2.9)
$$Y_n^{(s)}(z) = (\lambda_-(z))^n(1+\epsilon_n), \quad \epsilon_n = o(1)$$

and dominant solution

(2.10)
$$Y_n^{(d)}(z) = (\lambda_+(z))^n(1+\epsilon_n'), \quad \epsilon_n' = o(1)$$

determined from

(2.11
$$W(Y_n^{(s)}(z), Y_n^{(d)}(z)) = A_{n_0} \prod_{k=n_0}^{n} b_k^2, \quad n \geq n_0$$

Furthermore the estimates (2.9) and (2.10) are uniform for z bounded away from $[-1,1]$.

Proof. Assume (2.6). Let $y_n = |Y_n^{(s)}(z)/X_n^{(s)}(z)|$. Then from (2.6) and the bound
(2.12)
$$|G_{nm}(z)X_m^{(s)}(z)/X_n^{(s)}(z)| = \frac{\left|1 - \left(\frac{\lambda_-(z)}{\lambda_+(z)}\right)^{m-n}\right|}{|\sqrt{z^2-1}|} \leq \frac{2}{|\sqrt{z^2-1}|}$$

we obtain

(2.13)
$$y_n \leq 1 + \sum_{m=n+1}^{\infty}(c_m y_m + d_m y_{m-1})$$

$$c_n = \frac{2|a_n|}{|\sqrt{z^2-1}|}, \; d_n = \frac{4|b_n^2 - \frac{1}{4}||z+\sqrt{z^2-1}|}{|\sqrt{z^2-1}|}$$

Lemma 1 establishes the absolute and uniform convergence of (2.7) for z bounded away from $[-1,1]$ and n sufficiently large since (2.7) is majorized by $|X_n^{(s)}(z)| \prod_{k=n+1}^{\infty} \frac{(1+c_k)}{(1-d_k)}$ for all sufficiently large n. This justifies (2.7) as a solution of (2.6) and the estimate (2.9).

Equation (2.11) is an identity for any pair of solutions of (2.5). For n_0 sufficiently large this first order difference equation for $Y_n^{(d)}$ may be solved in terms of $Y_n^{(s)}$ to obtain

$$Y_{n+1}^{(d)}(z) = A_{n_0} Y_{n+1}^{(s)}(z) \sum_{r=n_0}^{n} \Big(\prod_{k=n_0}^{r} b_k^2 \Big) / Y_r^{(s)}(z) Y_{r+1}^{(s)}(z)$$
$$n \geq n_0 .$$

A choice of normalization constant A_{n_0} and (2.9) yields (2.10). □

We now consider the boundary values of the minimal solution to obtain solutions on the cut.

Theorem 2b. *(Asymptotics on the cut). If $z = x \in (-1,1)$, then (2.5) has linearly independent solutions given by the boundary values of (2.7) with*

(2.14)
$$Y_n^{(s)}(x \pm i0) = (\lambda_{\mp}(x))^n (1+o(1)), \; \lambda_{\pm}(x) = x \pm i\sqrt{1-x^2}$$

Furthermore the estimate (2.14) is uniform for x in a closed subinterval of $(-1,1)$.

Proof. Take the $z = x \pm i\epsilon$, $\epsilon \to 0+$ boundary values of (2.6) and (2.7). The estimates (2.12) and (2.13) continue to hold uniformly for x bounded away from $-1, 1$. Thus (2.14) follows from the boundary values of (2.9). □

3. POLYNOMIAL SOLUTIONS

Let $Q_n(z), P_n(z)$ be polynomial solutions of (2.5) of the first and second kind respectively ($Q_0 = P_1 = 1$, $Q_1 = z - a_0$, $P_2 = z - a_1$). We express these in terms of the solutions obtained in §2. Thus for $z \in \mathbb{C}\backslash[-1,1]$

$$(3.1) \qquad P_n(z) = \frac{Y_0^{(s)}(z)Y_n^{(d)}(z) - Y_0^{(d)}(z)Y_n^{(s)}(z)}{b_0^2 W(Y_{-1}^{(s)}(z), Y_{-1}^{(d)}(z))}$$

$$(3.2) \qquad Q_n(z) = \frac{Y_{-1}^{(s)}(z)Y_n^{(d)}(z) - Y_{-1}^{(d)}(z)Y_n^{(s)}(z)}{W(Y_{-1}^{(s)}(z), Y_{-1}^{(d)}(z))}$$

Note that b_0 is arbitrary and can be chosen equal to 1.

From the asymptotics off the cut and additional assumptions on the properties of the zeros of $Y_{-1}^{(s)}(z)$ we have a Stieltjes type representation for $\lim P_n(z)/Q_n(z)$ as $n \to \infty$.

Theorem 3. *Let $\{z_k\}_{k=1}^N$, $N \leq \infty$ be the zero set of $Y_{-1}^{(s)}(z)$. If $z \notin [-1,1] \cup \{z_k\}$, then*

$$(3.3) \qquad \lim_{n\to\infty} \frac{P_n(z)}{Q_n(z)} = \frac{Y_0^{(s)}(z)}{b_0^2 Y_{-1}^{(s)}(z)}$$

Furthermore if the zeros of $Y_{-1}^{(s)}(z)$ are simple and have no limit points in $(-1,1)$ and $Y_{-1}^{(s)}(x \pm i0) \neq 0$, $x \in (-1,1)$ then

$$(3.4) \qquad \frac{Y_0^{(s)}(z)}{b_0^2 Y_{-1}^{(s)}(z)} = \sum_{k=1}^{N} \frac{R_k}{z - z_k} + \int_{-1}^{1} \frac{d\alpha(x)}{z-x}$$

where $d\alpha(x) = \alpha'(x)dx$ if $x \in (-1,1)$ and

(3.5)
$$\alpha'(x) = \frac{1}{2\pi i}\left(\frac{Y_0^{(s)}(x-i0)}{b_0^2 Y_{-1}^{(s)}(x-i0)} - \frac{Y_0^{(s)}(x+i0)}{b_0^2 Y_{-1}^{(s)}(x+i0)} \right)$$

$$= \frac{\sqrt{1-x^2}}{2\pi b_0^4 (\prod_{k=1}^{\infty} 4b_k^2) Y_{-1}^{(s)}(x-i0) Y_{-1}^{(s)}(x+i0)}, \; x \in (-1,1).$$

Proof. Take the ratio of (3.1) and (3.2) and use the asymptotics for $Y_n^{(s)}, Y_n^{(d)}$ (Theorem 2a) to obtain

$$\frac{P_n(z)}{Q_n(z)} = \frac{Y_0^{(s)}(z)}{b_0^2 Y_{-1}^{(s)}(z)} \left(1 + O\left(\left(\frac{\lambda_-(z)}{\lambda_+(z)} \right)^n \right) \right)$$

(Pincherle's theorem) which establishes (3.3). From Weierstrass' theorem on the uniform convergence of a sequence of analytic functions applied to the iterated solution of (2.6), one has $Y_n^{(s)}(z)$ analytic for $z \in \mathbb{C}\backslash[-1,1]$ and all sufficiently large n. By backward recurrence one has $Y_0^{(s)}(z)$ and $Y_{-1}^{(s)}(z)$ also analytic for $z \in \mathbb{C}\backslash[-1,1]$. Also $P_n(z)/Q_n(z) = \frac{1}{z} + O(\frac{1}{z^2})$. From Cauchy's theorem, the assumption on the zeros of $Y_{-1}^{(s)}(z)$ and $Y_{-1}^{(s)}(x \pm i0) \neq 0$, $x \in (-1,1)$ one obtains (3.4) and (3.5). Finally (3.6) follows from the Wronskian (Casorati) identity

$$W(Y_{-1}^{(s)}(x+i0), Y_{-1}^{(s)}(x-i0))$$
$$= W(Y_n^{(s)}(x+i0), Y_n^{(s)}(x-i0)) / \prod_{k=0}^{n} b_k^2$$

and the asymptotics on the cut (Theorem 2b). Note that $\alpha'(x)$ is continuous since $Y_{-1}^{(s)}(x \pm i0)$ is continuous (uniform convergence of the boundary values of (2.5) for n sufficiently large plus backward recurrence). □

The additional assumptions for the zeros of $Y_{-1}^{(s)}(z)$ are convenient but not essential. The convenience here is in mimicking the corresponding positive definite formulas. For the positive definitive case these assumptions are superfluous since one has some control over the zero set. In particular the zeros of $Y_{-1}^{(s)}(z)$ are then simple and can occur only in $(-\infty,-1] \cup [1,\infty)$ with possible accumulation points only at ± 1. Furthermore the zeros of $Y_{-1}^{(s)}(z)$ and $Y_0^{(s)}(z)$ must interlace.

Corollary 4. *(Orthogonality). With the assumptions of Theorem 3 one has*

(3.7)
$$\sum_{k=1}^{N} Q_n(z_k)Q_m(z_k)R_k + \int_{-1}^{1} Q_n(x)Q_m(x)d\alpha(x) = \Big(\prod_{k=1}^{n} b_k^2\Big)\delta_{nm}\,.$$

Proof. Let $n \geq m$. Then from Cauchy's theorem (as in (3.4) and (3.5) of Theorem 3) one has

(3.8)
$$\frac{Q_m(z)Y_n^{(s)}(z)}{b_0^2 Y_{-1}^{(s)}(z)} = \sum_{k=1}^{N} \frac{Q_n(z_k)Q_m(z_k)}{z-z_k}R_k + \int_{-1}^{1} \frac{Q_n(x)Q_m(x)d\alpha(x)}{z-x}\,,$$

where we have made use of the fact that

(3.9) $\quad Q_n(z_k) = Y_n^{(s)}(z_k)/Y_0^{(s)}(z_k) \quad$ (from (3.2))

and

(3.10)
$$Q_n(x) = \frac{Y_{-1}^{(s)}(x+i0)Y_n^{(s)}(x-i0) - Y_{-1}^{(s)}(x-i0)Y_n^{(s)}(x+i0)}{W\big(Y_{-1}^{(s)}(x+i0), Y_{-1}^{(s)}(x-i0)\big)}$$

Now

$$\frac{Y_n^{(s)}(z)}{Y_{-1}^{(s)}(z)} = \frac{Y_0^{(s)}(z)}{Y_{-1}^{(s)}(z)}\frac{Y_1^{(s)}(z)}{Y_0^{(s)}(z)}\cdots\frac{Y_n^{(s)}(z)}{Y_{n-1}^{(s)}(z)} \underset{z\to\infty}{\sim} \frac{b_0^2}{z}\frac{b_1^2}{z}\cdots\frac{b_n^2}{z} = b_0^2 z^{-n-1}\prod_{k=1}^{n} b_k^2$$

Thus from the $O(1/z)$ behaviour of (3.8) one obtains (3.7). □

Corollary 5. *(Nevai's theorem [6, p. 143]) One has the following polynomial asymptotics:*

$$Q_n(z) = \frac{\big(\lambda_+(z)\big)^n Y_{-1}^{(s)}(z)}{\sqrt{z^2-1}} b_0^2\Big(\prod_{j=1}^{\infty} 4b_j^2\Big)\big(1+o(1)\big),$$
$$z \in \mathbb{C}\backslash[-1,1],\ z \neq z_k\,.$$

(3.12)
$$Q_n(z_k) = -\frac{\big(\lambda_-(z_k)\big)^n Y_{-1}^{(d)}(z_k)}{\sqrt{z_k^2-1}} b_0^2\Big(\prod_{j=1}^{\infty} 4b_j^2\Big)\big(1+o(1)\big),$$
$$z_k \in \mathbb{C}\backslash[-1,1],$$

and with the additional assumptions of Theorem 3

(3.13)
$$Q_n(x) = \Big(\frac{1}{2}\Big)^n \left(\frac{2\big(\prod_{j=1}^{\infty} 4b_j^2\big)}{\pi\sqrt{1-x^2}\,\alpha'(x)}\right)^{\frac{1}{2}} \sin\big(n\phi(x) + \delta(x) + o(1)\big),$$
$$x \in (-1,1),$$

$$\phi(x) = \arccos(x),$$
$$\delta(x) = \arccos\left(\frac{Y_{-1}^{(s)}(x+i0) + Y_{-1}^{(s)}(x-i0)}{2\big(Y_{-1}^{(s)}(x+i0)Y_{-1}^{(s)}(x-i0)\big)^{1/2}}\right)$$

Proof. From (2.9), (2.10), (3.2) and the Wronskian identity one obtains (3.11) and (3.12). From (2.14), (3.5), (3.10) and the Wronskian identity one obtains (3.13). □

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