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CONSTRUCTIVE APPROXIMATION

Weighted Analogues of Capacity, Transfinite Diameter, and Chebyshev Constant

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Abstract. For an arbitrary closed subset E of the complex plane, the notions of logarithmic capacity, transfinite diameter, and Chebyshev constant of E with respect to an admissible weight w on E are introduced. For the w-modified capacity, an electrostatics problem for logarithmic potentials in the presence of an external field is analyzed. This leads to an extremal measure whose support is the "smallest" compact set where the sup norm of weighted polynomials "live." The introduction of a weight w has the advantage that the classical quantities mentioned in the title can be considered for unbounded sets E. Some of the theorems presented are generalizations of the authors' previous results for the case when $E \subset \mathbf{R}$.

1. Introduction

In classical potential theory, three seemingly different numbers are associated with every compact subset of the complex plane $\mathbf{C}$; its logarithmic capacity, transfinite diameter, and Chebyshev constant. While there is no immediately clear connection among these, a fundamental theorem due partly to Fekete and partly to Szegö states that these numbers are, in fact, equal (see [17]). This theorem has a variety of important applications in approximation theory and complex analysis. Various generalizations and analogues of these quantities have also been studied.

In our investigations of weighted polynomial approximation we were led to introduce analogues of the notions of capacity and Chebyshev constant modified with an appropriate weight function so that these quantities can be defined even for unbounded subsets of the real line (see [10]). This enabled us to provide a method unifying certain important aspects of the theory of incomplete polynomials (initiated by Lorentz [4]) and the theory of orthogonal polynomials on the whole real line. Among the more significant applications of this unified approach is the proof (due to D. S. Lubinsky and the authors) of the "Freud conjecture" concerning orthogonal polynomials on $\mathbf{R}$ (see [5]). Various other versions and special cases of the weighted analogues of capacity were also studied independently

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by Leja and his students [13] and by Gonchar, Rakhmanov, and Stahl, among others [1], [11], [14], [16] and have found important applications in the theory of orthogonal polynomials, best rational approximation, and Padé approximation.

In this paper we study the notions of weight-modified logarithmic capacity, Chebyshev constant, and transfinite diameter for subsets of the complex plane, which need not be bounded. We also investigate properties of the analogue of the equilibrium measure and its logarithmic potential. While some of these results are direct extensions of our theorems in [10], there are some significant differences which we point out. The introduction of the modified transfinite diameter was, to the authors' knowledge, first due to Leja and further studied by Siciak [13], but only for the case of nonvanishing weights w on compact sets E. The important new feature of our results is that these restrictions are absent.

In Section 2 we give the precise definitions of the w-modified capacity, Chebyshev constant, and transfinite diameter of a set $E \subset \mathbf{C}$, respectively denoted by $\operatorname{cap}(w, E)$, $\operatorname{cheb}(w, E)$, and $\tau(w, E)$. In Section 3 we study properties of the analogue of the equilibrium measure and also establish some elementary facts about $\operatorname{cap}(w, E)$. In Section 4 we prove some basic polynomial inequalities and use them to establish the connection between $\operatorname{cap}(w, E)$ and $\operatorname{cheb}(w, E)$. Finally, in Section 5, we prove that $\operatorname{cap}(w, E)$ and $\tau(w, E)$ are, in fact, equal.

2. Notation and Definitions

Let $E \subset \mathbf{C}$ be a closed set and let $w: E \to [0, \infty)$ be an arbitrary bounded, Borel measurable function such that, if E is unbounded, $|zw(z)| \to 0$ as $z \to \infty$, $z \in E$. Let $\mathcal{M}(E)$ denote the class of all positive unit Borel measures whose support is contained in E. Also, for any function $f: E \to \mathbf{C}$, we set

$$\|f\|_E := \sup_{z \in E} |f(z)|. \tag{2.1}$$

Further, for each integer $n \geq 0$, we let $\mathcal{P}_n$ denote the class of all algebraic polynomials of degree at most n.

To define the w-modified capacity of E we first define the *weighted logarithmic energy* of $\sigma \in \mathcal{M}(E)$ by

$$I_w(\sigma) := \iint \log \frac{1}{|z - t| w(z) w(t)} \, d\sigma(z) \, d\sigma(t). \tag{2.2}$$

Set

$$V_w := V(w, E) := \inf_{\sigma \in \mathcal{M}(E)} I_w(\sigma). \tag{2.3}$$

Definition 2.1. For any closed set $E \subset \mathbf{C}$, the *w-modified capacity* of E is given by

$$\operatorname{cap}(w, E) := \exp(-V_w). \tag{2.4}$$

We note that in (2.2) we have adopted the more standard notation for the energy which is different from that used in [10].

The w-modified transfinite diameter of E is essentially a discretization of the definition for cap(w, E). Thus, for an integer $n \geq 2$, we set

$$\delta_n(w, E) := \sup_{z_1, \ldots, z_n \in E} \left\{ \prod_{1 \leq i < j \leq n} |(z_i - z_j)w(z_i)w(z_j)| \right\}^{2/n(n-1)}. \tag{2.5}$$

Definition 2.2. For any closed set $E \subset \mathbf{C}$, the *w-modified transfinite diameter* of E is defined by

$$\tau(w, E) := \inf_{n \geq 2} \delta_n(w, E). \tag{2.6}$$

Finally, we define cheb(w, E), the Chebyshev constant of E. To this end, we first consider the "extremal errors"

$$\mathscr{E}_n(w, E) := \inf_{P \in \mathscr{P}_{n-1}} \|[w(z)]^n(z^n - P(z))\|_E, \qquad n = 0, 1, 2, \ldots, \tag{2.7}$$

where $\mathscr{P}_{-1} := \{0\}$. Using a standard compactness argument, it can be shown that there exist monic polynomials $T_n(w, E; z) := z^n + \cdots \in \mathscr{P}_n$, which we call *$w$-Chebyshev polynomials*, such that

$$\|[w(z)]^n T_n(w, E; z)\|_E = \mathscr{E}_n(w, E), \qquad n = 0, 1, 2, \ldots. \tag{2.8}$$

Note that, for integers $n, m \geq 0$,

$$\begin{aligned} \mathscr{E}_{n+m}(w, E) &\leq \|w^n T_n(w, E) \cdot w^m T_m(w, E)\|_E \\ &\leq \|w^n T_n(w, E)\|_E \cdot \|w^m T_m(w, E)\|_E \\ &= \mathscr{E}_n(w, E) \cdot \mathscr{E}_m(w, E). \end{aligned} \tag{2.9}$$

It follows from this (see p. 73 of [17]) that the sequence $\{\mathscr{E}_n(w, E)^{1/n}\}_1^\infty$ converges and so we can give

Definition 2.3. For any closed set $E \subset \mathbf{C}$, the *w-modified Chebyshev constant* of E is defined by

$$\text{cheb}(w, E) := \lim_{n \to \infty} [\mathscr{E}_n(w, E)]^{1/n}. \tag{2.10}$$

When E is a compact subset of $\mathbf{C}$ and $w(z) = 1$ on E, that is, w is the characteristic function χ_E, then cap(w, E), $\tau(w, E)$, and cheb(w, E) are, respectively, the logarithmic capacity, transfinite diameter, and Chebyshev constant of E in the classical sense. In this case we omit w in the notation. Thus $\text{cap}(E) := \text{cap}(\chi_E, E)$, $\tau(E) = \tau(\chi_E, E)$, and $\text{cheb}(E) := \text{cheb}(\chi_E, E)$. As previously mentioned, it is well known that, for any compact set E,

$$\text{cap}(E) = \tau(E) = \text{cheb}(E). \tag{2.11}$$

Under mild conditions on w, we show in Theorem 5.1 that $\text{cap}(w, E) = \tau(w, E)$. In general, however, cap(w, E) and cheb(w, E) are different. For example, when $E = \mathbf{R}$ and $w(x) := \exp(-|x|^\alpha)$, $\alpha > 0$, then it is known that (see [9] and [10])

$$\text{cheb}(w, E) = \tfrac{1}{2}(1/e\lambda_\alpha)^{1/\alpha}, \qquad \lambda_\alpha := \Gamma(\alpha)/2^{\alpha-2}\{\Gamma(\alpha/2)\}^2, \tag{2.12}$$

while

$$\text{cap}(w, E) = \exp(-1/2\alpha)\cdot\text{cheb}(w, E). \tag{2.13}$$

Some further computations of cap(w, E) and cheb(w, E) are given later in the paper.

While the definitions of cap(w, E), $\tau(w, E)$, and cheb(w, E) do not require severe conditions on w, we are able to study the properties of these quantities only for the class of "admissible" weight functions. The following definition, which is a slight generalization of the admissibility criterion in [10], was suggested by V. Totik.

Definition 2.4. Let $E \subset \mathbf{C}$ be closed and let $w: E \to [0, \infty)$. We say that w is an *admissible weight* function if each of following properties holds:

(i) w is upper semicontinuous.
(ii) $E_0 := \{z \in E\colon\ w(z) > 0\}$ has positive (inner logarithmic) capacity,[1] i.e., $\text{cap}(E_0) > 0$.
(iii) If E is unbounded, then $|zw(z)| \to 0$, as $z \to \infty$, $z \in E$.

We conclude this section with some further notation and terminology. We say that a property holds *quasi-everywhere* (q.e.) on a set B if the subset A of B where it does not hold is of capacity zero. If K is compact and $\text{cap}(K) > 0$, then ν_K denotes the unique unit *equilibrium measure* for K with the property that

$$\int_K \log|z - t|\, d\nu_K(t) = \log[\text{cap}(K)] \qquad \text{q.e. on } K \tag{2.14}$$

(see p. 60 of [17]).

Throughout the rest of this paper we assume that $w: E \to [0, \infty)$ is an admissible weight function and we set

$$Q_w(z) := \log[1/w(z)]. \tag{2.15}$$

Finally, if $K \subset E$ is compact and $\text{cap}(K) > 0$, then we define the *F-functional* of K by the formula

$$F(K) := F(w, K) := \log[\text{cap}(K)] - \int Q_w\, d\nu_K. \tag{2.16}$$

This functional was introduced by the authors in [10].

3. Capacity and Equilibrium Measure

For w-modified capacities, we have the following analogue of the classical theorems concerning the equilibrium measure (see Chapter III of [17] and Theorem 2.3 of [10]).

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[1] Since $E_0 \subset E$, we then have $\text{cap}(E) > 0$. Hence admissible weights are only considered for sets having positive capacity.

Theorem 3.1. *Let w be an admissible weight on the closed set E. Then the following properties hold:*

(a) *The quantity V_w defined in* (2.3) *is finite.*
(b) *There exists a unique element $\mu_w := \mu(w, E) \in \mathcal{M}(E)$ such that*

$$I_w(\mu_w) = V_w; \tag{3.1}$$

moreover, μ_w has finite logarithmic energy.
(c) *$\mathcal{S}_w := \mathcal{S}(w, E) := \operatorname{supp}(\mu_w)$ is compact, $\mathcal{S}_w \subset E_0$, and $\operatorname{cap}(\mathcal{S}_w) > 0$, where E_0 is given in Definition* 2.4(ii).
(d) *The inequality*

$$\int \log(1/|z-t|)\, d\mu_w(t) + Q_w(z) \ge V_w - \int Q_w\, d\mu_w \tag{3.2}$$

holds q.e. on E, where $Q_w(z) := \log[1/w(z)]$.
(e) *The inequality*

$$\int \log(1/|z-t|)\, d\mu_w(t) + Q_w(z) \le V_w - \int Q_w\, d\mu_w \tag{3.3}$$

holds for all $z \in \mathcal{S}_w$.

The measure μ_w is called the *extremal measure* associated with w and the important quantity appearing on the right-hand sides of (3.2) and (3.3) is denoted by F_w; that is

$$F_w := V_w - \int Q_w\, d\mu_w. \tag{3.4}$$

Since the proof of Theorem 3.1 is only a minor modification of the proof of Theorem 2.3 in [10], we omit the details.

The set $\mathcal{S}_w$ happens to maximize the F-functional (see (2.16)) over compact subsets of E. This property is often useful in computations. The next result elaborates more on this property. Before we state the theorem, it is useful to introduce certain geometric notions.

If S is any compact subset of $\mathbf{C}$, then $\bar{\mathbf{C}} \setminus S$, where $\bar{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$, consists of at most countably many (disjoint) open, connected components. The unbounded component $D_\infty(S)$ is the component containing ∞, and the boundary $\partial_\infty(S)$ of $D_\infty(S)$ is called the *outer boundary* of S. If $A, B \subset \mathbf{C}$ and $D_\infty(A) \subset D_\infty(B)$, then we say that *$A$ surrounds B*.

Theorem 3.2. *Let $\mathcal{S}_w$ be defined as in Theorem* 3.1(c).

(a) *For every compact set $K \subset E$ with $\operatorname{cap}(K) > 0$,*

$$F(K) \le F(\mathcal{S}_w), \tag{3.5}$$

where F is defined in (2.16).
(b) *If equality holds in* (3.5), *then K surrounds $\mathcal{S}_w$.*

(c) *The quantity* F_w *defined in* (3.4) *satisfies*

$$F_w = -F(\mathscr{S}_w). \tag{3.6}$$

If $E \subset \mathbf{R}$, then in view of part (b) of the above theorem, equality in (3.5) entails $K \supset \mathscr{S}_w$; a fact which we proved in [10]. However, the equilibrium measure of a compact set $K \subset \mathbf{C}$ is also the equilibrium measure of $\partial_\infty(K)$ and $\operatorname{cap}(K) = \operatorname{cap}(\partial_\infty(K))$ (see Theorem III.31 of [17]). Thus we see from the definition of the F-functional that

$$F(K) = F(\partial_\infty(K)).$$

So, in general, equality in (3.5) cannot imply the stronger statement "$\mathscr{S}_w \subset K$."

Proof of Theorem 3.2. Parts (c) and (a) are proved using (3.2) and (3.3) in exactly the same way as in [10].

To prove (b) suppose that $F(K) = F(\mathscr{S}_w)$. Proceeding as on p. 83 of [10], we see that

$$\iint \log|z - t| \, d\nu_K(z) \, d\mu_w(t) = \log[\operatorname{cap}(K)], \tag{3.7}$$

where ν_K is the equilibrium measure of K. It is well known that, for all $t \in \mathbf{C}$,

$$u(t) := \int \log|z - t| \, d\nu_K(z) \geq \log[\operatorname{cap}(K)]. \tag{3.8}$$

Since $\operatorname{supp}(\nu_K) \subset \partial_\infty(K)$, the function $u(t)$ is superharmonic in $D_\infty(K)$. Since $D_\infty(K)$ is an unbounded, open, connected set, (3.8) and the minimum principle for superharmonic functions implies that

$$u(t) > \log[\operatorname{cap}(K)], \qquad t \in D_\infty(K). \tag{3.9}$$

Now suppose that $z_0 \in \mathscr{S}_w \cap D_\infty(K)$. Then there exists an open disk D with $z_0 \in D \subset D_\infty(K)$. Since $z_0 \in \mathscr{S}_w$, we have $\mu_w(D) > 0$. So, from (3.8) and (3.9) it follows that

$$\begin{aligned} \int u(t) \, d\mu_w(t) &= \int_D u(t) \, d\mu_w(t) + \int_{\mathscr{S}_w \setminus D} u(t) \, d\mu_w(t) \\ &> \mu_w(D) \log[\operatorname{cap}(K)] + (1 - \mu_w(D)) \log[\operatorname{cap}(K)] \\ &= \log[\operatorname{cap}(K)]. \end{aligned}$$

But this contradicts (3.7). Hence $D_\infty(K) \subset \overline{\mathbf{C}} \setminus \mathscr{S}_w$. Necessarily, then, $D_\infty(K) \subset D_\infty(\mathscr{S}_w)$. ■

We now pause in our discussion and illustrate how Theorems 3.1 and 3.2 can be used in computations. For the case when $E \subset \mathbf{R}$, our paper [10] contains several examples. Here we concentrate on the case when $E \not\subset \mathbf{R}$.

Example 3.1. Suppose E is a compact set, and $w: E \to [0, \infty)$ is an admissible weight satisfying

$$w(z) \leq 1 \quad \text{for} \quad z \in E \qquad \text{and} \qquad w(z) = 1 \quad \text{for} \quad z \in \partial_\infty(E). \tag{3.10}$$

Let ν_E be the equilibrium measure for E (so that ν_E is supported on $\partial_\infty(E)$) and let $\sigma \in \mathcal{M}(E)$ be arbitrary. Then

$$(3.11)\quad \iint \log\{|z-t|w(z)w(t)\}^{-1}\, d\nu_E(z)\, d\nu_E(t)$$
$$= \iint \log(1/|z-t|)\, d\nu_E(z)\, d\nu_E(t) \le \iint \log(1/|z-t|)\, d\sigma(z)\, d\sigma(t)$$
$$\le \iint \log\{|z-t|w(z)w(t)\}^{-1}\, d\sigma(z)\, d\sigma(t).$$

Thus, by the uniqueness of the solution to the minimal energy problem (see Theorem 3.1(b)), we have $\mu_w = \nu_E$, $\mathcal{S}_w = \operatorname{supp}(\nu_E)$, and $\operatorname{cap}(w, E) = \operatorname{cap}(E)$.

An important special case is when $E := \{z \in \mathbf{C}\colon |z| \le 1\}$ and $w(z) = |z|^\theta$, $\theta > 0$. In contrast to the "real case" of incomplete polynomials, we have shown in Example 3.1 that $\mathcal{S}_w$ does not depend upon θ. Indeed, $\mathcal{S}_w$ is the unit circle.

Example 3.2. Let $E := \mathbf{C}$ and let w be a continuous, radially symmetric admissible function, i.e., $w(z) = w(t)$ if $|z| = |t|$. Suppose further that $xQ_w'(x)$ is increasing and continuous on $[0, \infty)$ and that $\lim_{x\to 0^+} xQ_w'(x) = 0$. Then, arguing as on p. 87 of [10], we see that $\mathcal{S}_w$ is an annulus (possibly with inner radius zero) centered at 0. Thus, in maximizing the F-functional, we need only consider sets of the form $\mathcal{T}_r := \{z \in \mathbf{C}\colon |z| = r\}$. Since

$$(3.12)\qquad F(\mathcal{T}_r) = \log r - Q_w(r),$$

it follows from Theorem 3.2(b) that $\mathcal{S}_w$ is contained in $\{z \in \mathbf{C}\colon |z| \le r_0\}$, where r_0 is the smallest positive solution of the equation

$$(3.13)\qquad r_0 Q_w'(r_0) = 1.$$

In the next section we show that, in fact, $\mathcal{S}_w = \{z \in \mathbf{C}\colon |z| \le r_0\}$. Knowing the support set $\mathcal{S}_w$, it is then easy to determine the extremal measure μ_w. Indeed, when $rQ'(r)$ is absolutely continuous on $[0, \infty)$, define the positive unit measure σ by

$$(3.14)\quad d\sigma = (1/2\pi)\{rQ_w''(r) + Q_w'(r)\}\, dr\, d\theta; \qquad 0 \le r \le r_0, \quad 0 \le \theta \le 2\pi.$$

Then it is a matter of elementary computation to verify that, for all $z \in \mathcal{S}_w$,

$$(3.15)\qquad \int_0^{r_0}\int_0^{2\pi} \log|z - re^{i\theta}|\, d\sigma(r,\theta) = Q_w(z) + \log r_0 - Q_w(r_0)$$
$$= Q_w(z) + F(\mathcal{S}_w).$$

From Theorems 3.1 and 3.2 we know that

$$\int \log|z-t|\, d\mu_w(t) = Q_w(z) + F(\mathcal{S}_w)$$

for quasi-all $z \in \mathcal{S}_w$. Thus, by the principle of domination for potentials (note that both σ and μ_w are unit measures having finite logarithmic energy), we have

$$\int \log|z-t|\,d\sigma(t) = \int \log|z-t|\,d\mu_w(t)$$

for all $z \in \mathbf{C}$. Hence (see Theorem II.25 of [17]) $\mu_w = \sigma$.

In particular, if $w(z) = \exp(-|z|^\lambda)$, $\lambda > 0$, we have

$$\mathcal{S}_w = \{z \in \mathbf{C}: |z| \le (1/\lambda)^{1/\lambda}\},$$

$$d\mu_w = (\lambda^2/2\pi)r^{\lambda-1}\,dr\,d\theta, \qquad 0 \le r \le (1/\lambda)^{1/\lambda}, \quad 0 \le \theta \le 2\pi,$$

and

$$\mathrm{cap}(w, \mathbf{C}) = (1/\lambda)^{1/\lambda}\exp(-3/2\lambda).$$

Thus, when $\lambda = 2$, the extremal measure μ_w is just the normalized two-dimensional Lebesgue measure for the disk $|z| \le \sqrt{2}/2$.

Another interesting example in the complex setting was considered by Luo and Nuttall in their study [7] of the anharmonic oscillator.

We conclude this section by listing some elementary properties of cap(w, E). In so doing we consider the w-modified capacity of a closed subset K of E which we define from the restriction $w|_K$ of w to K. To simplify notation we write cap(w, K) instead of cap($w|_K$, K) and $\mu(w, K)$ instead of $\mu(w|_K, K)$. Note, however, that although $w|_K$ satisfies properties (i) and (iii) of Definition 2.4 (with E replaced by K), property (ii) may be violated. If the latter occurs, we set cap(w, K) = 0.

Theorem 3.3. *Let $E \subset \mathbf{C}$ be closed and let w: $E \to [0, \infty)$ be an admissible weight function.*

(a) *If $K \subset E$ is closed, then* cap(w, K) = 0 *if and only if w vanishes q.e. on K.*

(b) *If K_1, K_2 are closed subsets of E and $K_1 \subset K_2$, then*

$$\mathrm{cap}(w, K_1) \le \mathrm{cap}(w, K_2).$$

(c) *If $\{w_n\}_1^\infty$ is a sequence of admissible weights on E satisfying $w_{n+1} \le w_n$, $n = 1, 2, \ldots$, and $w = \lim_{n\to\infty} w_n$, then*

$$\mathrm{cap}(w, E) = \lim_{n\to\infty} \mathrm{cap}(w_n, E) \tag{3.16}$$

and, in the weak-star topology,

$$\lim_{n\to\infty} \mu(w_n, E) = \mu(w, E). \tag{3.17}$$

(d) *If $K_n \subset E$ is closed, $K_n \supset K_{n+1}$, $n = 1, 2, \ldots$, and $K := \bigcap_{n=1}^\infty K_n$, then*

$$\mathrm{cap}(w, K) = \lim_{n\to\infty} \mathrm{cap}(w, K_n) = \inf_{n \ge 1} \mathrm{cap}(w, K_n).$$

Moreover, if cap(w, K) > 0, *then, in the weak-star topology,*

$$\lim_{n\to\infty} \mu(w, K_n) = \mu(w, K).$$

(e) *If* $\{w_n\}$ *is a sequence of admissible weights on E satisfying* $w_n \le w_{n+1}$, $n = 1, 2, \ldots,$ *and* $v(z) := \sup_n w_n(z)$ *is admissible on E, then*

$$\operatorname{cap}(v, E) = \lim_{n\to\infty} \operatorname{cap}(w_n, E) = \sup_{n\ge 1} \operatorname{cap}(w_n, E) \tag{3.18}$$

and, in the weak-star topology,

$$\lim_{n\to\infty} \mu(w_n, E) = \mu(v, E). \tag{3.19}$$

(f) *Let M be a positive constant such that*

$$w(z)w(t)|z - t| \le M, \qquad z, t \in E.$$

If C_n, $n = 1, 2, \ldots,$ *are closed subsets of E and* $C := \bigcup_{n=1}^{\infty} C_n$ *is closed, then*

$$[\log(M/\operatorname{cap}(w, C))]^{-1} \le \sum_{n=1}^{\infty} [\log(M/\operatorname{cap}(w, C_n))]^{-1}. \tag{3.20}$$

Proof. Statements (a) and (b) are clear from the definitions, so we proceed with the proof of (c).

It is clear from the definition of weighted capacity that

$$\operatorname{cap}(w, E) \le \operatorname{cap}(w_{n+1}, E) \le \operatorname{cap}(w_n, E),$$

and so

$$\lim_{n\to\infty} \operatorname{cap}(w_n, E) \ge \operatorname{cap}(w, E). \tag{3.21}$$

Let $\mu_n := \mu(w_n, E)$. A careful examination of the proof of Theorem 3.1 shows that there is a compact set $K \subset E$, which depends only on w_1, such that each μ_n is supported on K, i.e., $\mathcal{S}(w_n, E) \subset K$, $n = 1, 2, \ldots$. Hence every subsequence of $\{\mu_n\}$ has a weak-star convergent subsequence and we may assume that the whole sequence $\{\mu_n\}$ converges to some $\sigma \in \mathcal{M}(K)$.

Since each w_n is upper semicontinuous on E, there exists a decreasing sequence $\{v_i^{(n)}\}_{i=1}^{\infty}$ of positive continuous functions that converges to w_n on E. Set

$$v_m := \min_{1\le i,\, j\le m} v_i^{(j)}.$$

Then $\{v_m\}$ is a decreasing sequence of positive continuous functions that converges to w. Let

$$G(z, t) := \log\{|z - t| w(z)w(t)\}^{-1}, \tag{3.22}$$

$$G_m(z, t) := \log\{|z - t| v_m(z)v_m(t)\}^{-1}, \tag{3.23}$$

and for $R > 0$ put

$$G_{R,m}(z, t) := \min(R, G_m(z, t)). \tag{3.24}$$

With the assumption that $\mu_n \xrightarrow{*} \sigma$, we get from the monotone convergence theorem that

$$(3.25)\qquad \begin{aligned} V(w, E) &\le \iint G(z, t)\, d\sigma(z)\, d\sigma(t) \\ &= \lim_{R\to\infty} \lim_{m\to\infty} \iint G_{R,m}(z, t)\, d\sigma(z)\, d\sigma(t) \\ &= \lim_{R\to\infty} \lim_{m\to\infty} \lim_{n\to\infty} \iint G_{R,m}(z, t)\, d\mu_n(z)\, d\mu_n(t) \\ &\le \lim_{R\to\infty} \liminf_{n\to\infty} \iint G_{R,n}(z, t)\, d\mu_n(z)\, d\mu_n(t) \\ &\le \liminf_{n\to\infty} \iint G_n(z, t)\, d\mu_n(z)\, d\mu_n(t) \\ &\le \liminf_{n\to\infty} \iint \log\{|z - t| w_n(z) w_n(t)\}^{-1}\, d\mu_n(z)\, d\mu_n(t) \\ &= \lim_{n\to\infty} V(w_n, E). \end{aligned}$$

Thus $\operatorname{cap}(w, E) \ge \lim_{n\to\infty} \operatorname{cap}(w_n, E)$ and, together with (3.21), this proves (3.16). The above argument also shows that $I_w(\sigma) = V(w, E)$ and so, by Theorem 3.1(b), we have $\sigma = \mu(w, E)$. Since σ is an arbitrary limit measure of $\{\mu_n\}$, (3.17) follows.

Part (d) is an easy consequence of part (c); when $\operatorname{cap}(w, K) > 0$, we set $w_n(z) := w(z)$ for $z \in K_n$, $w_n(z) := 0$ for $z \in E \backslash K_n$.

To prove part (e) we first observe that, from the definition of the weighted capacities, we have

$$(3.26)\qquad \sup_n \operatorname{cap}(w_n, E) \le \operatorname{cap}(v, E).$$

With $\mu_v := \mu(v, E)$, the monotone convergence theorem gives

$$(3.27)\qquad \begin{aligned} \log[\operatorname{cap}(v, E)] &= \iint \log\{|z - t| v(z) v(t)\}\, d\mu_v(z)\, d\mu_v(t) \\ &= \sup_n \iint \log\{|z - t| w_n(z) w_n(t)\}\, d\mu_v(z)\, d\mu_v(t) \\ &\le \sup_n \{\log[\operatorname{cap}(w_n, E)]\}. \end{aligned}$$

Along with (3.26), this proves (3.18). To prove (3.19), we note that each $\mu(w_n, E)$ is supported on a fixed compact set K such that if $(z, t) \notin K \times K$, then

$$\log\{v(z) v(t) |z - t|\} < -V(w_1, E) - 1.$$

The rest of the proof is similar to that of (3.17).

The proof of part (f) is very similar to that of Theorem III.17 in [17];

therefore we merely sketch it. Let G be defined as in (3.22). Then

$$G(z, t) + \log M \geq 0, \qquad z, t \in E.$$

Also note that $M \geq \text{cap}(w, C)$ is immediate from the definition of weighted capacity. Since (3.20) is trivial if $\text{cap}(w, C) = 0$, we assume hereafter that this weighted capacity is positive. If $\mu := \mu(w, C)$ and $\mathcal{S} := \mathcal{S}(w, C)$, then, in view of Theorem 3.1(d), (e), we get, for quasi-all $z \in \mathcal{S}$,

$$\begin{aligned} \log(M/\text{cap}(w, C)) &= \log M + V(w, C) \\ &= \int [G(z, t) + \log M]\, d\mu(t) \\ &\geq \int_{C_n} [G(z, t) + \log M]\, d\mu(t). \end{aligned} \tag{3.28}$$

Since μ has finite logarithmic energy, this holds μ-a.e. on $\mathcal{S} \cap C_n$. Thus, if $\mu(\mathcal{S} \cap C_n) = \mu(C_n) \neq 0$, then integrating (3.28) with respect to $\mu|_{C_n}$ yields

$$\begin{aligned} \log(M/\text{cap}(w, C)) &\geq \mu(C_n)[V(w, C_n) + \log M] \\ &= \mu(C_n) \log(M/\text{cap}(w, C_n)). \end{aligned} \tag{3.29}$$

Since $\sum_n \mu(C_n) \geq \mu(C) = 1$, it is easy to derive (3.20) from (3.29). ■

4. Bounds for Weighted Polynomials

In this section we state a fundamental inequality concerning weighted polynomials which implies, roughly speaking, that the sup norm of such quantities "lives" on the support of the extremal measure introduced in Theorem 3.1. We then use the techniques in [10] and [15] to establish the connection between the w-modified Chebyshev constant and capacity. There are no new techniques presented here, but the reader will find that the examples that we present are striking in their simplicity in contrast to the corresponding examples in the real case.

In what follows, E is a closed subset of $\mathbf{C}$, $w: E \to [0, \infty)$ is admissible, μ_w denotes $\mu(w, E)$, $\mathcal{S}_w$ denotes $\mathcal{S}(w, E)$, and F_w is given by (3.4) (see also (3.6)).

Theorem 4.1. *For any positive integer n, if $P_n \in \mathcal{P}_n$ and*

$$|[w(z)]^n P_n(z)| \leq M \qquad \textit{q.e. on } \mathcal{S}_w, \tag{4.1}$$

then

$$|P_n(z)| \leq M \exp\left(n\left[\int \log|z - t|\, d\mu_w(t) + F_w\right]\right), \qquad \forall z \in \mathbf{C}. \tag{4.2}$$

Furthermore, (4.2) *implies*

$$|[w(z)]^n P_n(z)| \leq M \qquad \textit{q.e. on } E. \tag{4.3}$$

Theorem 4.1 follows from Theorem 3.1(d), (e) and the "principle of domination" for measures with finite energy [3, Theorem 1.27]. The details are exactly the same as in [9] and [10] and hence are omitted.

Remark. Inequality (4.3) implies that, if we ignore sets of capacity zero, the sup norm of $w^n P_n$ taken over E is the same as its sup norm taken over $\mathcal{S}_w$. To be more precise, let $\|f\|_K^*$ denote the smallest number that is an upper bound for $|f|$ q.e. on K. Then Theorem 4.1 implies the following.

Corollary 4.2. *For any positive integer n and any* $P_n \in \mathcal{P}_n$,

$$\|w^n P_n\|_E^* = \|w^n P_n\|_{\mathcal{S}_w}^*. \tag{4.4}$$

Furthermore, if w is continuous on E and E is of positive capacity at each of its points, i.e., $\operatorname{cap}(\{\zeta \in E\colon |z-\zeta| < \delta\}) > 0$ *for every* $z \in E$ *and every* $\delta > 0$, *then* $\|w^n P_n\|_E^* = \|w^n P_n\|_E$ *and, consequently,*

$$\|w^n P_n\|_E = \|w^n P_n\|_{\mathcal{S}_w}. \tag{4.5}$$

As we show in Theorem 4.6, under slightly more restrictive assumptions on E and w, Theorem 4.1 is sharp in the sense that there is a sequence of polynomials $\{P_n^* \in \mathcal{P}_n\}$ such that

$$\lim_{n\to\infty} \|w^n P_n^*\|_E^{1/n} = 1$$

and

$$\lim_{n\to\infty} |P_n^*(z)|^{1/n} = \exp\left[\int \log|z-t|\, d\mu_w(t) + F_w\right], \qquad z \in \mathbf{C}\backslash\mathcal{S}_w. \tag{4.6}$$

Furthermore, Corollary 4.2 is sharp in the sense that $\mathcal{S}_w$ is the "smallest" set that works in (4.5). Before proving these facts, we establish the connection between cheb(w, E) and cap(w, E).

Theorem 4.3. *The w-modified Chebyshev constant satisfies*

$$\operatorname{cheb}(w, E) = \exp\left(\int Q_w \, d\mu_w\right) \operatorname{cap}(w, E) = \exp(-F_w). \tag{4.7}$$

The proof of this theorem is essentially the same as given by Stahl [15] and the authors [10] for the real case. In the following lemma we summarize certain technical facts that are needed in the proof of Theorem 4.3.

Lemma 4.4.

(a) [17, Theorem III.27] *Let* $K \subset \mathbf{C}$ *be a compact set with* $\operatorname{cap}(K) = 0$. *Then there exists a* $\tau \in \mathcal{M}(K)$ *such that*

$$\int \log|z-t|\, d\tau(t) = -\infty, \qquad \forall z \in K. \tag{4.8}$$

(b) *Let $K \subset \mathbf{C}$ be compact, $\sigma \in \mathcal{M}(K)$. Then there is a triangular scheme of points $\{t_{k,1}, \dots, t_{k,n_k}\}$ in K, not necessarily distinct, with the following property: Let $\nu_k \in \mathcal{M}(K)$ be the discrete unit measure defined by*

$$\nu_k(B) := n_k^{-1} |\{j\colon t_{k,j} \in B\}|$$

for any Borel set $B \subset \mathbf{C}$. Then, in the weak-star topology,

$$\lim_{k \to \infty} \nu_k = \sigma.$$

(c) *The sequence $\{\mathcal{E}_n^{1/n}(w, E)\}$ converges (see* (2.7) *and* (2.10)).

We have already proved part (c) in Section 2. The proof of part (b) is a standard argument using rectangular grids.

Proof of Theorem 4.3. The right-hand equality in (4.7) is immediate from (3.4) and (2.4). Thus we need only show that $\operatorname{cheb}(w, E) = \exp(-F_w)$.

Let $\varepsilon > 0$ and set

$$E_\varepsilon := \left\{z \in E\colon \int \log|z - t|\, d\mu_w(t) \geq Q_w(z) - F_w + \varepsilon\right\}. \tag{4.9}$$

In view of Theorem 3.1(d) and (3.5), $\operatorname{cap}(E_\varepsilon) = 0$. Also, since $\operatorname{supp}(\mu_w)$ is compact and w is admissible, we have, in the case when E is unbounded,

$$\int \log|z - t|\, d\mu_w(t) - Q_w(z) \to -\infty \qquad \text{as} \quad z \to \infty, \quad z \in E.$$

Thus, by the upper semicontinuity of the function in this last display, we see that E_ε is a compact set. Lemma 4.4(a) then yields a measure $\tau_\varepsilon \in \mathcal{M}(E_\varepsilon)$ such that

$$\int \log|z - t|\, d\tau_\varepsilon(t) = -\infty, \qquad \forall z \in E_\varepsilon. \tag{4.10}$$

Since w is admissible,

$$\sup_{z \in E}\left\{\int \log|z - t|\, d\tau_\varepsilon(t) - Q_w(z)\right\} =: c_\varepsilon \in \mathbf{R}. \tag{4.11}$$

Let $\alpha \in (0, 1)$ be arbitrary, and let $\sigma_{\alpha,\varepsilon} := (1 - \alpha)\mu_w + \alpha\tau_\varepsilon \in \mathcal{M}(E_\varepsilon \cup \mathcal{S}_w)$. Then

$$\int \log|z - t|\, d\sigma_{\alpha,\varepsilon}(t) \leq Q_w(z) - (1 - \alpha)F_w + \alpha c_\varepsilon + (1 - \alpha)\varepsilon, \qquad \forall z \in E. \tag{4.12}$$

Next, by Lemma 4.4(b), the measure $\sigma_{\alpha,\varepsilon}$ is the weak-star limit of a sequence of measures $\nu_k = \nu_{k,\alpha,\varepsilon}$ associated with some triangular scheme $\{t_{k,j}\colon j = 1, \dots, n_k,\ k = 1, 2, \dots\} \subset E_\varepsilon \cup \mathcal{S}_w$. Let

$$P_k(z) := (z - t_{k,1}) \cdots (z - t_{k,n_k}), \qquad k = 1, 2, \dots. \tag{4.13}$$

Since w is admissible, there is a fixed compact set $K \subset E$ and points $\{\zeta_k\} \subset K$, such that

$$\|w^{n_k} P_k\|_E = \|w^{n_k} P_k\|_K = |w^{n_k}(\zeta_k) P_k(\zeta_k)|. \tag{4.14}$$

By taking a further subsequence if necessary, we may assume that $\zeta_k \to \zeta_0 \in K$. Then it follows from the "principle of descent" [3, Theorem 1.3], the lower semicontinuity of Q_w, and (4.12) that

$$(4.15)\quad \limsup_{k\to\infty}(1/n_k)\log \mathscr{E}_{n_k}(w, E) \le \limsup_{k\to\infty}(1/n_k)\log\|w^{n_k}P_k\|_K$$
$$= \limsup_{k\to\infty}\left\{\int \log|\zeta_k - t|\, d\nu_k(t) - Q_w(\zeta_k)\right\}$$
$$\le \int \log|\zeta_0 - t|\, d\sigma_{\alpha,\varepsilon}(t) - Q_w(\zeta_0)$$
$$\le -(1-\alpha)F_w + \alpha c_\varepsilon + (1-\alpha)\varepsilon.$$

But $\alpha \in (0, 1)$ and $\varepsilon > 0$ are arbitrary; thus on first letting $\alpha \to 0$ and then $\varepsilon \to 0$, we get

$$\limsup_{k\to\infty}(1/n_k)\log \mathscr{E}_{n_k}(w, E) \le -F_w.$$

Since $\{(1/n)\log \mathscr{E}_n(w, E)\}$ is a convergent sequence, we have proved that

$$(4.16)\qquad \operatorname{cheb}(w, E) \le \exp(-F_w).$$

Next, we apply Theorem 4.1 to $T_n(w, E; z)$ (see (2.8)) to get

$$(4.17)\quad |T_n(w, E; z)| \le \mathscr{E}_n(w, E)\exp\left[n\int \log|z - t|\, d\mu_w(t) + nF_w\right],\qquad \forall z \in \mathbf{C}.$$

Letting $z \to \infty$ in the last inequality yields

$$(4.18)\qquad \mathscr{E}_n(w, E) \ge \exp(-nF_w),$$

and so $\operatorname{cheb}(w, E) \ge \exp(-F_w)$. Together with (4.16), this completes the proof. ■

For future reference we state the result of inequality (4.18) as

Corollary 4.5. *For every* $n = 1, 2, \ldots,$

$$\mathscr{E}_n(w, E) \ge \exp(-nF_w).$$

Example 4.1. Suppose E and w are as in Example 3.1 and let χ_E be the characteristic function for E. Then it is easy to see that

$$(4.19)\qquad T_n(w, E; z) = T_n(\chi_E, E; z);\qquad \mathscr{E}_n(w, E) = \mathscr{E}_n(\chi_E, E),$$

i.e., the weighted case reduces to the classical case. An extensive literature then exists describing the asymptotic behavior of $\mathscr{E}_n(\chi_E, E)$ (see [18]). In particular, if $\partial_\infty(E)$ is a smooth Jordan curve, then

$$(4.20)\qquad \lim_{n\to\infty}\mathscr{E}_n(w, E)/\exp(-nF_w) = \lim_{n\to\infty}\mathscr{E}_n(\chi_E, E)/[\operatorname{cap}(E)]^n = 1.$$

Example 4.2. Suppose w and E are as in Example 3.2. Let r_n be defined by

$$r_n w^{n+1}(r_n) = \max_{z \in \mathbf{C}} |zw^{n+1}(z)|, \qquad r_n > 0. \tag{4.21}$$

Then clearly

$$r_n Q'_w(r_n) = 1/(n+1). \tag{4.22}$$

Since $xQ'(x)$ is an increasing function on $(0, \infty)$ and tends to 0 as $x \to 0$, then $r_n \to 0$. In view of (4.5) in Corollary 4.2, we see that $r_n \in \mathcal{S}(w, \mathbf{C}) = \mathcal{S}_w$, $n = 0, 1, 2, \dots$. As we proved in Section 3, the set $\mathcal{S}_w$ is an annular region contained in $\{z \in \mathbf{C}: |z| \le r_0\}$, and having an outer radius equal to r_0. Thus, $\mathcal{S}_w = \{z \in \mathbf{C}: |z| \le r_0\}$. Recall that in Example 3.2 we showed

$$\exp(-F_w) = \exp(F(\mathcal{S}_w)) = r_0 w(r_0) = \max_{z \in \mathbf{C}} |zw(z)|. \tag{4.23}$$

It follows from Corollary 4.5 that

$$\exp(-nF_w) \le \mathscr{E}_n(w, \mathbf{C}) \le \max_{z \in \mathbf{C}} |z^n w^n(z)| = \exp(-nF_w).$$

Thus,

$$T_n(w, \mathbf{C}; z) = z^n, \qquad \mathscr{E}_n(w, \mathbf{C}) = \exp(-nF_w), \qquad n = 0, 1, \dots. \tag{4.24}$$

In contrast, the case when $E = \mathbf{R}$ and $Q(x) = |x|^\alpha$, $\alpha > 1$, is significantly more difficult (see [9], [5], and [6]).

We conclude this section by proving that, with suitable assumptions on E and w, Theorem 4.1 is sharp in the sense explained earlier.

Theorem 4.6. *Suppose that w is continuous on E and E is of positive capacity at each of its points.*

(a) *There exists a sequence of integers $m_1 < m_2 < \cdots$ and polynomials $P_k^* \in \mathscr{P}_{m_k}$ such that*

$$\|w^{m_k} P_k^*\|_E = 1, \qquad k = 1, 2, \dots, \tag{4.25}$$

and

$$\lim_{k \to \infty} (1/m_k) \log|P_k^*(z)| = \int \log|z - t| \, d\mu_w(t) + F_w, \qquad \forall z \in \mathbf{C} \backslash \mathcal{S}_w. \tag{4.26}$$

(b) *Let $B \subset E$ be a compact set with the property that, for every integer $n \ge 1$ and $P_n \in \mathscr{P}_n$,*

$$\|w^n P_n\|_E = \|w^n P_n\|_B. \tag{4.27}$$

Then B surrounds $\mathcal{S}_w$.

Proof of Theorem 4.6. (a) We first remark that the w-Chebyshev polynomials $T_n(w, E; z)$, suitably normalized, need not satisfy (4.26) since these polynomials may have zeros that accumulate outside $\mathcal{S}_w$. To construct the desired sequence we modify the proof of Theorem 4.3 as follows.

In the definitions of E_ε and c_ε in (4.9) and (4.11), replace the set E by the set $\mathscr{S}_w$ so that $E_\varepsilon \subset \mathscr{S}_w$. Then (4.12) holds for all $z \in \mathscr{S}_w$, $\sigma_{\alpha,\varepsilon} \in \mathscr{M}(\mathscr{S}_w)$, and the zeros $t_{k,j}$ of $P_k = P_{k,\alpha,\varepsilon} \in \mathscr{P}_{n_k}$ in (4.13) all lie in $\mathscr{S}_w$. Furthermore, from (4.5) of Corollary 4.2,

$$\|w^{n_k} P_{k,\alpha,\varepsilon}\|_E = \|w^{n_k} P_{k,\alpha,\varepsilon}\|_{\mathscr{S}_w}, \tag{4.28}$$

and inequality (4.15) yields

$$\limsup_{k\to\infty} (1/n_k) \log\|w^{n_k} P_{k,\alpha,\varepsilon}\|_E \le -(1-\alpha)F_w + \alpha c_\varepsilon + (1-\alpha)\varepsilon. \tag{4.29}$$

Next, we note that, for fixed ε and α, the zero measures $\nu_{k,\alpha,\varepsilon}$ associated with the $P_{k,\alpha,\varepsilon}$ converge to $\sigma_{\alpha,\varepsilon}$ as $k \to \infty$, and $\sigma_{\alpha,\varepsilon}$ converges to μ_w as $\alpha \to 0$. Hence from (4.29) we can select a sequence of integers $\{m_k\}$, and constants $\alpha_k \to 0$, $\varepsilon_k \to 0$ such that

$$\limsup_{k\to\infty} (1/m_k) \log\|w_k^{m_k} P_{k,\alpha_k,\varepsilon_k}\|_E \le \exp(-F_w), \tag{4.30}$$

where $P_{k,\alpha_k,\varepsilon_k} \in \mathscr{P}_{m_k}$ and the zero measures of $P_{k,\alpha_k,\varepsilon_k}$ converge to μ_w as $k \to \infty$. From (4.30) and Corollary 4.5 we get

$$\lim_{k\to\infty} \|w^{m_k} P_{k,\alpha_k,\varepsilon_k}\|_E^{1/m_k} = \exp(-F_w).$$

Setting

$$P_k^*(z) := P_{k,\alpha_k,\varepsilon_k}(z)/\|w^{m_k} P_{k,\alpha_k,\varepsilon_k}\|_E,$$

it is easy to see that P_k^* satisfies (4.25) and (4.26).

(b) The property (4.27) implies that w is admissible on B and that $\operatorname{cheb}(w, B) = \operatorname{cheb}(w, E)$. Using (4.7), we get $F(\hat{B}) = F(\mathscr{S}(w, E))$ for some subset $\hat{B}$ of B; namely $\hat{B} = \mathscr{S}(w, B)$. Then, by Theorem 3.2(b), $\hat{B}$ surrounds $\mathscr{S}_w$. Hence B surrounds $\mathscr{S}_w$. ∎

5. Transfinite Diameter

The main result of this section is the following.

Theorem 5.1. *Let $E \subset \mathbf{C}$ be closed and let $w: E \to [0, \infty)$ be an admissible weight. With $\delta_n := \delta_n(w, E)$ defined as in (2.5), the w-modified transfinite diameter satisfies*

$$\tau(w, E) = \lim_{n\to\infty} \delta_n = \operatorname{cap}(w, E). \tag{5.1}$$

For the proof of this result we need the following lemmas.

Lemma 5.2. *The sequence $\{\delta_n\}_{n=2}^{\infty}$ is decreasing.*

Proof. Let $z_i \in E$ be chosen so that

$$\delta_{n+1}^{n(n+1)/2} = \prod_{1\le i<j\le n+1} [|z_i - z_j| w(z_i) w(z_j)], \tag{5.2}$$

and set

$$D(z, t) := |z - t| w(z) w(t). \tag{5.3}$$

Then, for any fixed k, $1 \le k \le n + 1$,

$$\delta_{n+1}^{n(n+1)/2} \le \delta_n^{n(n+1)/2} \prod_{\substack{1 \le i \le n+1 \\ i \ne k}} D(z_i, z_k). \tag{5.4}$$

Multiplying the inequalities (5.4) for $k = 1, \ldots, n + 1$ and using (5.2) we get

$$\delta_{n+1}^{n(n+1)^2/2} \le \delta_{n+1}^{n(n+1)} \delta_n^{n(n^2-1)/2}, \tag{5.5}$$

which implies that $\delta_n \ge \delta_{n+1}$. ∎

Lemma 5.3. *Let* $\mu_w = \mu(w, E)$, $\mathcal{S}_w = \mathcal{S}(w, E)$, *and* F_w *be given by* (3.4). *Then the set*

$$\mathcal{S}_w^* := \left\{ z \in E : \int \log|z - t| \, d\mu_w(t) - Q_w(z) + F_w \ge 0 \right\} \tag{5.6}$$

is compact. Furthermore, for any positive integer n and any $P_n \in \mathcal{P}_n$,

$$\|w^n P_n\|_E = \|w^n P_n\|_{\mathcal{S}_w^*}. \tag{5.7}$$

Proof. That $\mathcal{S}_w^*$ is compact follows from the facts that $\int \log|z - t| \, d\mu_w(t)$ and $-Q_w(z)$ are upper semicontinuous and, in the case when E is unbounded,

$$\int \log|z - t| \, d\mu_w(t) - Q_w(z) \to -\infty \qquad \text{as} \quad z \to \infty, \quad z \in E$$

(recall Definition 2.4(iii)).

To establish (5.7), we note that from Theorem 4.1 we have, for any $P_n \in \mathcal{P}_n$,

$$|[w(z)]^n P_n(z)| \le \|w^n P_n\|_E \exp\left(n \left[\int \log|z - t| \, d\mu_w(t) - Q_w(z) + F_w \right] \right)$$

for $z \in E$. Hence if $z \in E \backslash \mathcal{S}_w^*$ and $P_n \not\equiv 0$,

$$|[w(z)]^n P_n(z)| < \|w^n P_n\|_E,$$

which implies (5.7). ∎

We now turn to the proof of Theorem 5.1 which follows by applying the classical argument to a suitable extension of the weight function w.

Proof of Theorem 5.1. If $z_1, \ldots, z_n \in E$, then

$$(n(n-1)/2) \log(1/\delta_n) \le \sum_{1 \le i < j \le n} \log\{|z_i - z_j| w(z_i) w(z_j)\}^{-1}. \tag{5.8}$$

If $\sigma \in \mathcal{M}(E)$ is arbitrary, then integrating both sides of (5.8) with respect to $d\sigma(z_1) \, d\sigma(z_2) \cdots d\sigma(z_n)$ we get (see Section 16.4 of [2])

$$\log(1/\delta_n) \le I_w(\sigma).$$

Hence, $\log(1/\delta_n) \le V(w, E)$, and so

$$\tau(w, E) \ge \text{cap}(w, E). \tag{5.9}$$

To prove the reverse inequality, we first note that (5.7) implies that

$$\delta_n(w, E) = \delta_n(w, \mathscr{S}_w^*), \qquad \tau(w, E) = \tau(w, \mathscr{S}_w^*).$$

Also, from Theorem 3.1(e), we see that $\mathscr{S}_w \subset \mathscr{S}_w^*$ and so it is clear from the relevant definitions that

$$\text{cap}(w, E) = \text{cap}(w, \mathscr{S}_w^*).$$

Thus, we can hereafter assume that $E = \mathscr{S}_w^*$ and, in particular, that E is compact and $w > 0$ on E (see Lemma 5.3).

Let us first assume that w is continuous on the compact set E. Using the Tietze extension theorem (see Theorem 20.4 of [12]), we can extend w to a positive continuous function on the set

$$\hat{E} := \{z \in \mathbf{C}\colon \text{dist}(z, E) \le 1\}. \tag{5.10}$$

Such an extension, which we continue to denote by w, is clearly admissible on $\hat{E}$. Next, for $n = 1, 2, \ldots$, let

$$E_n := \{z \in \mathbf{C}\colon \text{dist}(z, E) \le (\pi n)^{-1/2}\}, \tag{5.11}$$

so that each E_n is compact, $E_n \supset E_{n+1}$ and $E = \bigcap E_n$. Also, for $0 < \delta \le 1$, set

$$\omega(Q_w, \delta) := \max\{|Q_w(z) - Q_w(\zeta)|\colon z, \zeta \in \hat{E}, |z - \zeta| \le \delta\}, \tag{5.12}$$

and note that Q_w is real-valued and continuous on $\hat{E}$ since w is positive and continuous there. Thus $\omega(Q_w, \delta) \to 0$ as $\delta \to 0$.

Next, we choose $z_1, \ldots, z_n \in E$ such that

$$\begin{aligned} \log \delta_n &= (2/n(n-1)) \sum_{1 \le i < j \le n} \log[|z_i - z_j| w(z_i) w(z_j)] \\ &= (2/n(n-1)) \sum_{1 \le i < j \le n} \log|z_i - z_j| - (2/n) \sum_{k=1}^{n} Q_w(z_k). \end{aligned} \tag{5.13}$$

Let

$$\begin{aligned} &\Delta_i := \{z \in \mathbf{C}\colon |z - z_i| \le (\pi n)^{-1/2}\}, \qquad i = 1, 2, \ldots, n, \\ &\rho_i(z) := \begin{cases} 1 & \text{if } z \in \Delta_i, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{5.14}$$

Denoting the two-dimensional Lebesgue measure of a Borel set B by $m_2(B)$, we define a measure $\sigma_n \in \mathscr{M}(E_n)$ by the formula

$$\sigma_n(B) = \int_B \rho \, dm_2, \tag{5.15}$$

where

$$\rho(z) := \sum_{i=1}^{n} \rho_i(z). \tag{5.16}$$

Since E is compact, we can now use the classical argument in Section III.5, p. 75, of [17] to conclude that

$$\iint \log|z-t|\, d\sigma_n(z)\, d\sigma_n(t) \geq (2/n(n-1)) \sum_{1\leq i<j\leq n} \log|z_i - z_j| + \varepsilon_{1,n}, \tag{5.17}$$

where

$$\varepsilon_{1,n} = O(\log n/n). \tag{5.18}$$

Moreover,

$$\int Q_w\, d\sigma_n = \sum_{i=1}^{n} \int_{\Delta_i} Q_w\, dm_2 = (1/n) \sum_{i=1}^{n} Q_w(z_i) + \varepsilon_{2,n}, \tag{5.19}$$

where

$$\varepsilon_{2,n} = O(\omega(Q_w, (\pi n)^{-1/2})). \tag{5.20}$$

The formulas (5.13), (5.17)–(5.20) imply that

$$\log \delta_n \leq \iint \log[|z-t| w(z)w(t)]\, d\sigma_n(z)\, d\sigma_n(t) - \varepsilon_n,$$

where $\varepsilon_n \to 0$. Thus,

$$\log(1/\delta_n) \geq V(w, E_n) + \varepsilon_n = \log(1/\mathrm{cap}(w, E_n)) + \varepsilon_n. \tag{5.21}$$

We let $n \to \infty$ in (5.21) and use Lemma 5.2 and Theorem 3.3(d) to conclude that

$$\mathrm{cap}(w, E) \geq \tau(w, E). \tag{5.22}$$

Together with (5.9), we have established Theorem 5.1 for the case when w is continuous on E.

In the general case when w is merely assumed to be upper semicontinuous on E, we can again take $E = \mathscr{S}_w^*$ so that E is compact. Let $\{w_n\}$ be a decreasing sequence of positive continuous functions such that $w_n \to w$ on E. Clearly, each w_n is admissible and $\tau(w_n, E) \geq \tau(w, E)$. As shown above,

$$\mathrm{cap}(w_n, E) = \tau(w_n, E), \qquad n = 1, 2, \ldots.$$

It then follows from Theorem 3.3(c) that

$$\mathrm{cap}(w, E) = \lim_{n\to\infty} \mathrm{cap}(w_n, E) = \lim_{n\to\infty} \tau(w_n, E) \geq \tau(w, E),$$

which together with (5.9) completes the proof. ■

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