Szegő Type Asymptotics for Minimal Blaschke Products

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ABSTRACT Let μ be a positive, finite Borel measure on $[0, 2\pi)$. For $0 < r < 1$, $0 < p < \infty$, let

$$E_{n,p}(d\mu; r) := \inf_{B_n} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left|B_n(re^{i\theta})\right|^p d\mu(\theta) \right\}^{1/p}$$

where the infimum is taken over all Blaschke products of order n having zeros in $|z| < 1$. Let B_n^* denote a minimal Blaschke product and let $G(\mu')$ denote the geometric mean of the derivative of the absolutely continuous part of μ. In the first part of the paper we present a self-contained proof of a result due to Parfenov; namely $E_{n,p} \sim r^n G(\mu')^{1/p}$ as $n \to \infty$. In the second part we describe the extension of the classical Szegő function $D(z)$ and prove that $B_n^*(z) \sim z^n \left\{G(\mu')^{1/p}/D(z)^{2/p}\right\}$ as $n \to \infty$, uniformly on compact subsets of the annulus $r < |z| < 1/r$. Some generalizations and applications are also discussed.

1 Introduction

Let B_n denote a monic Blaschke product of order n with zeros in $|z| < 1$:

$$B_n(z) \quad \prod_{k=1}^{n} \frac{z - \alpha_k}{1 - \overline{\alpha}_k z} \qquad |\alpha_k| < 1, \quad k = 1, \ldots, n.$$

Let μ be an arbitrary positive, finite Borel measure on $[0, 2\pi)$ whose support contains infinitely many points. For $0 < r < 1$, $0 < p < \infty$, define

$$E_{n,p}(d\mu; r) := \inf_{B_n} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left|B_n(re^{i\theta})\right|^p d\mu(\theta) \right\}^{1/p} \tag{1.1}$$

A standard argument shows that the infimum in (1.1) is attained for some

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B_n^*, but the question of uniqueness of this minimal Blaschke product remains open. In the sequel B_n^* will denote any minimal Blaschke product of order n, that is

$$E_{n,p}(d\mu;r) \quad \left\{\frac{1}{2\pi}\int_0^{2\pi} |B_n^*(re^{i\theta})|^p\, d\mu(\theta)\right\}^{1/p}$$

Our aim is to describe the asymptotic behavior (as $n \to \infty$) of $E_{n,p}$ and $B_n^*(z)$. Since $|(z-\alpha_k)/(1-\overline{\alpha}_k z)|$ represents the hyperbolic distance between z and α_k, the results to be presented may be viewed as the extension to the non-Euclidean setting of the classical strong Szegő theory. So let us first recall some basic facts of this theory.

Let

$$\epsilon_{n,p}(d\mu) :: \inf_{P_n} \left\{\frac{1}{2\pi}\int_0^{2\pi} |P_n(e^{i\theta})|^p\, d\mu(\theta)\right\}^{1/p} \qquad p > 0, \qquad (1.3)$$

where the infimum is taken over all *monic* algebraic polynomials $P_n(z) = z^n + \cdots$ of degree n. This infimum is attained for the unique monic polynomial which we denote by $\varphi_{n,p}(z)$.

Given any $f \in L_1[0, 2\pi]$, $f \geq 0$ a.e., define its *geometric mean* $G(f)$ by

$$G(f) :: \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log f(\theta) d\theta\right\}$$

The case $\displaystyle\int \log f = -\infty$ is not excluded - we then set $G(f) = 0$. Returning to the measure μ, let $\mu = \mu_a + \mu_s$ be its canonical decomposition into the absolutely continuous and the singular parts (with respect to the Lebesgue measure $d\theta$). We denote by $\mu'(\theta)$ the Radon-Nikodym derivative $d\mu_a/d\theta$ of μ_a with respect to $d\theta$. Since, by the definition, $\mu' \in L_1[0, 2\pi]$, we may consider $G(\mu')$. If $\log \mu' \in L_1[0, 2\pi]$ (or, equivalently, if $G(\mu') > 0$) we say that μ satisfies the *Szegő condition*. We then define the *Szegő function* of μ by

$$D(d\mu; z) := \exp\left\{\frac{1}{4\pi}\int_0^{2\pi} \log \mu'(\theta)\frac{e^{i\theta}+z}{e^{i\theta}-z} d\theta\right\} \qquad |z| > 1. \qquad (1.5)$$

This function has the following properties (cf. [Sz, p. 276], but notice that $D(d\mu; z)$ in (1.5) and $D(z)$ defined in [Sz, p. 277] are related by $D(d\mu; z) = \overline{D(1/\overline{z})}$) :

(i) $D(d\mu; z)$ is analytic and non-vanishing in $|z| > 1$;

(ii) $D(d\mu; \infty) = G(\mu')^{1/2}$;

(iii) $\lim_{\rho \to 1+} D(d\mu; \rho e^{i\theta}) =: D(d\mu; e^{i\theta})$ exists for a.e. θ in $[0, 2\pi]$ and $|D(d\mu; e^{i\theta})|^2 = \mu'(\theta)$ a.e. on $[0, 2\pi]$.

The following results (due to Szegő, Kolmogorov and Krein) describe the behavior of $\epsilon_{n,p}$ and $\varphi_{n,2}(z)$.

Theorem 1.1 *For every* $0 < p < \infty$,

$$\lim_{n\to\infty} \epsilon_{n,p} = G(\mu')^{1/p}.$$

Theorem 1.2 *If* μ *satisfies the Szegő condition, then*

$$\lim_{n\to\infty} z^{-n}\varphi_{n,2}(z) = G(\mu')^{1/2}/D(d\mu; z), \quad |z| > 1$$

with the limit being uniform for $|z| \geq R > 1$.

The proofs of these results can be found in [GS, Ch. III]. The main ingredient in the proof of Theorem 1.1 is the relation

$$\inf_{f\in C[0,2\pi],\, f>0} \left\{ G(f)^{-1}\frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\mu(\theta)\right\} = G(\mu'),$$

to which (1.6) is easily reduced. (This reduction is carried out for $p = 2$, but a similar argument applies to any $p > 0$.) The method of proof of Theorem 1.2 is a purely L_2 argument. Yet, it can be modified to deal with any $p > 1$. This was done by Geronimus and more recently by X. Li and K. Pan:

Theorem 1.3 ([G], [LP, Thm 2.2]) *If* μ *satisfies the Szegő condition, then for* $p > 1$

$$\lim_{n\to\infty} z^{-n}\varphi_{n,p}(z) = G(\mu')^{1/p}/\{D(d\mu; z)\}^{2/p} \quad |z| > 1, \tag{1.9}$$

with the limit being uniform for $|z| \geq R > 1$.

We return now to our Blaschke product setting. The following result (essentially due to O. Parfenov) is analogous to Theorem 1.1.

Theorem 1.4 ([Pa1, Thm 2]) *For every* $0 < p < \infty$, $0 < r < 1$,

$$\lim_{n\to\infty} r^{-n}E_{n,p}(d\mu; r) = G(\mu')^{1/p} \tag{1.10}$$

and, moreover,

$$r^{-n}E_{n,p}(d\mu; r) \geq G(\mu')^{1/p} \quad n = 0, 1, 2, \dots . \tag{1.11}$$

In his proof, Parfenov also utilizes the relation (1.8). Below we give a direct proof of Theorem 1.4.

In applications the following version of Theorem 1.4 is sometimes more convenient. This version will enable us to consider the weighted L_∞-norm.

Theorem 1.5 *Let* $w(\theta) \in L_p[0, 2\pi]$. *Assume* $w \geq 0$ *a.e. and for* $p = \infty$ *assume additionally that* $w(\theta)$ *is upper semi-continuous. For* $0 < p \leq \infty$, $0 < r < 1$, *define*

$$E_{n,p}(w; r) := \inf_{B_n} \left\{ \frac{1}{2\pi}\int_0^{2\pi} |B_n(re^{i\theta})w(\theta)|^p d\theta \right\}^{1/p} \tag{1.12}$$

(with obvious modification for $p = \infty$). Then

$$\lim_{n\to\infty} r^{-n} E_{n,p}(w;r) = G(w) \tag{1.13}$$

and, moreover,

$$r^{-n} E_{n,p}(w;r) \geq G(w), \quad n = 0,1,2,\dots . \tag{1.14}$$

Remark 1. The infimum in (1.12) (like that in (1.1)) is attained for some B_n^*. The uniqueness of such a minimal monic Blaschke product is known only for $p = \infty, w \equiv 1$, in which case D.J. Newman showed that $B_n^*(z) = z^n$.

Remark 2. The existence of the limits in (1.10) and (1.13) is trivial; for if B_n^* is minimal, then on choosing $B_{n+1}(z) = zB_n^*(z)$ we obviously obtain $E_{n+1,p} \leq rE_{n,p}$. Hence the sequence $r^{-n}E_{n,p}$ is monotonically decreasing and thus converges.

We turn now to the description of the behavior of $B_n^*(z)$. It is easy to see (cf. [FM, Cor. 14] and the method of proof of Theorem 2 in [Sa]) that the zeros of any B_n^* lie in the disk $|z| < r$ (and, consequently, the poles of B_n^* lie in $|z| > 1/r$). Hence one may expect that the analogue of the limit relation (1.7) holds in the annulus $r < |z| < 1/r$. But first we must answer the question: what is the "Szegő function" for this setting? In a forthcoming paper we will describe this modification of the classical Szegő function as well as the asymptotics for $B_n^*(z)$ in full generality ($\log \mu' \in L_1[0, 2\pi]$). Here we confine ourselves to a simpler situation, namely $\log \mu' \in C[0, 2\pi]$.

Theorem 1.6 *Let $f(\theta)$ be a positive, continuous 2π-periodic function. For $0 < r < 1$ there exists a unique function $D(f;r;z) =: D(z)$ (the Szegő function of f for the annulus $r < |z| < 1/r$) that satisfies the following conditions:*
(i) *$D(z)$ is analytic and non-vanishing in $r < |z| < 1/r$ and satisfies there*

$$D(z)\overline{D(1/\bar{z})} = G(f). \tag{1.15}$$

In particular,

$$|D(z)|^2 = G(f) \quad \text{for} \quad |z| = 1; \tag{1.16}$$

(ii) *$|D(z)|$ is continuous in $r \leq |z| \leq 1/r$ and*

$$|D(re^{i\theta})|^2 = f(\theta), \quad 0 \leq \theta \leq 2\pi; \tag{1.17}$$

(iii) *$\log D(z)$ is single-valued in $r < |z| < 1/r$ and there is a branch of $\log D(z)$ that satisfies*

$$\frac{1}{2\pi} \int_{|z|=1} \log D(z)|dz| \quad \text{is real}. \tag{1.18}$$

Remark 3. Notice that (1.18) and (1.16) imply that the integral in (1.18) is equal to $(1/2)\log G(f)$. This corresponds to the normalization (ii) of the classical Szegö function defined above.

Remark 4. An integral representation for $D(f;r;z)$ (similar to (1.5)) can also be given. We shall do this in a future paper.

Remark 5. Given $\alpha > 0$, we denote by $D(z)^{\alpha}$ the function $\exp(\alpha \log D(z))$, where the branch of $\log D(z)$ is chosen to satisfy (1.18).

Remark 6. For the general case, namely $\log f \in L_1[0, 2\pi]$, we define $D(f;r;z)$ in the same fashion, except that property (ii) is replaced by

(ii) $D(z)$ is an *outer* function in the Hardy space H_2 of the annulus $r < |z| < 1$, and its limiting values on $|z| = r$ satisfy (1.17).

We can now formulate our main result.

Theorem 1.7 (a) *Let μ' be a positive, continuous 2π-periodic function and let $D(d\mu;r;z)$ denote the Szegö function of $\mu'(\theta)$ for the annulus $r < |z| < 1/r$. Given $0 < p < \infty$, let B_n^* denote a Blaschke product that realizes the infimum in* (1.1). *Then*

$$\lim_{n\to\infty} z^{-n} B_n^*(z) \quad G(\mu')^{1/p}/\{D(d\mu;r;z)\}^{2/p} \tag{1.19}$$

uniformly on compact subsets of the annulus $r < |z| < 1/r$.

(b) *Let $w(\theta)$ be a positive, continuous 2π-periodic function and let $D(w;r;z)$ denote the Szegö function of $w(\theta)$ for the annulus $r < |z| < 1/r$. Given $0 < p \leq \infty$, let B_n^* denote a Blaschke product that realizes the infimum in* (1.12). *Then*

$$\lim_{n\to\infty} z^{-n} B_n^*(z) = G(w)/\{D(w;r;z)\}^2 \tag{1.20}$$

uniformly on compact subsets of the annulus $r < |z| < 1/r$.

We remark that the method of the proof of Theorem 1.7 is a new one and it can be applied to the classical polynomial situation. This will enable us to extend Theorem 1.3 to any $p > 0$:

Theorem 1.8 *If μ satisfies the Szegö condition, then for $p > 0$*

$$\lim_{n\to\infty} z^{-n}\varphi_{n,p}(z) = G(\mu')^{1/p}/\{D(d\mu;z)\}^{2/p}$$

locally uniformly for $|z| > 1$.

This paper is organized as follows. In Section 2 we prove some auxiliary results. In Section 3 we prove Theorems 1.4 and 1.5. The Szegö function is discussed in Section 4. In Section 5, Theorems 1.7 and 1.8 are proven. Finally, in Section 6, we consider a more general situation and discuss the relation between $E_{n,p}$ and the n-widths of certain classes of analytic functions.

2 Auxiliary Results

(a) *The proof of the lower bounds* (1.11) *and* (1.14)

Let $B_n(z) = \prod_{k=1}^{n}(z-\alpha_k)/(a-\overline{\alpha}_k z)$, $|\alpha_k|<1$, $k=1,2,\ldots,n$. Let $d\mu = \mu'(\theta)d\theta + d\mu_s$ be the canonical decomposition of μ. Since μ is positive, so is μ_s. Hence

$$\frac{1}{2\pi}\int_0^{2\pi} |B_n(re^{i\theta})|^p d\mu \geq \frac{1}{2\pi}\int_0^{2\pi} |B_n(re^{i\theta})|^p \mu'(\theta)d\theta. \tag{2.1}$$

Assuming $\int \log \mu' > -\infty$ (otherwise (1.11) is obvious) and using the Jensen inequality, we obtain

$$\log\left\{\frac{1}{2\pi}\int_0^{2\pi} |B_n(re^{i\theta})|^p \mu'(\theta)d\theta\right\}^{1/p}$$

$$\geq \frac{1}{2\pi}\int_0^{2\pi} \log|B_n(re^{i\theta})|d\theta + \frac{1}{p}\cdot\frac{1}{2\pi}\int_0^{2\pi}\log\mu'(\theta)d\theta$$

$$= \sum_{k=1}^{n}\frac{1}{2\pi}\int_0^{2\pi}\{\log|re^{i\theta}-\alpha_k| - \log|1-\overline{\alpha}_k re^{i\theta}|\}d\theta + \frac{1}{p}\log G(\mu').$$

Since $|\alpha_k|<1$, $\log|1-\overline{\alpha}_k z|$ is harmonic in $|z|<1$, and so the mean value theorem yields

$$\int_0^{2\pi}\log|1-\overline{\alpha}_k re^{i\theta}|d\theta = 0.$$

Furthermore,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\log|re^{i\theta}-\alpha_k|d\theta &= \log r + \log^{+}\left|\frac{\alpha_k}{r}\right| \\ &\geq \log r \end{aligned} \tag{2.4}$$

(see Lemma 14.4.1 in [H]). Inserting (2.3), (2.4) into (2.2) and using (2.1) we obtain the lower bound (1.11). Applying it with $d\mu = [w(\theta)]^p d\theta$, we also obtain the lower bound (1.14) for $p<\infty$. The case $p=\infty$ then follows by passing to the limit as $p\uparrow\infty$.

(b) *Annihilating the singular part of a measure*

In [N, Lemma 4], P. Nevai introduced a simple but very effective device to deal with a singular part of a measure. Following is a version of his result with one ingredient added.

Lemma 2.1 *Let σ be a positive, finite Borel measure on $[0,2\pi)$ that is singular with respect to $d\theta$. Then there is a sequence $\{h_n\}$ of continuous 2π-periodic functions such that*

$$\tfrac{1}{n} \leq h_n(\theta) \leq 1, \quad 0\leq\theta\leq 2\pi, \tag{2.5}$$

$$h_n(\theta) \to 1 \text{ a.e. with respect to } d\theta,$$
$$\lim_{n\to\infty} \int_0^{2\pi} h_n(\theta) d\sigma(\theta) = 0, \tag{2.7}$$
$$\lim_{n\to\infty} G(h_n) = 1.$$

Proof. Let $S \subset [0, 2\pi)$ be a Borel set such that $\int_S d\theta = 0$ and $\sigma(S) = \sigma([0, 2\pi))$. Let $\{S_n\}$ be a decreasing sequence of open subsets of $[0, 2\pi)$ containing S such that

$$\int_{S_n} d\theta < \frac{1}{n} \text{ and } \int_{[0,2\pi)\setminus S_n} d\sigma = 0,$$

and set $\hat{S} := \bigcap_{n=1}^{\infty} S_n$. For every n, let C_n be a compact set such that $C_n \subset S_n$ and

$$\int_{[0,2\pi)\setminus C_n} d\sigma < \frac{1}{n} \tag{2.10}$$

(such C_n exists since a finite Borel measure is regular). Let h_n be a continuous function from $[0, 2\pi]$ into $[1/n, 1]$ such that

$$h_n(\theta) = \frac{1}{n} \text{ on } C_n, \quad h_n(\theta) = 1 \text{ on } [0, 2\pi] \setminus S_n$$

Then $h_n \to 1$ on $[0, 2\pi] \setminus \hat{S}$. Hence (2.6) holds. Also, by (2.10) and (2.11),

$$\begin{aligned} 0 \le \int_0^{2\pi} h_n(\theta) d\sigma(\theta) &\le \int_{[0,2\pi)\setminus C_n} d\sigma(\theta) + \frac{1}{n} \int_{C_n} d\sigma(\theta) \\ &\le \frac{1}{n} + \frac{1}{n}\|\sigma\|. \end{aligned}$$

Whence, (2.7) holds. Furthermore, by (2.9),

$$\begin{aligned} 0 \ge \int_0^{2\pi} \log h_n(\theta) d\theta &= \int_{S_n} \log h_n(\theta) d\theta \ge (-\log n) \int_{S_n} d\theta \\ &\ge \frac{\log n}{n} \end{aligned}$$

Thus, $\lim_{n\to\infty} \int_0^{2\pi} \log h_n(\theta) d\theta = 0$ and (2.8) follows.

If h_n satisfies $h_n(0) = h_n(2\pi)$, we are done. If not, redefine h_n by setting $\bar{h}_n(\theta) = h_n(\theta) \ell_n(\theta)$, where

$$\ell_n(\theta) := \begin{cases} 1, \ 0 \le \theta < 2\pi - 1/n \\ h_n(0) + (2\pi - \theta)[1 - h_n(0)]n, \ \ 2\pi - \frac{1}{n} \le \theta \le 2\pi. \end{cases}$$

By construction, $h_n(2\pi) = 1$ and so $\tilde{h}_n(0) = \tilde{h}_n(2\pi)$. Moreover, the $\tilde{h}_n$'s obviously satisfy (2.5) with $1/n$ replaced by $1/n^2$, as well as (2.6), (2.7) and (2.8). ∎

(c) *A special class of weights*

In this paragraph we prove Theorem 1.5 for a special class of weights. These will be used later to approximate arbitrary $\mu'(\theta)$.

Example. Let

$$w(\theta) := \lambda|B_M(re^{i\theta})|, \tag{2.12}$$

where $\lambda > 0$ and

$$B_M(z) := \prod_{k=1}^{M} \frac{z-\alpha_k}{1-\overline{\alpha}_k z} \qquad |\alpha_k| < r, \quad k = 1, \ldots, M. \tag{2.13}$$

For $n \geq 1$ define a Blaschke product B_{nM} of order nM by

$$B_{nM}(z) := \frac{1}{B_M(z)} \prod_{k=1}^{M} \frac{z^{n+1} - \alpha_k^{n+1}}{1 - \overline{\alpha}_k^{n+1} z^{n+1}}. \tag{2.14}$$

Since $|\alpha_k| < r$, we have

$$B_{nM}(z) = \frac{1}{B_M(z)} z^{(n+1)M}(1 + o(1)), \quad r \leq |z| \leq 1, \tag{2.15}$$

where $o(1) \to 0$ as $n \to \infty$, uniformly in $r \leq |z| \leq 1$. In particular, we obtain

$$|B_{nM}(re^{i\theta})| = \lambda r^{nM} \frac{r^M}{w(\theta)}(1 + o(1)). \tag{2.16}$$

Since $|\alpha_k| < r$, equations (2.3) and (2.4) imply that

$$\frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{re^{i\theta} - \alpha_k}{1 - \overline{\alpha}_k re^{i\theta}} \right| d\theta = \log r$$

and therefore $G(w) = \lambda r^M$. Thus (2.16) may be rewritten as

$$|B_{nM}(re^{i\theta})| = r^{nM} \frac{G(w)}{w(\theta)}(1 + o(1)), \tag{2.17}$$

uniformly for $\theta \in [0, 2\pi]$. In view of (1.14) and the existence of the limit (1.13), this implies Theorem 1.5 for $w(\theta)$ of the form (2.12).

(d) *Approximation of continuous weights*

The passage from $w(\theta)$ of the form (2.12) to general continuous weights is furnished by the following

Lemma 2.2 ([Pal, Lemma 1]) *Let $w(\theta)$ be a non-negative continuous 2π-periodic function. Then $w(\theta)$ can be approximated on $[0, 2\pi]$ arbitrarily closely in the uniform norm, by functions of form (2.12), that is by $\lambda|B_M(re^{i\theta})|$, where $\lambda > 0$ and B_M has all its zeros in $|z| < r$.*

Proof. Since $w \geq 0$ is continuous and 2π-periodic, it may be approximated uniformly on $[0, 2\pi]$ by positive trigonometric polynomials. Such a polynomial may be written as $|g(e^{i\theta})|^2$, where $g(z)$ is a polynomial in z, whose zeros all lie in the unit disk $|z| < 1$. So it suffices to approximate $|e^{i\theta} - \beta|$, $|\beta| < 1$, on $[0, 2\pi]$ by functions of the form (2.12) or, equivalently, to approximate $|z - \alpha|$, $|\alpha| < r$, on $|z| = r$ by functions of the form $\lambda|B_M(z)|$, where all zeros of B_M lie in $|z| < r$.

For $m = 1, 2, \ldots$ let

$$\lambda_m B_m(z) := r^{-2m} \prod_{k=0}^{m} \frac{z - \alpha r^{2k}}{1 - \overline{\alpha} r^{2k} z}.$$

Since $|\alpha| < r$, the product in (2.18) represents a Blaschke product zeros in $|z| < r$. It is readily verified that for $|z| = r$ we have

$$\lambda_m |B_m(z)| \quad \frac{|z - \alpha|}{|1 - \overline{\alpha} r^{2m} z|}.$$

Since $r < 1$, $\lambda_m |B_m(z)| \to |z - \alpha|$ as $m \to \infty$, uniformly on $|z| \quad r$. This completes the proof. ∎

3 Proof of Theorems 1.4 and 1.5

We start with the proof of Theorem 1.4. Let $d\mu = \mu'(\theta)d\theta + d\mu_s$, where $\mu' \in L_1[0, 2\pi]$, and $\mu_s' = 0$ a.e.

STEP 1. We first show that it suffices to assume that for some $a > 0$,

$$\mu'(\theta) \geq a, \qquad 0 \leq \theta < 2\pi, \tag{3.1}$$

(and consequently, $\int \log \mu'(\theta)d\theta > -\infty$). Indeed, assume that Theorem 1.4 holds for such μ. Given any μ, define

$$f_n(\theta) := \begin{cases} \mu'(\theta), & \text{if} \quad \mu'(\theta) > 1/n, \\ 1/n, & \text{if} \quad \mu'(\theta) \leq 1/n. \end{cases}$$

By the Monotone Convergence Theorem,

$$\lim_{n \to \infty} \int_0^{2\pi} \log f_n(\theta) d\theta = \int_0^{2\pi} \log \mu'(\theta) d\theta$$

(the latter integral may be equal to ∞) and consequently

$$\lim_{n\to\infty} G(f_n) = G(\mu').$$

Let $d\mu_n := f_n(\theta)d\theta + d\mu_s$. Since $\mu'(\theta) \le f_n(\theta)$, and Theorem 1.4 holds (by assumption) for each $d\mu_n$, we have

$$\lim_{m\to\infty} r^{-m}E_{m,p}(d\mu;r) \le \lim_{m\to\infty} r^{-m}E_{m,p}(d\mu_n;r) \quad G(f_n),$$

Passing to the limit as $n \to \infty$ and applying (3.2) yields $\lim_{m\to\infty} r^{-m}E_{m,p}(d\mu;r) \le G(\mu')$. Since the reverse inequality has already been proved, Theorem 1.4 holds for μ. Thus, from now on we assume that (3.1) holds.

STEP 2. Fix any $\epsilon > 0$. Then there is a trigonometric polynomial Q_ϵ that satisfies

$$\int_0^{2\pi} |\mu'(\theta) - Q_\epsilon(\theta)|d\theta < \epsilon. \tag{3.3}$$

In view of (3.1) we may also assume (see e.g. [Sz, Thm 1.5.3]) that

$$Q_\epsilon(\theta) \ge a, \qquad 0 \le \theta \le 2\pi. \tag{3.4}$$

Then, $|\log\mu' \quad \log Q_\epsilon| \le a^{-1}|\mu' - Q_\epsilon|$, and so we obtain

$$\lim_{\epsilon\to 0} G(Q_\epsilon) = G(\mu'). \tag{3.5}$$

Notice also that, for any $\varphi \in L_\infty[0,2\pi]$, we get from (3.3)

$$\frac{1}{2\pi}\int_0^{2\pi} |\varphi|d\mu \le \frac{1}{2\pi}\int_0^{2\pi} |\varphi|Q_\epsilon d\theta + \frac{1}{2\pi}\epsilon\|\varphi\|_\infty + \frac{1}{2\pi}\int_0^{2\pi} |\varphi|d\mu_s \tag{3.6}$$

STEP 3. For $k = 1,2,3,\ldots$, define

$$f_{k,\epsilon}(\theta) := Q_\epsilon(\theta)/h_k(\theta), \tag{3.7}$$

where the h_k are constructed by Lemma 2.1 (for the measure $\sigma = \mu_s$). The assertion (2.8) of that lemma then gives

$$\lim_{k\to\infty} G(f_{k,\epsilon}) = G(Q_\epsilon). \tag{3.8}$$

STEP 4. Applying Lemma 2.2 to the continuous, 2π-periodic weight $w(\theta) = [f_{k,\epsilon}(\theta)]^{1/p}$ we can find a sequence $\{w_{\ell,k,\epsilon}(\theta)\}_{\ell=1}^{\infty}$ that satisfies

$$\lim_{\ell\to\infty} w_{\ell,k,\epsilon}(\theta) \quad [f_{k,\epsilon}(\theta)]^{1/p} \qquad \text{uniformly on } [0,2\pi], \tag{3.9}$$

where

$$w_{\ell,k,\epsilon}(\theta) = \lambda_\ell|B_{M_\ell}(re^{i\theta})|, \tag{3.10}$$

and all the zeros of B_{M_ℓ} lie in $|z| < r$. From (3.9) and the fact that $f_{k,\epsilon}(\theta) \geq ka > 0$ on $[0, 2\pi]$ we obtain

$$\lim_{\ell \to \infty} w^p_{\ell,k,\epsilon}(\theta) = f_{k,\epsilon}(\theta), \quad \text{uniformly on } [0, 2\pi] \tag{3.11}$$

and, also, that

$$\lim_{\ell \to \infty} G(w^p_{\ell,k,\epsilon}) = G(f_{k,\epsilon}). \tag{3.12}$$

STEP 5. Applying the result of Section 2(c) (see (2.17)) to the weight (3.10) we construct, for each $n = 1, 2, 3, \ldots$, a Blaschke product B_{nM_ℓ} such that

$$r^{-nM_\ell}|B_{nM_\ell}(re^{i\theta})| = G(w_{\ell,k,\epsilon})w^{-1}_{\ell,k,\epsilon}(\theta)(1 + o(1)), \tag{3.13}$$

where $o(1) \to 0$ as $n \to \infty$, uniformly on $[0, 2\pi]$.

Now set

$$L := \lim_{n \to \infty} r^{-n} E_{n,p}(d\mu; r). \tag{3.14}$$

Applying (3.6) with $\varphi =$ [left hand side of (3.13) $]^p$ and letting $n \to \infty$ we obviously obtain

$$L \;<\; G^p(w_{\ell,k,\epsilon}) \left[\frac{1}{2\pi} \int_0^{2\pi} w^{-p}_{\ell,k,\epsilon}(\theta) Q_\epsilon(\theta) d\theta + \frac{\epsilon}{2\pi} \|w^{-p}_{\ell,k,\epsilon}\|_\infty \right.$$
$$\left. + \frac{1}{2\pi} \int_0^{2\pi} w^{-p}_{\ell,k,\epsilon}(\theta) d\mu_s \right]$$

Now, let $\ell \to \infty$. Then (3.11) and (3.12) imply that

$$L \leq G(f_{k,\epsilon}) \left[\frac{1}{2\pi} \int_0^{2\pi} h_k(\theta) d\theta + \frac{\epsilon}{2\pi} \|f^{-1}_{k,\epsilon}\|_\infty + \frac{1}{2\pi} \int_0^{2\pi} \frac{h_k(\theta)}{Q_\epsilon(\theta)} d\mu_s \right]$$

Since $h_k \leq 1$, we obtain from (3.7) and (3.4) that

$$\|f^{-1}_{k,\epsilon}\|_\infty \leq \tfrac{1}{a}.$$

Hence, letting $k \to \infty$ and applying (2.6), (2.7) of Lemma 2.1 and (3.8), we get

$$L \leq G(Q_\epsilon) \left[1 + \frac{\epsilon}{2\pi a} \right].$$

Finally, let $\epsilon \to 0$ and apply (3.5) to obtain $L \leq G(\mu')$. Since the reverse inequality has already been proved, the proof of Theorem 1.4 is complete. ∎

Proof of Theorem 1.5. Applying Theorem 1.4 with $d\mu = [w(\theta)]^p d\theta$, we obtain Theorem 1.5 for $p < \infty$.

For $p = \infty$ assume first that $w(\theta)$ is continuous and 2π-periodic. Then we repeat the proof of Theorem 1.4 omitting steps 2 and 3. We thus obtain a sequence $\{w_\ell\}$ of functions of the form (3.10) that satisfies

$$\lim_{\ell \to \infty} w_\ell(\theta) = w(\theta), \quad \text{uniformly on } [0, 2\pi]$$

From this we proceed to (3.13) (with w_ℓ instead $w_{\ell,k,\epsilon}$) and obviously obtain

$$\lim_{n \to \infty} r^{-n} E_{n,\infty}(w; r) \leq G(w_\ell).$$

Letting $\ell \to \infty$ and recalling the assumption $w(\theta) \geq a > 0$ of Step 1 we obtain that $\lim_{n\to\infty} r^{-n} E_{n,\infty}(w, r) \leq G(w)$. The reverse inequality was proved in Section 2(a).

If w is merely upper semi-continuous and 2π-periodic we find a decreasing sequence $\{f_k\}$ of continuous 2π-periodic functions that converges to w pointwise. The Monotone Convergence Theorem implies that $\lim_{k\to\infty} G(f_k) = G(w)$. Since $w \leq f_k$, we obtain

$$\lim_{n \to \infty} r^{-n} E_{n,\infty}(w; r) \leq \lim_{n \to \infty} r^{-n} E_{n,\infty}(f_k, r) = G(f_k)$$

The result now follows by passing to the limit as $k \to \infty$.

Finally, if $w(0) \neq w(2\pi)$ we consider instead $\tilde{w}$ defined by $\tilde{w}(0) = \tilde{w}(2\pi) = \max\{w(0), w(2\pi)\}$ and $\tilde{w}(\theta) = w(\theta)$ for $0 < \theta < 2\pi$. Then $\tilde{w}$ is upper semi-continuous, 2π-periodic and satisfies $E_{n,\infty}(\tilde{w}; r) = E_{n,\infty}(w; r)$. Thus, the previous case applies. ∎

4 The Szegő Function for the Annulus

In this section we give the proof of Theorem 1.6.

Lemma 4.1 *Let $u(z)$ be harmonic in $r < |z| \leq 1$ and continuous in $r \leq |z| \leq 1$. If*

$$\int_{|z|=1} u \, ds = \frac{1}{r} \int_{|z|=r} u \, ds \tag{4.1}$$

(ds denotes the element of arc length), then $u(z)$ has a single-valued conjugate $v(z)$ in $r < |z| \leq 1$.

Proof. Let $\rho := \sqrt{x^2 + y^2}$. For $\epsilon > 0$ small enough, $u(z)$ and $\log 1/\rho$ are harmonic in $r + \epsilon \leq |z| \leq 1$. By Green's theorem we then have:

$$\left(\int_{|z|=1} + \int_{|z|=r+\epsilon} \right) \left\{ u \frac{\partial}{\partial n} \log \frac{1}{\rho} - (\log \frac{1}{\rho}) \frac{\partial u}{\partial n} \right\} ds = 0, \tag{4.2}$$

where $\partial/\partial n$ denotes differentiation along the inward normal with respect to the annulus $r+\epsilon<|z|<1$. Since

$$\frac{\partial}{\partial n}\log\frac{1}{\rho}=\begin{cases}1 & \text{on } |z|=1\\ -(r+\epsilon)^{-1} & \text{on } |z|=r+\epsilon\end{cases}$$

we obtain from (4.2) that

$$\int_{|z|=1} u\,ds \quad \frac{1}{r+\epsilon}\int_{|z|=r+\epsilon} u\,ds \quad \left(\log\frac{1}{r+\epsilon}\right)\int_{|z|=r+\epsilon}\frac{\partial u}{\partial n}ds.$$

Since u is harmonic, the integral in the right-hand side of (4.3) is independent of ϵ. Letting $\epsilon\to 0$ in (4.3) and using the continuity of u in $r\le|z|\le 1$ we obtain (see (4.1)) that

$$\int_{|z|=r_1}\frac{\partial u}{\partial n}ds=0,\quad r<r_1\le 1$$

Since the last integral represents (up to the factor $\log 1/r_1$) the period about $|z|=r$ of a harmonic conjugate of $u(z)$ (cf. [F, pp. 79-80]), the result follows. ∎

Proof of Theorem 1.6. Let $f(\theta)$ be a positive, continuous 2π-periodic function. Then $\log f(\theta)$ is continuous and 2π-periodic and $G(f)>0$. Let $u(z)$ be the solution of the Dirichlet problem in $r<|z|<1$, with boundary values

$$u(e^{i\theta})=\tfrac{1}{2}\log G(f),\quad u(re^{i\theta})=\tfrac{1}{2}\log f(\theta). \tag{4.4}$$

Since $u=$ const. on $|z|=1$, $u(z)$ has a harmonic extension (by the reflection principle) to $r<|z|<1/r$. Next, (4.4) and the definition (1.4) of $G(f)$ yield

$$\int_0^{2\pi}u(e^{i\theta})d\theta=\int_0^{2\pi}u(re^{i\theta})d\theta.$$

Hence $u(z)$ satisfies (4.1) of Lemma 4.1. Applying this lemma, pick any single-valued harmonic conjugate $v(z)$. Since u is harmonic in $r<|z|<1/r$, so is v. Let

$$\gamma:=-\frac{1}{2\pi}\int_{|z|=1}v(z)|dz| \tag{4.5}$$

and define

$$D(z):=e^{i\gamma}e^{u(z)+iv(z)},\quad r<|z|<1/r. \tag{4.6}$$

By its construction, $D(z)$ obviously satisfies (i) and (ii) of Theorem 1.6 (the relation (1.15) follows by the reflection principle). Defining the (single-valued) branch of $\log D(z)$ by

$$\log D(z):=u(z)+i(\gamma+v(z)), \tag{4.7}$$

we obtain by (4.5), that (iii) of Theorem 1.6 is also satisfied

The uniqueness of such D is easily established. For if D_1, D_2 both satisfy the conditions (i), (ii) of Theorem 1.6, the Maximum Principle (for the harmonic function $\log|D_1/D_2|$) yields that D_1/D_2 is a unimodular constant. Then (iii) of Theorem 1.6 yields that (for a suitable branch of log) the integral $\int_{|z|=1}\log(D_1/D_2)|dz|$ is real. Hence $D_1 = D_2$. This completes the proof of Theorem 1.6. ∎

Example. Let

$$w(\theta) := \left|B_M(re^{i\theta})\right|, \quad B_M(z) := \prod_{k=1}^{M}\frac{z-\alpha_k}{1-\overline{\alpha}_k z}, \tag{4.8}$$

where the zeros $\alpha_1, \ldots, \alpha_M$ of B_M lie in $|z| < r$ (recall the example of Section 2 (c)). We claim that there is a single-valued branch of $\log(B_M(z)/z^M)$ in $r < |z| < 1/r$ that satisfies

$$\int_{|z|=1}\log\frac{B_M(z)}{z^M}|dz| = 0. \tag{4.9}$$

Indeed, consider the branches

$$\log\frac{z-\alpha_k}{z} \qquad \sum_{j=1}^{\infty}\left(\frac{\alpha_k}{z}\right)^j\frac{1}{j}, \quad |z| > r, \tag{4.10}$$

and

$$\log(1 \quad \overline{\alpha}_k z) :: \quad \sum_{j=1}^{\infty}(\overline{\alpha}_k z)^j\frac{1}{j}, \quad |z| < 1/r, \tag{4.11}$$

and define

$$\log\frac{B_M(z)}{z^M} := \sum_{k=1}^{M}\left\{\log\frac{z-\alpha_k}{z} \quad \log(1-\overline{\alpha}_k z)\right\} \quad r < |z| < 1/r. \tag{4.12}$$

Notice that the Laurent expansion (in $r < |z| < 1/r$) of $\log(B_M(z)/z^M)$ does not contain a constant term. Hence (4.9) follows.

We also know that, for the case considered, $G(w) = r^M$. Thus (4.8) and (4.9) imply that the function

$$\left\{r^M B_M(z)/z^M\right\}^{1/2} := r^{M/2}\exp\left\{\tfrac{1}{2}\log\left(B_M(z)/z^M\right)\right\}$$

is the Szegő function $D(w; r; z)$ of $w(\theta)$ for the annulus $r < |z| < 1/r$. Recalling (2.15) of Section 2 we obtain that the (asymptotically) minimal Blaschke product of order nM satisfies

$$\begin{aligned} B_{nM}(z) &= z^{nM}\frac{r^M}{r^M B_M(z)/z^M}(1+o(1)) \\ & \quad z^{nM}\frac{G(w)}{\{D(w;r;z)\}^2}(1+o(1)), \end{aligned}$$

where $o(1) \to 0$ as $n \to \infty$, uniformly on compact subsets of $r < |z| < 1/r$. This illustrates Theorem 1.7 for $p = 1$.

5 Proof of Theorems 1.7 and 1.8

We start with the proof of part (a) of Theorem 1.7. Under our assumptions on $\mu'(\theta)$ one can define the Szegő function $D(\mu'; r; z) =: D(z)$. Set

$$\varphi_n(z) :: \frac{B_n^*(z)}{} \cdot \frac{D^{2/p}(z)}{G^{1/p}(\mu')},$$

and observe that φ_n is analytic in $r < |z| < 1/r$ and $|\varphi_n|$ is continuous in the closed annulus. Since (by (1.16))

$$|\varphi_n(z)| = 1 \quad \text{for} \quad |z| = 1,$$

it suffices to prove that $\lim_{n\to\infty} \varphi_n(z) = 1$, uniformly on compact subsets of $\Omega := \{z : r < |z| \le 1\}$.

From the proof of the lower bound (see Section 2(a)), we know that

$$^{-np}\frac{1}{2\pi}\int_0^{2\pi} B_n^*(re^{i\theta})|^p \mu'(\theta)d\theta \ge G(\mu').$$

This and (1.10) imply that

$$\lim_{n\to\infty} r^{-np}\frac{1}{2\pi}\int_0^{2\pi} |B_n^*(re^{i\theta})|^p \mu'(\theta)d\theta = G(\mu'),$$

or, equivalently (by (5.1) and (1.17))

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} |\varphi_n(re^{i\theta})|^p d\theta \quad 1$$

We have already mentioned in the Introduction that the zeros of B_n^* lie in $|z| < r$. Hence there is a branch of $\log(B_n^*(z)/z^n)$ that is single-valued in Ω and satisfies

$$\int_{|z|=1} \log\frac{B_n^*(z)}{z^n}|dz| = 0$$

(see the Example at the end of Section 4). Also, (see Remark 3 in Section 1), there is a branch of $\log D^{2/p}(z)$ that satisfies

$$\frac{1}{2\pi}\int_{|z|=1} \log D^{2/p}(z)|dz| \quad \log G(\mu')^{1/p}$$

Thus we can define the single-valued branch of $\log\varphi_n(z)$ in Ω, such that

$$\int_{|z|=1} \log\varphi_n(z)|dz| = 0. \tag{5.4}$$

For any $n = 1, 2, \ldots$, fix this branch and define

$$\varphi_n^p(z) := \exp(p \log \varphi_n(z)), \quad z \in \Omega.$$

Given $\zeta \in \Omega$, we apply the Cauchy formula for $r + \epsilon <$
small enough) to $\varphi_n^p(z)$ and deduce (see (5.2)) that

$$|\varphi_n^p(\zeta)| \le c \left\{ \frac{1}{1 - |\zeta|} + \frac{1}{|\zeta| - r - \epsilon} \int_{|z| = r + \epsilon} |\varphi_n(z)|^p |dz \right.$$

where c is a constant independent of ϵ and n. Since $|\varphi_n|$ is continuous in $r \le |z| \le 1$ we may pass to the limit as $\epsilon \to 0$ and then use (5.3) to obtain that $\{\varphi_n^p\}$ (and, consequently, $\{\varphi_n\}$) form a normal family in Ω. Choose any convergent subsequence $\{\varphi_n\}_{n \in \Lambda}$:

$$\lim_{\substack{n \to \infty \\ n \in \Lambda}} \varphi_n(z) =: \varphi(z).$$

It remains to show that $\varphi(z) \equiv 1$ in Ω.

For this purpose we introduce the function

$$g_n(z) := |\varphi_n(z)|^p - p \log |\varphi_n(z)| - 1.$$

Let us examine some properties of g_n. Since $|\varphi_n|$ is continuous in $\overline{\Omega}$ and $|\varphi_n| > 0$ in $\overline{\Omega}$ (recall that the zeros of B_n^* lie in $|z| < r$), we obtain that g_n is continuous in $\overline{\Omega}$ and

$$\lim_{\epsilon \to 0+} \int_0^{2\pi} g_n\left((r + \epsilon)e^{i\theta}\right) d\theta = \int_0^{2\pi} g_n(re^{i\theta}) d\theta. \tag{5.8}$$

Next, since φ_n is analytic and nonvanishing in Ω, it follows that $|\varphi_n|^p$ is subharmonic in Ω and $\log |\varphi_n|$ is harmonic in Ω. Thus, g_n is subharmonic in Ω.

By the logarithmic convexity of the integral means of subharmonic function (cf. [HK, Theorem 2.12]) we may write for a given $r < \rho < 1$ and $\epsilon > 0$ small enough:

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} g_n(\rho e^{i\theta}) d\theta \quad \le \quad & \frac{\log 1/\rho}{\log\{1/(r + \epsilon)\}} \frac{1}{2\pi} \int_0^{2\pi} g_n\left((r + \epsilon)e^{i\theta}\right) d\theta \\ & + \frac{\log\{\rho/(r + \epsilon)\}}{\log\{1/(r + \epsilon)\}} \frac{1}{2\pi} \int_0^{2\pi} g_n(e^{i\theta}) d\theta \qquad (5.9) \\ & \frac{\log 1/\rho}{\log\{1/(r + \epsilon)\}} \frac{1}{2\pi} \int_0^{2\pi} g_n\left((r + \epsilon)e^{i\theta}\right) d\theta, \end{aligned}$$

where, in the last step, we used the property (see (5.2)) that

$$g_n(e^{i\theta}) = 0. \tag{5.10}$$

Next, (5.4) and (5.3) yield

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} g_n(re^{i\theta})d\theta = 0. \tag{5.11}$$

Passing in (5.9) to the limit first as $\epsilon \to 0^+$ and then as $n \to \infty$, $n \in \Lambda$, we obtain (by (5.8), (5.11), (5.6) and (5.7)) that

$$\frac{1}{2\pi}\int_0^{2\pi} g(\rho e^{i\theta})d\theta \le 0, \quad r < \rho \le 1, \tag{5.12}$$

where

$$g(z) := |\varphi(z)|^p - p\log|\varphi(z)| - 1, \quad z \in \Omega. \tag{5.13}$$

The elementary inequality

$$x - \log x - 1 > 0 \quad \text{for} \quad x > 0,\ x \neq 1 \tag{5.14}$$

implies that $g(z) \ge 0$ in Ω. Then (5.12) yields $g(z) = 0$, $z \in \Omega$. By (5.13), (5.14) we thus obtain:

$$|\varphi(z)| = 1, \quad z \in \Omega,$$

and therefore

$$\varphi(z) = e^{i\gamma}, \quad z \in \Omega \tag{5.15}$$

for some $-\pi \le \gamma < \pi$.

It remains to show that $\gamma = 0$. This follows directly from (5.6), (5.4). Indeed, the branch of log we fixed above for $n = 1, 2, \ldots$ can be written in the form

$$\log\varphi_n(z) = \int_a^z \frac{\varphi_n'(\zeta)}{\varphi_n(\zeta)}d\zeta + \operatorname{Log}\varphi_n(a) + 2\pi i k_n, \tag{5.16}$$

where Log denotes the principal branch, k_n is an integer and a is a fixed point. As $n \to \infty$, $n \in \Lambda$, the integral in (5.16) approaches 0 and $\operatorname{Log}\varphi_n(a) \to i\gamma$ (by (5.15)). Thus, (5.4) yields: $k_n = 0$ for $n \ge N$ and $\gamma = 0$. The proof of part (a) of Theorem 1.7 is now complete.

Proof of part (b) of Theorem 1.7. For $0 < p < \infty$, apply Theorem 1.7(a) with $d\mu(\theta) := w^p(\theta)d\theta$. For $p = \infty$, set

$$\varphi_n(z) := \frac{B_n^*(z)}{z^n}\frac{D^2(z)}{G(w)}.$$

Then (5.2) holds. Also, (5.3) holds with $p = 1$ and with the equality sign replaced by $\le$. Proceeding as in the proof of Theorem 1.7(a), we get the result. ∎

Remark 7. The same proof applies for the general case, namely $\log\mu' \in L_1[0, 2\pi]$. In this case $D(\mu'; r; z)$ belongs to the Hardy space H_2 in the annulus Ω (see Remark 6 in Section 1) and therefore $\varphi_n^2 \in H_2(\Omega)$. Hence, in (5.5), we may pass to the limit, as $\epsilon \to 0$. Also, D is an *outer* function and therefore

$$\lim_{\epsilon \to 0+} \int_0^{2\pi} \log \left| D\left((r+\epsilon)e^{i\theta}\right)\right| d\theta = \int_0^{2\pi} \log \left| D(re^{i\theta})\right| d\theta. \tag{5.17}$$

Thus, (5.8) is valid. The rest of the proof remains unchanged.

Proof of Theorem 1.8. We give the sketch of the proof and leave the details to the reader. Set

$$\varphi_n(z) := \frac{\varphi_{n,p}(z)}{z^n} \frac{D(d\mu; z)^{2/p}}{G(\mu')^{1/p}},$$

where $\varphi_{n,p}$ is the minimal polynomial and $D(d\mu; z)$ is defined by (1.5). Since $\varphi_{n,p}$ is monic and has its zeros in $|z| < 1$, we obtain by the properties of $D(d\mu; z)$, that

$$\varphi_n(\infty) = 1 \tag{5.18}$$

and that there is a branch of $\log \varphi_n(z)$ in $|z| > 1$ that satisfies (5.4). Since φ_n satisfies (5.3) with $r = 1$, the normality of $\{\varphi_n\}$ follows. As before, define g_n and use the monotonicity (rather than logarithmic convexity) of its integral means, to obtain

$$\int_0^{2\pi} g_n(\rho e^{i\theta}) d\theta \leq \int_0^{2\pi} g_n\left((1+\epsilon)e^{i\theta}\right) d\theta,$$

for $\rho > 1$ and $\epsilon > 0$ small enough (cf. [HK, Theorem 2.12], for the case $r_1 = 0$). Passing to the limit, first as $\epsilon \to 0$ and then as $n \to \infty$, $n \in \Lambda$, and using (5.4) and (5.3), we deduce as before that $\varphi(z) = e^{i\gamma}$, $|z| > 1$. Since $\varphi(\infty) = 1$, we get $\gamma = 0$ and the result follows. ∎

6 Generalizations. Application to n-widths.

Let us consider a more general case, when the circle $|z| = r$ is replaced by a compact set K in the open unit disk Δ. Given such a K and given a positive, finite Borel measure μ on K, we set for $0 < p < \infty$

$$E_{n,p}(d\mu; K) := \inf_{B_n} \left\{ \|B_n\|_{L_p(d\mu; K)} \right\}, \tag{6.1}$$

where the infimum is taken over all Blaschke products of order n, with zeros in Δ.

We shall need some basic notions from the potential theory (cf. [T, pp. 94-104]). Let

$$V := \inf_{\sigma} \int\int_K \log \left| \frac{z-\zeta}{1-\bar{\zeta} z} \right|^{-1} d\sigma_z d\sigma_\zeta =: \inf_{\sigma} I(\sigma),$$

where the infimum is taken over all probability measures on K. Then $0 < V \leq \infty$. Provided $V < \infty$, there exists a unique probability measure ν on K (the *equilibrium distribution* for K) such that $V = I(\nu)$. The equilibrium potential u for K is defined by

$$u(z) := \int_K \log \left| \frac{z-\zeta}{1-\bar{\zeta}z} \right|^{-1} d\nu_\zeta.$$

It satisfies

$$\begin{aligned} u(z) &\leq V \quad \text{for} \quad z \in \Delta \\ u(z) &= V \quad \nu \text{ a.e. for } z \text{ in } K. \end{aligned} \tag{6.4}$$

We set $c(K;\Delta) := 1/V$, and call $c(K;\Delta)$ the *capacity of* K *relative to* Δ (cf. [W] and notice that the capacity in [T] is defined to be e^{-V}).

Proceeding as in Section 2(a) and using (6.2), (6.3), we obtain (cf. [FM2]) the lower bound

$$E_{n,p}(d\mu; K) \geq G(\mu')^{1/p} e^{-n/c(K;\Delta)}$$

where

$$G(\mu') := \exp \int_K \left(\log \frac{d\mu}{d\nu} \right) d\nu,$$

and $d\mu/d\nu$ denotes the Radon-Nikodym derivative with respect to ν of the part of μ that is absolutely continuous with respect to ν. The crude upper bound, namely

$$E_{n,p}(d\mu; K) \leq e^{-n/(1+\epsilon)c(K;\Delta)}, \quad n > n(\epsilon), \quad \epsilon > 0,$$

can also be obtained. Together with (6.5) this gives (cf. [FM1,2])

$$\lim_{n\to\infty} E_{n,p}^{1/n}(d\mu; K) = e^{-1/c(K;\Delta)}$$

Now, let us turn our attention to another quantity. For $1 \leq q \leq \infty$, let A_q denote the restriction to K of the unit ball of the usual Hardy space H_q in Δ:

$$A_q := \{ f|_K : f \in H_q(\Delta), \quad \|f\|_q \leq 1 \}. \tag{6.8}$$

For $1 \leq p < \infty$, the Kolmogorov n-width of A_q in the space $L_p(d\mu; K)$ is defined by

$$d_n(A_q, L_p(d\mu; K)) := \inf_{X_n} \sup_{f \in A_q} \inf_{x \in X_n} \|f - x\|_{L_p(d\mu;K)},$$

where X_n denotes an arbitrary n-dimensional subspace of $L_p(d\mu; K)$. Fisher and Micchelli have proved (see [FM1]) that

$$d_n(A_\infty, L_p(d\mu; K)) = E_{n,p}(d\mu; K) \tag{6.10}$$

In view of (6.7) this gives the n-th root asymptotics for d_n. A similar result holds for weighted L_∞ norms. These asymptotics were established by Widom [W] (a simpler proof is given in [FM1]), but many special cases were known earlier (see [LT] for the history of this problem).

The first result concerning the strong asymptotics of d_n was established by Parfenov. He considered the case $p = q = 2$, K is a smooth closed curve, and $d\mu = w|dz|$, where $w \in C(K)$, $w > 0$, and $|dz|$ is the arc length on K. Although stated in different terms, the result of Parfenov reads (cf. [Pa3] for $w = 1$ and [Pa2] for general w):

$$\lim_{n\to\infty} e^{-n/c(K;\Delta)} d_n(A_2, L_2(d\mu;K)) = G(\mu')^{1/2} \tag{6.11}$$

Since $A_\infty \subset A_2$, we have $d_n(A_\infty, L_2(d\mu;K)) \le d_n(A_2, L_2(d\mu;K))$ (see (6.8), (6.9)). Therefore (6.5), (6.10) and (6.11) yield:

Theorem 6.1 *Let K be a simple closed Jordan curve of the class $C^{1+\epsilon}$, $\epsilon > 0$. Let $d\mu = w|dz|$, where $w \in C(K)$, $w > 0$, and $|dz|$ denotes the arc length on K. Then*

$$\lim_{n\to\infty} e^{-n/c(K;\Delta)} E_{n,2}(d\mu;K) = G(\mu')^{1/2} \tag{6.12}$$

To describe the behavior of minimal Blaschke products B_n^* we have to first define the appropriate Szegő function. Let Ω denote the doubly connected domain bounded by a curve K and by its reflection about $|z| = 1$. The Szegő function $D(d\mu;K;z)$ of μ' for the "annulus" Ω is defined as in Theorem 1.6 (with obvious alterations) and with (1.18) replaced by

$$\int_K \log D(z)d\nu \text{ is real}$$

Having defined $D(d\mu;K;z)$, we observe that the relation (6.12) implies that the zeros of $\{B_n^*\}_{n=1}^\infty$ have no limit points outside K. Hence the method of the proof of Theorem 1.7 applies and we obtain

Theorem 6.2 *Assume the conditions of Theorem 6.1. Let B_n^* denote a Blaschke product that realizes the infimum in (6.1), for $p = 2$. Let u be the equilibrium potential for K defined by (6.2) and let v be its conjugate. Then*

$$\lim_{n\to\infty} e^{(u(z)+iv(z))n} B_n^*(z) = \frac{G(\mu')}{D(d\mu;K;z)},$$

uniformly on compact subsets of Ω.

The details of the proof as well as some generalizations will be given in a future paper.

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