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An extension of a row convergence theorem for vector Padé approximants

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Abstract

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A row convergence theorem of de Montessus' type is established for vector Padé approximants to a vector-valued meromorphic function f. While in a previous paper the authors established such a theorem for the case when f has simple poles, the essential feature of the present paper is to treat the situation when f has multiple poles.

Keywords: Vector Padé approximant, rational approximation, de Montessus, row convergence, multiple pole.

1. Introduction

We are concerned with approximation of vector functions defined by their power series. We will assume that our vector function $f(z)$ is analytic at the origin, so that it has a Maclaurin expansion

$$f(z) = c_0 + c_1 z + \cdots \quad + c_n z^n + \tag{1.1}$$

where $c_i \in \mathbb{C}^d$, $i = 0, 1, \ldots$. The series (1.1) converges in a neighbourhood of the origin and $f : \mathbb{C} \to \mathbb{C}^d$. Rational fractions of the form

$$r(z \quad \frac{p(z)}{q(z)} \tag{1.2}$$

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normally exist and approximate $\boldsymbol{f}(z)$ in the sense that

$$\frac{\boldsymbol{p}(z)}{q(z)}=\boldsymbol{f}(z)+\mathrm{O}(z^{n+1}$$

$$\partial\{\boldsymbol{p}\}\leqslant n, \qquad \partial\{q\}=2k,$$

$$q(z)\mid \boldsymbol{p}(z)\cdot\boldsymbol{p}^*(z). \tag{1.5}$$

The vector Padé approximant $\boldsymbol{r}(z)$ is normally uniquely defined by (1.1)–(1.5) and the asterisk * denotes complex conjugation [3]. These vector approximants (previously called generalised inverse, vector-valued Padé approximants, or GIPAs) share many of the properties of Padé approximants:

(i) If all the coefficients $\{\boldsymbol{c}_i\}$ have the same direction, i.e.,

$$\boldsymbol{c}_i=\boldsymbol{\lambda} d_i, \quad i=0,1,2,\ldots,$$

with $\boldsymbol{\lambda}$ fixed, $\boldsymbol{\lambda}\in\mathbb{C}^d$, $d_i\in\mathbb{C}$, and $r(z)$ is an ordinary Padé approximant of $\Sigma_{i=0}^{\infty}d_iz^i$, then $\boldsymbol{\lambda}r(z)$ is the vector Padé approximant of equivalent type for $\boldsymbol{f}(z)$.

(ii) Identical entries in the table of vector Padé forms occur in blocks which are exactly square [6].

(iii) Entries in a normal table of vector Padé approximants can be constructed pointwise using Wynn's vector ϵ-algorithm based on Moore–Penrose generalised inverses [5,16].

In this paper, we establish a row convergence theorem which is analogous to de Montessus' theorem for ordinary Padé approximants. De Montessus' theorem asserts that a particular row (fixed denominator degree) sequence of Padé approximants converges to a given meromorphic function in a disk punctured by the poles of the function [2,15]. To formulate mathematically the developments of these ideas, we begin by defining

$$D_\mu:=\{z:\ |z\ <\mu$$

as a disk of arbitrary radius μ. In our previous paper [10] (see also the Erratum at the end of this paper), we considered convergence of a row sequence of vector Padé approximants to

$$\boldsymbol{f}(z)\quad \frac{\boldsymbol{g}(z)}{Q(z)}, \tag{1.7}$$

where $\boldsymbol{g}(z)$ is analytic in a given disk D_ρ and $Q(z)$ is a polynomial whose zeros $\{\zeta_i\}_{i=1}^{\nu}$ have respective multiplicity m_i, and where

$$0<|\zeta_i|<\rho, \quad i=1,2,\ldots,\nu.$$

In the *first* row convergence theorem of [10], we discussed the case in which, inter alia, $Q(z)$ is a real polynomial of the form

$$Q(z)=\prod_{i=1}^{\nu}(z-\zeta_i)^{m_i}$$

and

$$\boldsymbol{g}(\zeta_i)\cdot\boldsymbol{g}^*(\zeta_i)\neq 0, \quad i=1,2,\quad,\nu. \tag{1.10}$$

We note that this theorem admits the case of higher-order poles, expressed in (1.9), as originally stated and under the distinctive condition (1.10). An example of a vector function of this kind is the generating function of Gauss–Seidel iterates, occurring in the iterative solution of a system of real linear equations [4].

It is our *second* row convergence theorem [10, Theorem 4.1] which is generalised here: we lift the earlier restriction to simple poles. We retain the representation

$$f(z) = \frac{g(z)}{Q(z)}$$

in (1.7), (1.8), but assume that (1.9) and (1.10) are *replaced* by

$$Q(z) := \prod_{i=1}^{\nu} (z - \zeta_i)^{m_i} (z \quad \zeta_i^*)^{m_i} \tag{1.11}$$

$$\left.\begin{aligned} &g^{(s)*}(\zeta_i) = \mathbf{0}, \quad s = 0, 1, \ldots, m_i - 1, \\ &g(\zeta_i) \neq \mathbf{0}, \end{aligned}\right\} \quad i = 1, 2, \quad , \nu \tag{1.12}$$

A characteristic property of (1.11), (1.12) is that

$$h(z) := \frac{g(z) \cdot g^*(z)}{Q(z)} \tag{1.13}$$

is real and analytic in $z < \rho$. Note the contrast between this property and (1.10). A vector whose components are the (complex-valued) response functions taken at different points on a linearly damped vibrating Newtonian system, as a function of angular frequency, is an example of this kind of vector function [14]. If $f(z)$ has the representation in (1.7), (1.11), (1.12) and

$$2k = \partial\{Q\}, \tag{1.14}$$

then our main result is that the row sequence of vector Padé approximants of type $[n/2k]$ converges to $(f(z), f^*(z))$, as detailed in Theorem 1 below.

Our proof turns on the requirement that the determinant of a certain confluent Gram–Cauchy matrix be nonzero, where the underlying Cauchy matrix does not have to be positive definite. A generalisation of the Schur product theorem underpins the necessary theory. The key property is that the Cauchy matrix is diagonally signed (complementary), and this holds for Hermitian Cauchy matrices formed from distinct elements [7]. Secondly, it was necessary to define confluent Cauchy matrices and show that they too are normally diagonally signed [8]. Thirdly, certain determinantal bounds for Cauchy matrices were established which allow generalisation of the property of diagonal signature of Hermitian Gram–Cauchy matrices to be extended to confluent Hermitian Gram–Cauchy matrices as well [9]. It was our conviction of the validity of Theorem 1 which led to these developments in matrix analysis [11].

2. Results

Theorem 1. *Let $f(z)$ be a vector function which is analytic in the disk D_ρ except for precisely ν poles $\zeta_1, \zeta_2, \ldots, \zeta_\nu$, having total multiplicity k. We assume that $\zeta_1, \zeta_1^*, \zeta_2, \zeta_2^*, \ldots, \zeta_\nu, \zeta_\nu^*$ are all distinct and that*

$$0 < |\zeta_i < \rho, \quad i = 1, 2, \ldots, \nu. \tag{2.1}$$

Let the pole of $f(z)$ *at* ζ_i *have multiplicity* m_i, *and define*

$$Q(z) := \prod_{i=1}^{\nu} (z-\zeta_i)^{m_i}(z-\zeta_i^*)^{m_i}$$

$$D_\rho^- := D_\rho - \bigcup_{i=1} \{\zeta_i, \zeta_i^*\}. \tag{2.3}$$

Let $(P_n(z), P_n^*(z); Q_n(z))$ *be vector Padé polynomials of type* $[n/2k]$ *for* $(f(z), f^*(z))$. *Let* E *be any compact subset of* $\mathbb{C}$, *and for any given* $\mu \in (0, \rho)$, *let* K *be any compact subset of* $D_\mu \backslash \bigcup_{i=1}^{\nu}\{\zeta_i, \zeta_i^*\}$. *Then*

$$\lim_{n\to\infty} \frac{P_n(z)}{Q_n(z)} = f(z), \quad z \in D_\rho^-, \tag{2.4}$$

and the rate of convergence of these approximants is governed by

$$\limsup_{n\to\infty} \left\| f - \frac{P_n}{Q_n} \right\|_K^{1/n} \leqslant \frac{\mu}{\rho}. \tag{2.5}$$

Additionally, if $Q_n(z)$ *is normalised to have leading coefficient unity*,

$$\lim_{n\to\infty} Q_n(z) = Q(z), \quad z \in \mathbb{C}, \tag{2.6}$$

and the rate of convergence is governed by

$$\limsup_{n\to\infty} \| Q_n - Q \|_E^{1/n} \leqslant \max_{1\leqslant i\leqslant \nu} \frac{|\zeta_i|}{\rho}.$$

Observations. By hypothesis, we may express

$$f(z) - \frac{a(z)}{b(z)}$$

with

$$b(z) := \prod_{i=1}^{\nu} (z-\zeta_i)^{m_i}.$$

The numerator $a(z)$ is analytic in $|z| < \rho$, and $a(\zeta_i) \neq 0$, $i = 1, \ldots, \nu$. We take

$$g(z) = a(z)b^*(z), \qquad h(z) = a(z) \cdot a^*(z). \tag{2.8}$$

Equations (2.8) are fully consistent with (1.11)–(1.13) and (2.2). The theorem asserts the existence (for n sufficiently large) and convergence of vector Padé approximants $(P_n(z), P_n^*(z))/Q_n(z)$ of type $[n/2k]$ for the vector function

$$f^{\mathrm{E}}(z) := \frac{(g(z), g^*(z))}{Q(z)},$$

where f^{E}: $\mathbb{C} \to \mathbb{C}^{2d}$. In fact, it also implies that $P_n(z) \to g(z)$ in D_ρ. The proof below is substantially self-contained; for brevity, a few details given in [10] are not repeated here.

Notation. The order of magnitude of a sequence $\{x_n\}$ is denoted by the symbol $\Theta(\alpha^n)$. statement that

$$x_n = \Theta(\alpha^n)$$

means that $\limsup_{n\to\infty} |x_n|^{1/n} \leqslant \alpha$.

Truncation of Maclaurin series between orders l, m inclusively is denoted by

$$\left[\sum_{j=0}^{\infty} c_j z^j\right]_l^m := \sum_{j=l}^{m} c_j z^j.$$

Proof. We use the nontrivial polynomials $q_n(z)$ and

$$\boldsymbol{p}_n^{\mathrm{E}}(z) = (\boldsymbol{p}_n(z),\ \boldsymbol{p}_n^*(z)) \tag{2.9}$$

which constitute the vector Padé polynomials of type $[n/2k]$ for $\boldsymbol{f}^{\mathrm{E}}(z)$. To construct $q_n(z)$, it is natural to begin with a system of homogeneous equations for its coefficients $q_{n,i}$, where $q_n(z) = \sum_{i=0}^{2k} q_{n,i} z^i$. If it happens that

$$q_{n,0} = q_{n,1} = \cdots = q_{n,\lambda_n - 1} = 0 \neq q_{n,\lambda_n}$$

for some $\lambda_n \geqslant 0$, Graves-Morris and Jenkins [6, Theorem 3.2] have shown that it is possible to derive vector Padé polynomials $(\boldsymbol{p}_n(z),\ q_n(z))$ and a polynomial $\pi_{2n-2k}(z)$ of degree $2n-2k$ at most for which

$$\boldsymbol{p}_n(z) - \boldsymbol{f}(z) q_n(z) = \mathrm{O}(z^{n+\sigma_n+1}), \tag{2.10}$$

$$\boldsymbol{p}_n(z) \cdot \boldsymbol{p}_n^*(z) = \pi_{2n-2k}(z) q_n(z), \tag{2.11}$$

where the integer σ_n is defined as

$$\sigma_n := \left[\tfrac{1}{2}(\lambda_n + 1)\right].$$

From (2.10) and (2.11), we have

$$\begin{aligned}
&(\boldsymbol{p}_n(z) - \boldsymbol{f}(z) q_n(z)) \cdot (\boldsymbol{p}_n^*(z) - \boldsymbol{f}^*(z) q_n(z)) \\
&\quad = q_n(z)\{\pi_{2n-2k}(z) - \boldsymbol{p}_n(z) \cdot \boldsymbol{f}^*(z) - \boldsymbol{p}_n^*(z) \cdot \boldsymbol{f}(z) + q_n(z) \boldsymbol{f}(z) \cdot \boldsymbol{f}^*(z)\} \\
&\quad = \mathrm{O}(z^{2(n+\sigma_n+1)}).
\end{aligned} \tag{2.12}$$

By multiplying (2.12) by $Q(z)/q_n(z)$ and omitting some details given previously [10], we have

$$\begin{aligned}
Q(z)\pi_{2n-2k}(z) - \boldsymbol{p}_n(z) \cdot \boldsymbol{g}^*(z) - \boldsymbol{p}_n^*(z) \cdot \boldsymbol{g}(z) + h(z) q_n(z) &= \mathrm{O}(z^{2n+2\sigma_n+2-\lambda_n}) \\
\Rightarrow\ &= \mathrm{O}(z^{2n+1}).
\end{aligned} \tag{2.13}$$

We can now use Hermite's formula to give precise form to the right-hand side of (2.13). We find that

$$Q(z)\pi_{2n-2k}(z) - \boldsymbol{p}_n(z) \cdot \boldsymbol{g}^*(z) - \boldsymbol{p}_n^*(z) \cdot \boldsymbol{g}(z) + h(z) q_n(z) = A_n(z) + C_n(z), \tag{2.14}$$

where, for any $\rho' < \rho$ and $z \in D_{\rho'}$,

$$A_n(z) := -\frac{z^{2n+1}}{2\pi \mathrm{i}} \int_{|t|=\rho'} \{\boldsymbol{p}_n(t) \cdot \boldsymbol{g}^*(t) + \boldsymbol{p}_n^*(t) \cdot \boldsymbol{g}(t)\} \frac{\mathrm{d}t}{(t-z)t^{2n+1}} \tag{2.15}$$

and

$$C_n(z) := \frac{z^{2n+1}}{2\pi \mathrm{i}} \int_{|t|=\rho'} \boldsymbol{a}(t) \cdot \boldsymbol{a}^*(t) q_n(t) \frac{\mathrm{d}t}{(t-z)t^{2n+1}}. \tag{2.16}$$

Next, we introduce the Hermite–Lagrange basis B for polynomials of degree $2k-1$ at most,

$$B := \{ B_{j,s}(z),\ B_{j,s}^*(z),\ j=1,2,\ldots,\nu,\ s=0,1,\ldots,m_j-1 \}$$

in which

$$\left.\begin{aligned} B_{j,s}^{(r)}(\zeta_l) &= B_{j,s}^{(r)*}(\zeta_l^*) = \delta_{jl}\delta_{rs}, \\ B_{j,s}^{(r)*}(\zeta_l) &= B_{j,s}^{(r)}(\zeta_l^*) = 0, \end{aligned}\right\} \quad \begin{aligned} & j,\ l=1,2,\ldots,\nu, \\ & s=0,1,\ldots,m_j-1, \\ & r=0,1,\ldots,m_l-1. \end{aligned} \tag{2.17}$$

We can now express

$$q_n(z) = \sum_{l=1}^{\nu} \sum_{r=0}^{m_l-1} \{ q_n^{(r)}(\zeta_l) B_{l,r}(z) + q_n^{(r)}(\zeta_l^*) B_{l,r}^*(z) \} + c_n Q(z). \tag{2.18}$$

We choose a normalisation for $q_n(z)$ in which $c_n \geqslant 0$ and

$$c_n + 2 \sum_{l=1}^{\nu} \sum_{r=0}^{m_l-1} | q_n^{(r)}(\zeta_l) | = 1, \tag{2.19}$$

so that $\{ q_n(z) \}$ is uniformly bounded on any compact set in $\mathbb{C}$.

From (2.16), we find that

$$C_n^{(s)}(z) = \Theta\left(\left(\frac{|z|}{\rho} \right)^{2n} \right) \quad \text{for } |z| < \rho,\ s = 0,1,2,\ldots, \tag{2.20}$$

as an estimate of each sth derivative of $C_n(z)$. To obtain a similar result for $A_n(z)$, we substitute (2.18) into (2.10):

$$\begin{aligned} p_n^*(t) &= [q_n(t) f^*(t)]_0^n \\ &= c_n [Q(t) f^*(t)]_0^n + \sum_{l=1}^{\nu} \sum_{r=0}^{m_l-1} \{ q_n^{(r)}(\zeta_l) [B_{l,r}(t) f^*(t)]_0^n \\ &\qquad + q_n^{(r)}(\zeta_l^*) [B_{l,r}^*(t) f^*(t)]_0^n \}. \end{aligned} \tag{2.21}$$

Substituting (2.21) into (2.15), we find that for $|z| < \rho$

$$\begin{aligned} A_n(z) &= \sum_{l=1}^{\nu} \sum_{r=0}^{m_l-1} \{ q_n^{(r)}(\zeta_l) \hat{A}_n^{(l,r)}(z) + q_n^{(r)}(\zeta_l^*) \hat{A}_n^{(l,r)*}(z) \} \\ &\quad - c_n \frac{z^{2n+1}}{2\pi i} \int_{|t|=\rho'} \{ \mathbf{g}(t) \cdot [Q(t) f^*(t)]_0^n \\ &\qquad + \mathbf{g}^*(t) \cdot [Q(t) f(t)]_0^n \} \frac{dt}{(t-z) t^{2n+1}}, \end{aligned} \tag{2.22}$$

where

$$\begin{aligned} \hat{A}_n^{(l,r)}(z) &= \frac{z^{2n+1}}{2\pi i} \int_{|t|=\rho'} \{ \mathbf{g}(t) \cdot [B_{l,r}(t) f^*(t)]_0^n \\ &\qquad + \mathbf{g}^*(t) \cdot [B_{l,r}(t) f(t)]_0^n \} \frac{dt}{(t-z) t^{2n+1}}. \end{aligned} \tag{2.23}$$

From (2.23), we can estimate

$$\hat{A}_n^{(l,r)}(z) = \Theta\left[\left[\frac{|z|^2}{\rho|\zeta_l|}\right]^n\right], \quad r = 0, 1, \ldots, m_l - 1 \tag{2.24}$$

Unless $q_n^{(r)}(\zeta_l^*) = 0$, and this case is easily included, (2.22) becomes

$$A_n(z) = \sum_{l=1}^{\nu} \sum_{r=0}^{m_l-1} q_n^{(r)}(\zeta_l^*) A_n^{(l,r)}(z) + \Theta\left(\left(\frac{|z|}{\rho}\right)^{2n}\right) \tag{2.25}$$

where

$$A_n^{(l,r)}(z) := \hat{A}_n^{(l,r)*}(z) + \hat{A}_n^{(l,r)}(z) \frac{q_n^{(r)}(\zeta_l)}{q_n^{(r)}(\zeta_l^*)} \tag{2.26}$$

Again, we estimate

$$A_n^{(l,r)}(z) = \Theta\left[\left[\frac{|z|^2}{\rho|\zeta_l|}\right]^n\right] \tag{2.27}$$

By differentiating (2.22), we can similarly deduce step-by-step [10] that

$$\left[\frac{\mathrm{d}}{\mathrm{d}z}\right]^s \left\{z^{-n-1} A_n^{(l,r)}(z)\right\} = \Theta\left[\left[\frac{|z|}{\rho|\zeta_l|}\right]^n\right], \quad s \quad 0, 1, 2, \tag{2.28}$$

For the next stage of the proof, we multiply each side of equation (2.14) by z^{-n-1}. Then we evaluate it, and each of its first $m_j - 1$ derivatives, at each ζ_j. The result is tersely summarised as

$$L_{j,s} = R_{j,s}, \quad j = 1, 2, \ldots, \nu, \ s = 0, 1, \ldots, m_j - 1, \tag{2.29}$$

if we define

$$L \quad := \left\{\left(Q\pi_{2n-2k} - \boldsymbol{p}_n \cdot \boldsymbol{g}^* - \boldsymbol{p}_n^* \cdot \boldsymbol{g} + q_n \boldsymbol{g} \cdot \boldsymbol{f}^*\right) z^{-n-1}\right\}^{(s)}(\zeta_j) \tag{2.30}$$

and

$$R_{j,s} := \left\{\left(A_n(z) + C_n(z)\right) z^{-n-1}\right\}^{(s)}(\zeta_j). \tag{2.31}$$

Now we show how equations (2.29)–(2.31) are manipulated to yield the estimates given in (2.44) below for $q^{(r)}(\zeta_l)$. From (2.30), we have

$$\begin{aligned} L_{j,s} &= \left\{\boldsymbol{g} \cdot (\boldsymbol{f}^* q_n - \boldsymbol{p}_n^*) z^{-n-1}\right\}^{(s)}(\zeta_j) \\ &= \left[\frac{\partial}{}\right]^s \left\{\frac{\boldsymbol{g}(z)}{2\pi\mathrm{i}} \int_{\Gamma_j} \boldsymbol{f}^*(t) \tilde{q}_n(t) \, \frac{\mathrm{d}t}{t \quad z}\right\}\Bigg|_{z=\zeta_j} \end{aligned} \tag{2.32}$$

where

$$\tilde{q}_n(t) := t^{-n-1} q_n(t), \tag{2.33}$$

and Γ_j is a simple contour enclosing 0, ζ_j but no singularities of $\boldsymbol{f}^*(t)$. We expand the contour in (2.32) to become $|t| = \rho'$ with $\max_l |\zeta_l| < \rho' < \rho$, and deduct the contributions of all the extra poles of the integrand thereby incorporated. Using the leading term expansion,

$$Q(t) = (t \quad \zeta_l^*)^{m_l} \; \frac{Q^{(m_l)}(\zeta_l^*)}{m_l!} + \text{higher-order terms},$$

we obtain

$$L_{j,s} = -\left[\frac{\partial}{\partial z}\right]^s \left\{ \mathbf{g}(z) \cdot \sum_{l=1}^{\nu} \frac{m_l}{Q^{(m_l)}(\zeta_l^*)} \left[\left[\frac{\partial}{\quad}\right]^{m_l-1} \frac{\tilde{q}_n(t)\mathbf{g}^*(t)}{t \quad z} \right]_{t=\zeta_l^*} \right\} \Bigg|_{z=\zeta_j} + \Theta(\rho^{-n})$$

$$\left[\frac{\partial}{\partial z}\right]^s \left\{ \mathbf{g}(z) \quad \sum_{l=1}^{\nu} \frac{m_l}{Q^{(m_l)}(\zeta_l^*)} \sum_{r=0}^{m_l-1} \begin{pmatrix} m_l - 1 \\ r \end{pmatrix} \tilde{q}_n^{(m_l-1-r)}(\zeta_l^* \right.$$

$$\left. \times \left[\left[\frac{\partial}{\partial t}\right]^r \frac{\mathbf{g}^*(\;}{t - \;} \right]_{t=\zeta_l^*} \right\} \Bigg|_{z=\zeta_j} + \Theta(\rho^{-n}) \tag{2.34}$$

We are led to define

$$\frac{m_l!}{(m_l \qquad 1)!} \frac{\tilde{q}_n^{(m_l-r-1)}(\zeta_l^*)}{Q^{(m_l)}(\zeta_l^*)} \tag{2.35}$$

and

$$\hat{M}_{s,r}^{(j,l)} := \frac{1}{r!s!} \left[\left[\frac{\partial}{\partial z}\right]^s \left[\frac{\partial}{\quad}\right] \frac{\mathbf{g}(z)\cdot\mathbf{g}^*(t)}{t - z} \right]_{t=\zeta_l^*, z=\zeta_j} \tag{2.36}$$

so that (2.34) takes the compact form

$$L_{j,s} = -s! \sum_{l=1}^{\nu} \sum_{r=0}^{m_l-1} \hat{M}_{s,r}^{(j,l)} \hat{q}_n^{(l,r)} + \Theta(\rho^{-n}), \quad j = 1, 2, \ldots, \nu, \; s \quad 0, 1, \ldots, m_j - 1 \tag{2.37}$$

A side effect of introducing parameters $\hat{q}_n^{(l,r)}$ instead of $q_n^{(r)}(\zeta_l^*)$ is that (2.25) and (2.28) must be re-expressed in terms of $\hat{q}_n^{(l,r)}$. It is straightforward but tedious to show that

$$\left[\frac{\mathrm{d}}{\quad}\right]^s \left\{ z^{-n-1} A_n(z) \right\} = \sum_{l=1}^{\nu} \sum_{r=0}^{m_l-1} \hat{q}_n^{(l,r)} \epsilon_A^{(l,r,s)}(z) + \Theta\left(\left(\frac{|z|}{\rho^2} \right)^n \right) \qquad s = 0, 1, 2, \tag{2.38}$$

where

$$\epsilon_A^{(l,r,s)}(z) := Q^{(m_l)}(\zeta_l^*) \frac{(m_l - r - 1)!}{m_l!} \sum_{i=0}^{r} \begin{pmatrix} m_l \quad 1 \quad i \\ m_l \quad 1 \quad r \end{pmatrix} \frac{(n+1)!}{(n+1+i-r)!} (\zeta_l^*)^{n+1+i-r}$$

$$\times \left\{ z^{-n-1} A^{(l, m_l-1-i)}(z) \right\}^{(s)} \tag{2.39}$$

As in (2.26)–(2.28), we use (2.39) to estimate

$$\epsilon_A^{(l,r,s)}(z) = \Theta\left(\left(\frac{|z|}{\rho} \right)^n \right). \tag{2.40}$$

We use (2.20) and (2.38) to simplify (2.31) as

$$R_{j,s} = \sum_{l=1}^{\nu} \sum_{r=0}^{m_l-1} \hat{q}_n^{(l,r)} \epsilon_A^{(l,r,s)}(\zeta_j) + \Theta(\rho^{-n}) \tag{2.41}$$

We substitute (2.37) and (2.41) into (2.29) to obtain the basic equations for $\hat{q}_n^{(l,r)}$:

$$\sum_{l=1}^{\nu} \sum_{r=0}^{m_l-1} \left(\hat{M}_{s,r}^{(j,l)} + \frac{\epsilon_A^{(l,r,s)}(\zeta_j)}{s!} \right) \hat{q}_n^{(l,r)} = \Theta(\rho^{-n}). \tag{2.42}$$

The coefficients $\hat{M}_{s,r}^{(j,l)}$ which occur in (2.42) are defined in (2.36) and constitute a block matrix $\hat{M}$. If we set $z = \mathrm{i}x$, $t = -\mathrm{i}y$, $\zeta_j = \mathrm{i}\eta_j$ and $\boldsymbol{g}(z) = \boldsymbol{u}(x)$ in (2.36), it becomes

$$\hat{M}_{s,r}^{(j,l)} = (-\mathrm{i})^s \mathrm{i}^{r+1} M_{s,r}^{(j,l)},$$

where

$$M_{s,r}^{(j,l)} := \frac{1}{r!s!} \left[\frac{\partial}{\partial x} \right]^s \left[\frac{\partial}{\partial y} \right] \frac{\boldsymbol{u}(x) \cdot \boldsymbol{u}^*(y)}{x+y} \bigg|_{x=\eta_j,\, y=\eta_l^*} \tag{2.43}$$

The entries $M_{s,r}^{(j,l)}$ in (2.43) constitute a block matrix M, which is a confluent Hermitian Gram–Cauchy matrix. It is known to be nonsingular provided only that $\eta_j + \eta_l^* \neq 0$ for all j, l and that $\boldsymbol{u}(\eta_j) \neq \boldsymbol{0}$ for all j [9]. This is the case, and hence the matrices M and $\hat{M}$ are nonsingular. It now follows from (2.43) that

$$\hat{q}_n^{(l,r)} = \Theta(\rho^{-n})$$

and from (2.33), (2.35) that

$$q_n^{(r)}(\zeta_l) = \Theta\left(\left(\frac{|\zeta_l|}{\rho} \right) \right) \qquad l = 1, 2, \quad , \nu, \ r = 0, 1, \quad , m_l - 1 \tag{2.44}$$

Equations (2.19), (2.44) imply that $c_n \to 1$ and that we can define

$$Q_n(z) := \frac{q_n(z)}{c_n} \tag{2.45}$$

for n sufficiently large. The convergence results (2.6) and (2.7) for $Q_n(z)$ now follow, and the remainder of the proof is identical to that of [10]. □

3. Conclusion

We have shown that the second row convergence theorem for vector Padé approximants for a vector function $\boldsymbol{f}$ includes in a natural way the case in which $\boldsymbol{f}$ has multiple poles. This result goes some way to proving the conjecture made by Graves-Morris and Saff [10] which, if true, would amalgamate the two row convergence theorems.

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Erratum

To: Row convergence theorems for generalised inverse vector-valued Padé approximants [10].

On p.63, Abstract, line 4, "de Montessus de Ballore", and subsequently on p.66.

On p.64, line 19, "McLeod", and subsequently on p.85, [7];
line -4, "$\theta(\alpha^n)$" should be "$\Theta(\alpha^n)$" and throughout the paper.

On p.71, equation (3.10) should be

$$\limsup_{n\to\infty} \| Q_n - Q^2 \|_E^{1/n} \leqslant \max_{1\leqslant i\leqslant k} \frac{|z_i|}{\rho}$$

equation (3.13) should be

$$q_{\cdot}(z)\{\pi_{2n-2k}(z) - \boldsymbol{p}_n(z)\cdot\boldsymbol{f}^*(z) \quad \boldsymbol{p}_n^*(z)\cdot\boldsymbol{f}(z) + q_n(z)\,\boldsymbol{f}(z)\cdot\boldsymbol{f}^*(z)\}$$
$$- \mathrm{O}(z^{2(n+\sigma_n+1)})$$