Some Characterization Theorems for Measures Associated with Orthogonal Polynomials on the Unit Circle

K. Pan and E. B. Saff

Dedicated to G. G. Lorentz on the occasion of his 80th Birthday

Abstract. Let $\{\phi_n\}$ be a system of polynomials orthonormal on the unit circle with respect to a measure $d\mu$. Regular measures, log integrable measures and analytic measures are characterized in terms of certain integrals involving the corresponding orthogonal polynomials on the unit circle.

§1 Introduction

Throughout this paper we assume that $d\mu$ is a finite positive Borel measure on the unit circle $\partial\Delta := \{z \in \mathbf{C} : \ |z| = 1\}$ whose support consists of infinitely many points. Let $\phi_n(z) = \phi_n(d\mu, z) := \kappa_n z^n + \cdots \in \mathcal{P}_n, \ \kappa_n > 0, \ n = 0, 1, \cdots,$ be the n-th orthonormal polynomial with respect to $d\mu$, that is,

$$\frac{1}{2\pi}\int_{\partial\Delta} \phi_m(z)\overline{\phi_n(z)}d\mu = \delta_{m,n}, \quad m, n = 0, 1, 2, \cdots.$$

Let $\mu = \mu_a + \mu_s$ be its canonical decomposition into the absolutely continuous and the singular parts (with respect to Lebesgue measure on the circle). We denote by $\mu'(\theta)$ the Radon-Nikodym derivative of μ_a with respect to $d\theta$. Then $\mu' \in L^1[0, 2\pi)$ and $\mu'(\theta) \geq 0$ a.e. and we define its geometric mean $G(\mu')$ by

$$G(\mu') := \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log\mu'(\theta)d\theta\right\}.$$

If $\log\mu' \in L^1[0, \ 2\pi)$ one can define the Szegö function

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$$D(z) = D(d\mu, z) := \exp\left\{\frac{1}{4\pi}\int_0^{2\pi} \log \mu'(\theta)\frac{e^{i\theta}+z}{e^{i\theta}-z}d\theta\right\}, \quad (|z| < 1);$$

it is known that (cf. [13, p. 276]) $D \in H^2(|z| < 1)$ and for almost every $\theta \in [0, 2\pi)$,

$$\lim_{r \nearrow 1} D(d\mu, re^{i\theta}) =: D(d\mu, e^{i\theta})$$

exists and $|D(d\mu, e^{i\theta})|^2 = \mu'(\theta)$ for almost every $\theta \in [0, 2\pi)$. If $\log \mu' \notin L^1[0, 2\pi)$ we define $D(d\mu, z) \equiv 0$.

Following Stahl and Totik [12] we call $d\mu$ a regular measure with respect to $\partial\Delta$ if $\lim_{n\to\infty} \kappa_n^{1/n} = 1$. By a theorem of Rahmanov (cf. [11] and [6]), we know that $\mu' > 0$ a.e. in $[0, 2\pi)$ implies that $d\mu$ is a regular measure.

We now state two characterization theorems that motivated the results of this paper. The first is due to P. Nevai.

Theorem 1.1 ([9]) *Let* $\Phi_n(z) := \phi_n(z)/\kappa_n$, $\phi_n^*(z) := z^n\overline{\phi_n(1/\overline{z})}$. *Then*

$$\lim_{n\to\infty} \Phi_n(0) = 0 \text{ iff } \lim_{n\to\infty} \inf_{l\geq 1} \int_0^{2\pi} \left|\frac{|\phi_n^*(z)|^2}{|\phi_{n+l}^*(z)|^2} - 1\right| d\theta = 0, \quad z = e^{i\theta}.$$

The next result, due to P. Nevai and Li and Saff, characterizes measures with positive derivative.

Theorem 1.2. ([7] and [4]) $\mu' > 0$ *a.e. in* $[0, 2\pi)$ *if and only if*

$$\lim_{n\to\infty} \sup_{l\geq 0} \int_0^{2\pi} \left|\frac{|\phi_n^*(z)|^2}{|\phi_{n+l}^*(z)|^2} - 1\right| d\theta = 0, \quad z = e^{i\theta}.$$

The main purpose of this paper is to obtain similar characterization theorems for regular measures, log integrable measures, and analytic measures.

§2 Main Results

In this section we state our main results. The proofs are given in Section 3.

Theorem 2.1. *The following statements are equivalent:*

(a) $d\mu$ *is regular with respect to* $\partial\Delta$,

(b) $\displaystyle\lim_{n\to\infty} \sup_{l\geq 0} \left[1 - (\kappa_n/\kappa_{n+l})^{1/n+l}\right] = 0,$

(c) $\displaystyle\lim_{n\to\infty} \sup_{l\geq 0} \int_0^{2\pi} \left|[\phi_n^*(z)/\phi_{n+l}^*(z)]^{1/n+l} - 1\right| d\theta = 0,$

(d) $\displaystyle\lim_{n\to\infty} \sup_{l\geq 0} \int_0^{2\pi} \left|[\phi_n^*(z)/\phi_{n+l}^*(z)]^{1/n+l} - 1\right|^2 d\theta = 0,$

(e) $\displaystyle\lim_{n\to\infty} \sup_{l\geq 0} \int_0^{2\pi} \left|[\phi_n^*(z)/\phi_{n+l}^*(z)]^{2/n+l} - 1\right| d\theta = 0,$

where $z = e^{i\theta}$ and the branch of the $(n+l)$-th root is taken to be positive at $z = 0$.

We remark that Li, Saff, and Sha [5] have shown that (a) is also equivalent to the condition $\lim_{n\to\infty} \|\phi_n\|_{\partial\Delta}^{1/n} = 1$, where $\|\cdot\|_{\partial\Delta}$ denotes the sup norm on $|z| = 1$.

The next result characterizes measures belonging to the Szegö class.

Theorem 2.2. *The following statements are equivalent:*

$$\text{(a)} \log \mu' \in L^1[0,\ 2\pi),$$
$$\text{(b)} \lim_{n\to\infty} \sup_{l\geq 0} (1 - \kappa_n/\kappa_{n+l}) = 0,$$
$$\text{(c)} \lim_{n\to\infty} \sup_{l\geq 0} \int_0^{2\pi} |\phi_n^*(z)/\phi_{n+l}^*(z) - 1|\, d\theta = 0,$$
$$\text{(d)} \lim_{n\to\infty} \sup_{l\geq 0} \int_0^{2\pi} |\phi_n^*(z)/\phi_{n+l}^*(z) - 1|^2\, d\theta = 0,$$
$$\text{(e)} \lim_{n\to\infty} \sup_{l\geq 0} \int_0^{2\pi} \left|[\phi_n^*(z)/\phi_{n+l}^*(z)]^2 - 1\right| d\theta = 0,$$

where $z = e^{i\theta}$.

Next, we are concerned with the characterization of analyticity on $|z| = 1$ for $[D(d\mu,\ z)]^{-1}$. Let

$$\varepsilon_n := 1 - \frac{\kappa_n}{\kappa_{n+1}},\quad n = 0, 1, \cdot$$

It is well known that $\log \mu' \in L^1[0,\ 2\pi)$ if and only if $\lim_{n\to\infty} \kappa_n =: \kappa < \infty$ (cf. [13, p. 291]), which, in turn, is equivalent to

$$\sum_{n=0}^{\infty} \varepsilon_n < \infty.$$

Furthermore, if this series converges like a geometric series, then $[D(d\mu, z)]^{-1}$ is analytic on $|z| = 1$. More precisely, for the quantities

$$\frac{1}{R_1} := \limsup_{n\to\infty} \left(1 - \frac{\kappa_n}{\kappa_{n+1}}\right)^{1/2n}$$

$$\frac{1}{R_2} := \limsup_{n\to\infty} \left[\int_0^{2\pi} \left|\frac{|\phi_n^*(z)|^2}{|\phi_{n+1}^*(z)|^2} - 1\right| d\theta\right]^{1/n}$$

and

$$R_3 := \sup\{r : D^{-1}(d\mu, z) \text{ is analytic on } |z| < r\};$$

we prove the following result.

Theorem 2.3. *If $d\mu$ is a finite positive Borel measure on $\partial\Delta$ with infinite support, then $R_1 = R_2$. Moreover, if there is a $j \in \{1,2,3\}$ such that $R_j > 1$, then $R_1 = R_2 = R_3$.*

§3 Proofs of Theorems

The proofs require the following lemmas.

Lemma 3.1. (cf. [6] and [10]) *For every system of orthogonal polynomials on the unit circle,*

$$|\Phi_{n+1}(0)| \le \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{|\phi_n^*(z)|^2}{|\phi_{n+1}^*(z)|^2} - 1\right| d\theta, \quad z = e^{i\theta}.$$

Proof: Here we present a short proof. Since all zeros of $\phi_n(z)$ lie in $|z| < 1$ (cf. [13, p. 292]), the function

$$\frac{z\phi_n(z)\phi_n^*(z) - \phi_{n+1}(z)\phi_{n+1}^*(z)}{\phi_{n+1}^{*2}(z)}$$

is analytic on $|z| \le 1$ and takes the value $-\Phi_{n+1}(0)$ at $z = 0$. Thus from Cauchy's integral formula, we have

$$\begin{aligned}
&|\Phi_{n+1}(0)| \\
&\le \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{z\phi_n(z)\phi_n^*(z) - \phi_{n+1}(z)\phi_{n+1}^*(z)}{\phi_{n+1}^{*2}(z)}\right| d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{|\phi_n^*(z)|^2}{|\phi_{n+1}^*(z)|^2} - 1\right| d\theta. \quad \blacksquare
\end{aligned}$$

Lemma 3.2. *Let $\{f_{n,l}\}_{n,l=0}^{\infty}$ be a sequence of functions in $H^2(|z| < 1)$ satisfying*

$$\frac{1}{2\pi}\int_0^{2\pi}|f_{n,l}(e^{i\theta})|^2 d\theta \le 1 \text{ and } f_{n,l}(0) \ge 0.$$

Then the following statements are equivalent:

$$\text{(a)} \lim_{n\to\infty}\sup_{l\ge 0}|f_{n,l}(0) - 1| = 0,$$

$$\text{(b)} \lim_{n\to\infty}\sup_{l\ge 0}\int_0^{2\pi}|f_{n,l}(z) - 1| d\theta = 0,$$

$$\text{(c)} \lim_{n\to\infty}\sup_{l\ge 0}\int_0^{2\pi}|f_{n,l}(z) - 1|^2 d\theta = 0,$$

$$\text{(d)} \lim_{n\to\infty}\sup_{l\ge 0}\int_0^{2\pi}|f_{n,l}^2(z) - 1| d\theta = 0,$$

where $z = e^{i\theta}$.

Proof: (a) $\Rightarrow$ (b): From Schwarz's inequality, we have

$$\begin{aligned}
&\left(\frac{1}{2\pi}\int_0^{2\pi} |f_{n,l}(z) - 1| d\theta\right)^2 \\
&\leq \frac{1}{2\pi}\int_0^{2\pi} |f_{n,l}(z) - 1|^2 d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} |f_{n,l}(z)|^2 d\theta - 2\mathrm{Re}\left\{\frac{1}{2\pi}\int_0^{2\pi} f_{n,l}(z) d\theta\right\} + 1 \\
&\leq 1 - 2f_{n,l}(0) + 1 = 2(1 - f_{n,l}(0)); \qquad (3.1)
\end{aligned}$$

thus if (a) holds, so does (b).

(b) $\Rightarrow$ (a): This follows immediately by applying Cauchy's integral formula.

Next notice that (a) $\Rightarrow$ (c) follows from (3.1) and that (c) $\Rightarrow$ (b) is trivial. Thus (c) $\Leftrightarrow$ (a).

(c) $\Rightarrow$ (d): Since

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} |f_{n,l}^2(z) - 1| d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} |f_{n,l}(z) - 1||f_{n,l}(z) + 1| d\theta \\
&\leq \left(\frac{1}{2\pi}\int_0^{2\pi} |f_{n,l}(z) - 1|^2 d\theta\right)^{1/2} \left(\frac{1}{2\pi}\int_0^{2\pi} |f_{n,l}(z) + 1|^2 d\theta\right)^{1/2} \\
&\leq \left(\frac{1}{2\pi}\int_0^{2\pi} |f_{n,l}(z) - 1|^2 d\theta\right)^{1/2} \left[\left(\frac{1}{2\pi}\int_0^{2\pi} |f_{n,l}(z)|^2 d\theta\right)^{1/2} + 1\right] \\
&\leq 2\left(\frac{1}{2\pi}\int_0^{2\pi} |f_{n,l}(z) - 1|^2 d\theta\right)^{1/2}
\end{aligned}$$

it follows that if (c) holds, so does (d).

(d) $\Rightarrow$ (c): Again, from Cauchy's integral formula and the fact that $0 \leq f_{n,l}(0) \leq 1$, we have

$$0 \leq 1 - f_{n,l}(0) \leq 1 - f_{n,l}^2(0) \leq \frac{1}{2\pi}\int_0^{2\pi} |f_{n,l}^2(z) - 1| d\theta,$$

and so (d) $\Rightarrow$ (a) $\Rightarrow$ (c). ∎

Proof of Theorem 2.1

(a) $\Rightarrow$ (b): If $d\mu$ is regular, then $\lim_{n\to\infty} \kappa_n^{1/n} = 1$ and so for any $\varepsilon \in (0,\ 1)$ there is an $N(\varepsilon)$ such that

$$(1-\varepsilon)^n \le \kappa_n \le (1+\varepsilon)^n, \quad n \ge N(\varepsilon).$$

From (3.2) we have

$$\left(\frac{\kappa_n}{\kappa_{n+l}}\right)^{1/n+l} \ge \frac{1-\varepsilon}{1+\varepsilon}, \quad \text{for } n \ge N(\varepsilon) \text{ and } l \ge 0.$$

Note that (cf. [2, p. 7])

$$\kappa_{n+1}^2 - \kappa_n^2 = |\phi_{n+1}(0)|^2, \quad n = 0, 1, 2, \cdots, \tag{3.4}$$

and so $\{\kappa_n\}_{n=0}^{\infty}$ is increasing. Thus, from (3.3), we have for $l \ge 0$ and $n \ge N(\varepsilon)$

$$0 \le 1 - \left(\frac{\kappa_n}{\kappa_{n+l}}\right)^{1/n+l} \le 1 - \frac{1-\varepsilon}{1+\varepsilon} < 2\varepsilon,$$

and so (b) follows.

(b) $\Rightarrow$ (a): Now assume that (b) holds. Then given an $\varepsilon \in (0,\ 1)$ we can find an integer $M(\varepsilon)$ such that, for any $l \ge 0$,

$$1 - \left(\frac{\kappa_M}{\kappa_{M+l}}\right)^{1/M+l} < \varepsilon.$$

Then we have

$$\kappa_{M+l}^{1/M+l} \le \kappa_M^{1/M+l}/(1-\varepsilon)$$

for all $l \ge 0$. By letting $l \to \infty$, we get

$$\limsup_{n\to\infty} \kappa_n^{1/n} \le 1/(1-\varepsilon), \quad \text{for } \varepsilon \in (0,\ 1),$$

and so, on letting $\varepsilon \to 0$, we obtain

$$\limsup_{n\to\infty} \kappa_n^{1/n} \le 1.$$

But note that κ_n is increasing and so

$$\liminf_{n\to\infty} \kappa_n^{1/n} \ge 1. \tag{3.7}$$

From (3.6) and (3.7) we see that $d\mu$ is regular; that is, (b) $\Rightarrow$ (a).

Next, let $f_{n,l}(z) := [\phi_n^*(z)/\phi_{n+\ell}^*(z)]^{1/n+l}$. Since all the zeros of $\phi_n(z)$ lie in $|z| < 1$, there exists a branch of $[\phi_n^*(z)/\phi_{n+l}^*(z)]^{1/n+l}$ analytic in $|z| \le 1$; we select the branch that is positive at $z = 0$ (recall that $\phi_n^*(0) = \kappa_n > 0$). According to Theorem 5.2.2 in [2, p. 198] we have for $l \ge 0$

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{|\phi_n^*(z)|^2}{|\phi_{n+l}^*(z)|^2}d\theta = \frac{1}{2\pi}\int_0^{2\pi}|\phi_n^*(z)|^2 d\mu(\theta) = 1, \quad z = e^{i\theta}.$$

Consequently, by Hölder's inequality, we have, for any $l \geq 0,\ n \geq 1$,

$$\frac{1}{2\pi}\int_0^{2\pi}|f_{n,l}(z)|^2 d\theta = \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\phi_n^*(z)}{\phi_{n+l}^*(z)}\right|^{2/n+l} d\theta$$
$$\leq \left(\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\phi_n^*(z)}{\phi_{n+l}^*(z)}\right|^2 d\theta\right)^{1/n+l} = \left(\frac{1}{2\pi}\int_0^{2\pi}|\phi_n^*(z)|^2\, d\mu\right)^{1/n+l} = 1.$$

Thus, by Lemma 3.2, the statements (b) through (e) are equivalent. ∎

Proof of Theorem 2.2

(a) $\Rightarrow$ (b): If $\log \mu' \in L^1[0,\ 2\pi)$, we have $\lim_{n\to\infty}\kappa_n = \kappa < \infty$. Given $\varepsilon \in (0,\ 1)$, we can find an integer $N(\varepsilon)$ such that

$$|\kappa_{n+l} - \kappa| < \varepsilon, \quad \text{for } n > N(\varepsilon) \ \text{ and } l \geq 0.$$

$$0 \leq 1 - \frac{\kappa_n}{\kappa_{n+l}} = \frac{\kappa_{n+l} - \kappa_n}{\kappa_{n+l}} \leq 2\varepsilon/\kappa_0$$

for $n > N(\varepsilon)$ and uniformly for $l \geq 0$, so we obtain (b).

(b) $\Rightarrow$ (a): If (b) holds, we can find an integer N such that

$$1 - \frac{\kappa_N}{\kappa_{N+l}} < 1/2 \quad \text{for all } \ l \geq 0,$$

and so $\kappa_{N+l} \leq 2\kappa_N$ for all $l \geq 0$. On letting $l \to \infty$, we get

$$\lim_{n\to\infty}\kappa_n = \kappa \leq 2\kappa_N < \infty,$$

which implies that $\log \mu' \in L[0,\ 2\pi)$.

Finally on taking $f_{n,l}(z) = \phi_n^*(z)/\phi_{n+l}^*(z)$ in Lemma 3.2, we know that statements (b) through (e) are equivalent. ∎

$$
\begin{aligned}
r_1 &:= \limsup_{n\to\infty} |\phi_n(d\mu,\ 0)|^{1/n},\\
r_2 &:= \inf_k \limsup_{n\to\infty} |z_{k,n}|,\\
r_3 &:= \inf\{r : \sup_n \max_{|z|=r^{-1}} |\phi_n^*(z)| < \infty\},\\
r_4 &:= \inf\{r : D^{-1}(d\mu,\ z) \text{ is analytic for } |z| < r^{-1}\} = 1/R_3,
\end{aligned}
$$

where $z_{k,n}$ denote the zeros of ϕ_n ordered in such a way that

$$|z_{n,n}| \le |z_{n-1,n}| \le \ldots \le |z_{1,n}| < 1$$

Lemma 3.3. (cf. [10]) *For every measure $d\mu$ we have $r_1 = r_2$. If there is a $j \in \{1,2,3,4\}$ such that $r_j < 1$, then $r_1 = r_3 = r_4$.*

Proof of Theorem 2.3

From (3.4) we get

$$1 - \frac{\kappa_n}{\kappa_{n+1}} \le 1 - \left(\frac{\kappa_n}{\kappa_{n+1}}\right)^2 = |\Phi_{n+1}(0)|^2. \tag{3.9}$$

Thus, from Lemma 3.1 and (3.9), we have

$$
\begin{aligned}
1/R_1^2 &= \limsup_{n\to\infty} \left(1 - \frac{\kappa_n}{\kappa_{n+1}}\right)^{1/n}\\
&\le \limsup_{n\to\infty} \left[\int_0^{2\pi} \left|\frac{|\phi_n^*(z)|^2}{|\phi_{n+1}^*(z)|^2} - 1\right| d\theta\right]^{2/n} = 1/R_2^2,
\end{aligned}
$$

and so $R_1 \ge R_2$.

On the other hand, from Schwarz's inequality, (3.1) and (3.8), we have

$$
\begin{aligned}
1/R_2 &= \limsup_{n\to\infty}\left[\int_0^{2\pi}\left|\frac{|\phi_n^*(z)|^2}{|\phi_{n+1}^*(z)|^2}-1\right|d\theta\right]^{1/n}\\
&\le \limsup_{n\to\infty}\left[\int_0^{2\pi}\left|\frac{|\phi_n^*(z)|}{|\phi_{n+1}^*(z)|}-1\right|^2 d\theta\right]^{1/2n}\\
&\times \limsup_{n\to\infty}\left[\int_0^{2\pi}\left|\frac{|\phi_n^*(z)|}{|\phi_{n+1}^*(z)|}+1\right|^2 d\theta\right]^{1/2n}\\
&\le \limsup_{n\to\infty}\left[\int_0^{2\pi}\left|\frac{\phi_n^*(z)}{\phi_{n+1}^*(z)}-1\right|^2 d\theta\right]^{1/2n}\\
&\times \limsup_{n\to\infty}\left[\left(\int_0^{2\pi}\left|\frac{\phi_n^*(z)}{\phi_{n+1}^*(z)}\right|^2 d\theta\right)^{1/2}+(2\pi)^{1/2}\right]^{1/n}\\
&= \limsup_{n\to\infty}\left(1-\frac{\kappa_n}{\kappa_{n+1}}\right)^{1/2n}\left(\limsup_{n\to\infty}(8\pi)^{1/2n}\right)\\
&= 1/R_1.
\end{aligned}
$$

Thus, we have $R_2 \ge R_1$, and so we conclude that $R_1 = R_2$.

Next assume that $R_1 > 1$. Since

$$
\limsup_{n\to\infty}\left(1-\frac{\kappa_n}{\kappa_{n+1}}\right)^{1/n} = 1/R_1^2, \tag{3.10}
$$

we have $\lim_{n\to\infty}\kappa_n/\kappa_{n+1} = 1$, and so

$$
\lim_{n\to\infty}\kappa_n^{1/n} = 1. \tag{3.11}
$$

Also note that, from (3.4), we have

$$
\begin{aligned}
|\phi_{n+1}(0)|^2 = \kappa_{n+1}^2-\kappa_n^2 &= \left(1-\frac{\kappa_n}{\kappa_{n+1}}\right)\kappa_{n+1}(\kappa_n+\kappa_{n+1})\\
&\le 2\kappa_{n+1}^2\left(1-\frac{\kappa_n}{\kappa_{n+1}}\right)
\end{aligned}
$$

Thus (3.10) and (3.11) yield

$$
\limsup_{n\to\infty}|\phi_n(0)|^{1/n} \le 1/R_1,
$$

and so the inequality $R_1 \le R_3$ follows from Lemma 3.3.

Again from (3.4),

$$
1-\frac{\kappa_n}{\kappa_{n+1}} = \frac{|\phi_{n+1}(0)|^2}{\kappa_{n+1}(\kappa_{n+1}+\kappa_n)} \le |\phi_{n+1}(0)|^2/2\kappa_0^2, \tag{3.12}
$$

and since we have proved $R_3 \geq R_1 > 1$, Lemma 3.3 implies that

$$1/R_1^2 = \limsup_{n\to\infty}\left(1 - \frac{\kappa_n}{\kappa_{n+1}}\right)^{1/n} \leq \limsup_{n\to\infty} |\phi_n(0)|^{2/n} = 1/R_3^2.$$

Thus $R_3 \leq R_1$ and so $R_1 = R_3$.

If $R_3 > 1$, then from (3.12) and Lemma 3.3, we have $R_1 \geq R_3 > 1$ and the equality $R_3 = R_2 = R_1$ follows from the case $R_1 > 1$. ■

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References

1. Freud, G., *Orthogonal Polynomials*, Pergaman Press, New York, 1971.
2. Geronimus, Ya. L., *Orthogonal Polynomials*, Consultants Bureau, New York, 1961.
3. Lubinsky, D. S., Jump distributions on [−1, 1] whose orthogonal polynomials have leading coefficients with given asymptotic behavior, Proc. Amer. Math. Soc., **104** (1988), 516-524.
4. Li, X. and E. B. Saff, On Nevai's characterization of measures with positive derivative, J. Approx. Theory, to appear.
5. Li, X., E. B. Saff, and Z. Sha, Behavior of best L_p polynomial approximants on the unit interval and on the unit circle, J. Approx. Theory, to appear.
6. Máté, A., P. Nevai, and V. Totik, Asymptotics for the ratio of leading coefficients of orthonormal polynomials on the unit circle, Constr. Approx. **1** (1985), 63-69.
7. Nevai, P., Characterization of measures associated with orthogonal polynomials on the unit circle, Rocky Mountain J. of Math. **19** (1989), 293-302.
8. Nevai, P., Orthogonal polynomials, recurrences, Jacobi matrices and measures, preprint.
9. Nevai, P., Weakly convergent sequences of functions and orthogonal polynomials, manuscript.
10. Nevai, P. and V. Totik, Orthogonal polynomials and their zeros, Acta Sci. Math. (Szeged), **53** (1989), 99-114.
11. Rahmanov, E. A., On the asymptotics of the ratio of orthogonal polynomials, II, Math. USSR Sbornik **46** (1983), 105-117.
12. Stahl, H. and V. Totik, *General Orthogonal Polynomials*, manuscript.
13. Szegö, G., *Orthogonal polynomials*, 4th ed., Amer. Math. Soc. Colloquium Publ. **23**, Rhode Island, Providence, 1975.

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K. Pan
Department of Mathematics
University of California, Riverside
Riverside, CA 92521

E. B. Saff
Institute for Constructive Mathematics
Department of Mathematics
University of South Florida
Tampa, FL 33620
dkgavaa@cfrvm.bitnet