PACIFIC JOURNAL OF MATHEMATICS
Vol. 45, No. 2, 1973

ON THE CONVERGENCE OF RATIONAL FUNCTIONS WHICH INTERPOLATE IN THE ROOTS OF UNITY

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Results are obtained on the existence and convergence of certain types of rational functions which interpolate in the roots of unity to a function f which is meromorphic in $|z| < 1$ and continuous on $|z| \leqq 1$. The theorems presented extend results of Fejér and Walsh and Sharma on interpolating polynomials.

In a recent paper [2] the first author investigated the convergence of certain sequences of rational functions which interpolate to a meromorphic function f. The results obtained in [2] apply, for example, when f is analytic on $|z| \leqq 1$, meromorphic in $|z| < \rho$, $\rho > 1$, and the points of interpolation are the roots of unity.

In this paper we study the convergence of rational functions which interpolate in the roots of unity to a function f which is meromorphic in $|z| < 1$ and continuous on $|z| \leqq 1$. The theorems presented extend those of Fejér [1] and Walsh and Sharma [4] concerning interpolating polynomials. The method of proof of Theorem 1 is basically that of [2].

A rational function $r_{n\nu}(z)$ is said to be *of type* (n, ν) if it is of the form

$$r_{n\nu}(z) = p_n(z)/q_\nu(z) , \qquad q_\nu(z) \not\equiv 0 ,$$

where $p_n(z)$ and $q_\nu(z)$ are polynomials of degrees at most n and ν respectively.

THEOREM 1. *Let $f(z)$ be meromorphic with precisely ν poles (multiplicity included) in D: $|z| < 1$ and otherwise finite and continuous on $|z| \leqq 1$. Let D' denote the domain obtained from D by deleting the ν poles of $f(z)$. Then for all n sufficiently large there exists a unique rational function $r_{n\nu}(z)$ of type (n, ν) which interpolates to $f(z)$ in the $n + \nu + 1$ roots of unity. Each $r_{n\nu}(z)$ for n large enough has precisely ν finite poles and as $n \to \infty$ these poles approach respectively the ν poles of $f(z)$ in D. The sequence $r_{n\nu}(z)$ converges to $f(z)$ throughout D', uniformly on any closed subset of D'.*

For the case $\nu = 0$ the above theorem is due to Fejér [1].

Proof. For any function g defined on $|z| = 1$ the unique polynomial of degree at most n which interpolates to g in the $n + 1$ roots of unity shall be denoted by $L_n(g; z)$.

Let $\alpha_1, \alpha_2, \cdots, \alpha_\nu$ be the ν poles of $f(z)$ in D and set

$$Q_0(z) = 1 , \quad Q_k(z) = \prod_{i=1}^{k} (z - \alpha_i) , \quad 1 \leqq k \leqq \nu ,$$

$$q_n(z) = Q_\nu(z) + \sum_{k=1}^{\nu} a_k^{(n)} Q_{k-1}(z) .$$

We shall show that for n sufficiently large the coefficients $a_k^{(n)}$ can be chosen so that $Q_\nu(z)$ divides the interpolating polynomial $L_{n+\nu}(q_n Q_\nu f; z)$. For simplicity we assume that the points α_j are distinct, i.e., $f(z)$ has only simple poles in D. The case of multiple poles is left to the reader.

Clearly $Q_\nu(z) \mid L_{n+\nu}(q_n Q_\nu f; z)$ if and only if

$$(1) \qquad \sum_{k=1}^{\nu} c_{jk}^{(n)} a_k^{(n)} = d_j^{(n)} , \qquad j = 1, 2, \cdots, \nu ,$$

where

$$c_{jk}^{(n)} = L_{n+\nu}(Q_{k-1} Q_\nu f; \alpha_j) , \qquad d_j^{(n)} = -L_{n+\nu}(Q_\nu^2 f; \alpha_j) .$$

For each k the function $Q_{k-1} Q_\nu f$ is analytic in D and continuous on $|z| \leqq 1$, and so Fejér's theorem implies that

$$\lim_{n\to\infty} c_{jk}^{(n)} = (Q_{k-1} Q_\nu f)(\alpha_j) , \quad \lim_{n\to\infty} d_j^{(n)} = -(Q_\nu^2 f)(\alpha_j) , \quad 1 \leqq j, k \leqq \nu ,$$

Since α_j is a simple pole of f we have

$$(Q_{k-1} Q_\nu f)(\alpha_j) = 0 , \qquad \text{for } k > j ,$$
$$(Q_{k-1} Q_\nu f)(\alpha_j) \neq 0 , \qquad \text{for } k = j .$$

Hence

$$\lim_{n\to\infty} \det [c_{jk}^{(n)}] = \prod_{l=1}^{\nu} (Q_{l-1} Q_\nu f)(\alpha_l) \neq 0 ,$$

which implies that for n sufficiently large the linear system (1) can be solved uniquely for the coefficients $a_j^{(n)}$. Furthermore since $d_j^{(n)} \to 0$ as $n \to \infty$, it follows from Cramer's rule that for each k, $1 \leqq k \leqq \nu$, we have $a_k^{(n)} \to 0$ as $n \to \infty$. Thus

$$(2) \qquad \lim_{n\to\infty} q_n(z) = Q_\nu(z) ,$$

uniformly on each bounded subset of the plane.

Now set $r_{n\nu}(z) \equiv L_{n+\nu}(q_n Q_\nu f; z)/q_n(z) Q_\nu(z)$. Then by our choice of the coefficients $a_k^{(n)}$ we have that $r_{n\nu}(z)$ is a rational function of type (n, ν). Also from (2) it follows that for n sufficiently large $q_n(z)$ is different from zero in the $n + \nu + 1$ roots of unity and so $r_{n\nu}(z)$ must interpolate to $f(z)$ in these points. It is easy to see that $r_{n\nu}(z)$ is uniquely determined by its interpolation property. From Fejér's theorem and (2) we have $r_{n\nu}(z) \to f(z)$ as $n \to \infty$ uniformly on any closed subset of D'.

Finally note that $r_{n\nu}(z)$ has ν formal poles, namely the zeros of $q_n(z)$, and as $n \to \infty$ these poles approach respectively the ν poles of $f(z)$ in D. Since

$$\lim_{n\to\infty} L_{n+\nu}(q_n Q_\nu f; z)/Q_\nu(z) = Q_\nu(z) f(z) ,$$

uniformly for z in a neighborhood of each α_j, it follows that for n sufficiently large no zero of the polynomial $L_{n+\nu}(q_n Q_\nu f; z)/Q_\nu(z)$ is a zero of $q_n(z)$. Thus the ν formal poles of $r_{n\nu}(z)$ are *actual* poles. This completes the proof of Theorem 1.

Walsh and Sharma [4] have shown that for any function $g(z)$ analytic in $|z| < 1$ and continuous on $|z| \leqq 1$, the sequence $L_n(g; z)$ converges to $g(z)$ on $|z| = 1$ in the mean of second order. Applying this result to each of the sequences $\{L_{n+\nu}(Q_{k-1} Q_\nu f; z)\}$, $1 \leqq k \leqq \nu + 1$, there follows from (2)

THEOREM 2. *The sequence $r_{n\nu}(z)$ of Theorem* 1 *converges to $f(z)$ in the mean of second order on $|z| = 1$.*

Theorems 1 and 2 are another illustration of the close analogy between approximation in the sense of least squares on $|z| = 1$ and interpolation in the roots of unity; compare [3, §§ 7.10, 9.1, 11.6], [4].

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November 10, 1971. The research of the authors was supported, in part, by NSF Grant GF-19275 and USAF Grant 69-1690 respectively.

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