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BEST POLYNOMIAL APPROXIMATION WITH LINEAR CONSTRAINTS

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ABSTRACT. Let A be a $(k+1)\times(k+1)$ nonzero matrix. For polynomials $p\in\mathcal{P}_n$, set $\underline{p} := \big(p(0), p'(0),\ldots,p^{(k)}(0)\big)^T$ and $B_n(A) := \{p\in\mathcal{P}_n : A\underline{p} = \underline{0}\}$. Let $E\subset\mathbf{C}$ be a compact set that does not separate the plane and f be a function continuous on E and analytic in the interior of E. Set $E_n(A,f) := \inf\{\|f-p\|_E : p\in B_n(A)\}$ and $E_n(f) := \inf\{\|f-p\|_E : p\in\mathcal{P}_n\}$. Our goal is to study approximation to f on E by polynomials from $B_n(A)$. We obtain necessary and sufficient conditions on the matrix A for the convergence $E_n(A,f)\longrightarrow 0$ to take place. These results depend on whether zero lies inside, on the boundary or outside E and yield generalizations of theorems of Clunie, Hasson and Saff for approximation by polynomials that omit a power of z. Let $p^*_{n,A}\in B_n(A)$ be such that $E_n(A,f) = \|f-p^*_{n,A}\|_E$. We also study the asymptotic behavior of the zeros of $p^*_{n,A}$ and the asymptotic relation between $E_n(f)$ and $E_n(A,f)$.

1. **Introduction and notation.** Let E be a compact set in the complex plane $\mathbf{C}$ containing infinitely many points and let $\|\cdot\|$ denote the uniform norm on E. For a function f, if the derivatives $f^{(i)}(0)$, $i=0,\ldots,k$, exist, define:

$$\underline{f} := \big(f(0), f'(0),\ldots,f^{(k)}(0)\big)^T.$$

Let $A := (a_{i,j})_{i,j=0}^k \neq 0$ be a given $(k+1)\times(k+1)$ matrix with complex constant entries. With $\mathcal{P}_n$ denoting the collection of all algebraic polynomials of degree at most n, we set

$$\alpha_{n,A}(f) := \inf\{\|p\| : p\in\mathcal{P}_n \text{ and } A\underline{p} = A\underline{f}\},\quad n\geq k.$$

We also define

$$B_n(A) := \{p\in\mathcal{P}_n : A\underline{p} = \underline{0}\},$$
$$C(E) := \{f : f \text{ continuous on } E\},$$
$$\mathcal{A}(E) := \{f\in C(E) : f \text{ analytic in the interior of } E\},$$
$$E_n(A,f) := \inf\{\|f-p\| : p\in B_n(A)\},$$
$$E_n(f) := \inf\{\|f-p\| : p\in\mathcal{P}_n\},$$
$$\mathcal{B}_n(f) := \{p\in B_n(A) : \|f-p\| = E_n(A,f)\}.$$

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Throughout we let $p^*_{nA} := p^*_{nA}(f)$ denote an arbitrary but fixed element of $\mathcal{B}_n(f)$, and we let $p^*_n := p^*_n(f)$ denote the unique polynomial in $\mathcal{P}_n$ satisfying $\|f - p^*_n(f)\| = E_n(f)$. As we shall show, the behavior of $E_n(A,f)$ depends on whether zero lies inside E, on the boundary of E or outside E. Our results generalize theorems of Clunie, Hasson and Saff [CHS] for approximation by polynomials that omit a single power of z. One important aspect of our investigation is the relation between $E_n(f)$ and $E_n(A,f)$. We also study the asymptotic behavior of the zeros of p^*_{nA}.

It is natural to consider the more general problem of approximation from the set $B_n(A,\underline{a}) := \{p \in \mathcal{P}_n : A\underline{p} = \underline{a}\}$. If $\underline{a} \neq \underline{0}$, then one can replace the function $f(z)$ by the new function $g(z) := f(z) - \sum_{i=0}^{k}(w_i/i!)z^i$, where $\underline{w} := (w_0, \ldots, w_k)^T$ is a solution of $A\underline{x} = \underline{a}$ (the existence of $\underline{w}$ is assumed; otherwise $B_n(A,\underline{a}) = \emptyset$). Then for each $n \geq k$, the polynomial $p^*_{nA}(g) + \sum_{i=0}^{k}(w_i/i!)z^i$ is a best approximation to f from $B_n(A,\underline{a})$. Thus, without loss of generality, we only need consider approximation from $B_n(A)$.

2. **Asymptotic behavior of $E_n(A,f)$.** For k a fixed nonnegative integer and $f \in \mathcal{A}(E)$, we shall examine the asymptotic behavior of $E_n(A,f)$ as $n \longrightarrow \infty$. We begin with some basic lemmas.

LEMMA 2. *If $f \in C(E)$ and $n \geq k$, then*

$$\alpha_{nA}(p^*_n) - E_n(f) \leq E_n(A,f) \leq \alpha_{nA}(p^*_n) + E_n(f).$$

PROOF. Note that

$$E_n(A,f) = \|f - p^*_{nA}\| \geq \|p^*_n - p^*_{nA}\| - \|f - p^*_n\|. \tag{2.2}$$

Since $A(\underline{p}^*_n - \underline{p}^*_{nA}) = A\underline{p}^*_n$, we have $\|p^*_n - p^*_{nA}\| \geq \alpha_{nA}(p^*_n)$ and so the lower estimate in (2.1) follows from (2.2).

Now let $q \in \mathcal{P}_n$ be such that $A\underline{q} = A\underline{p}^*_n$ and $\|q\| = \alpha_{nA}(p^*_n)$. Then we have

$$E_n(A,f) \leq \|f - p^*_n + q\| \leq E_n(f) + \|q\| = E_n(f) + \alpha_{nA}(p^*_n).$$

■

LEMMA 2.2. *If $f \in C(E)$, then*

$$\lim_{n\to\infty} E_n(A,f) = 0$$

if and only if

$$\lim_{n\to\infty} E_n(f) = 0 \text{ and } \lim_{n\to\infty} \alpha_{nA}(p^*_n) = 0.$$

PROOF. If $E_n(A,f) \longrightarrow 0$, then clearly $E_n(f) \longrightarrow 0$ as $n \longrightarrow \infty$. From the lower estimate in Lemma 2.1 we then deduce that $\lim_{n\to\infty} \alpha_{nA}(p^*_n) = 0$.

The sufficiency of the conditions follows immediately from the upper estimate in Lemma 2.1. ■

Multiplying the inequalities in (2.1) by $\alpha^{-1}_{nA}(p^*_n)$ we immediately get

LEMMA 2.3. *Let $f \in C(E)$. Suppose $\alpha_{nA}(p_n^*) \neq 0$ for all n large and*

3) $$\lim_{n\to\infty} [\alpha_{nA}(p_n^*)]^{-1} E_n(f) = 0.$$

$$E_n(A, f) \cong \alpha_{nA}(p_n^*) \quad as \ n \to \infty.$$

Here we use the notation $a_n \cong b_n$ to mean $a_n / b_n \to 1$ as $n \to \infty$.

To give conditions under which (2.3) is satisfied we need some further notation. We denote by K the unbounded component of $\overline{\mathbf{C}} \setminus E$ and by $g_K(z, \infty)$ the Green function with pole at infinity for K. We say that K is regular if for each point $z_0 \in \partial K$, the boundary of K, we have

$$\lim_{z\to z_0} g_K(z, \infty) = 0, \quad z \in K.$$

The following result is known as the Bernstein-Walsh lemma.

LEMMA 2.4 ([W, §4.6]). *Let E be a compact set whose complement K is connected and regular. If the polynomial $p \in \mathcal{P}_n$ satisfies the inequality $|p(z)| \leq L$ for z on E, then*

$$|p(z)| \leq L \exp\big(n g_K(z, \infty)\big), \quad z \in K.$$

We can now establish

THEOREM 2.5. *Suppose E is a compact set whose complement $\overline{\mathbf{C}} \setminus E$ is connected and regular. Assume that $f(z)$ is analytic on E and $0 \in E$. If $A\underline{f} \neq \underline{0}$, then the asymptotic formula (2.4) holds.*

PROOF. It is well-known (*cf.* [W, §4.7]) that since f is analytic on E,

(2.5) $$\limsup_{n\to\infty} E_n^{1/n}(f) < 1$$

and $\{p_n^*\}_0^\infty$ converges uniformly to f on some open set containing E. The latter property implies that

(2.6) $$\lim_{n\to\infty} p_n^{*(j)}(0) = f^{(j)}(0), \quad j = 0, \ldots, k.$$

Next, define

$$\beta_{nA} := \sup\left\{ \max_{0\leq i\leq k} \left| \sum_{j=0}^{k} a_{i,j} p^{(j)}(0) \right|, \|p\| \leq 1 \text{ and } p \in \mathcal{P}_n \right\}$$

We claim that

(2.8) $$\limsup_{n\to\infty} \beta_{nA}^{1/n} \leq 1.$$

In fact, if we define for each $\delta > 1$, the level curve

$$\Gamma_\delta := \{z : g_K(z, \infty) = \log \delta\},$$

then 0 is surrounded by Γ_δ since $0 \in E$. Now by the Cauchy integral formula, we have for all $p \in \mathcal{P}_n$,

$$p^{(j)}(0) = \frac{j!}{2\pi i}\int_{\Gamma_\delta} \frac{p(z)}{z^{j+1}}\,dz, \quad j = 0, 1, \ldots \text{ and } \delta > 1$$

So for $p \in \mathcal{P}_n$ with $\|p\| \leq 1$, we obtain from Lemma 2.4 that

$$|p^{(j)}(0)| \leq \frac{j!}{2\pi}\delta^n \frac{\text{length}(\Gamma_\delta)}{\text{dist}(0, \Gamma_\delta)^{j+1}}, \quad j = 0, 1,$$

According to the definition of $\beta_{n,A}$ in (2.7), we therefore get

$$\limsup_{n\to\infty} \beta_{n,A}^{1/n} \leq \delta,$$

and by letting $\delta \longrightarrow 1^+$ we have verified the claim (2.8).

Since $A\underline{f} \neq \underline{0}$, there is an i_0, $0 \leq i_0 \leq k$, such that

$$\sum_{j=0}^{k} a_{i_0,j} f^{(j)}(0) \neq 0.$$

For $n \geq k$, let $q_n \in \mathcal{P}_n$ satisfy $\|q_n\| = \alpha_{n,A}(p_n^*)$ and $A\underline{q}_n = A\underline{p}_n^*$. Then from (2.6) and (2.9) it follows that, for n large, $\|q_n\| \neq 0$ and so

$$\left|\sum_{j=0}^{k} a_{i_0,j} q_n^{(j)}(0)\right| \Big/ \|q_n\| \leq \beta_{n,A}.$$

Thus, for n large,

$$\alpha_{n,A}^{-1}(p_n^*) \leq \beta_{n,A}\left\{\left|\sum_{j=0}^{k} a_{i_0,j} p_n^{*(j)}(0)\right|\right\}^{-1} \tag{2.10}$$

Furthermore, from (2.6) we have, for n large,

$$\left|\sum_{j=0}^{k} a_{i_0,j} p_n^{*(j)}(0)\right| \geq \frac{1}{2}\left|\sum_{j=0}^{k} a_{i_0,j} f^{(j)}(0)\right|,$$

and so from (2.8), (2.10) and (2.11) we get

$$\limsup_{n\to\infty}[\alpha_{n,A}^{-1}(p_n^*)]^{1/n} \leq 1 \tag{2.12}$$

Combining (2.5) and (2.12) yields

$$\lim_{n\to\infty} \alpha_{n,A}^{-1}(p_n^*) E_n(f) = 0,$$

and so the theorem follows from Lemma 2.3. ∎

3. **Approximation with linear constraints.** It is well-known that, by Mergelyan's theorem, $E_n(f) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $f \in \mathcal{A}(E)$ if and only if the compact set E does not separate the plane; that is, $\overline{\mathbf{C}} \setminus E$ is connected. In this section, we shall study the conditions on the matrix A that imply $E_n(A,f) \longrightarrow 0$ as $n \longrightarrow \infty$.

THEOREM 3.1. *Let $f \in \mathcal{A}(E)$ and assume $\overline{\mathbf{C}} \setminus E$ is connected and $0 \notin E$. Then*

$$\lim_{n\to\infty} E_n(A,f) = 0. \tag{3.1}$$

PROOF. Since $0 \notin E$, the function $f(z)z^{-(k+1)} \in \mathcal{A}(E)$. Using Mergelyan's theorem, we have that for any $\varepsilon > 0$ there is a $p_{n-k-1} \in \mathcal{P}_{n-k-1}$ such that

$$\|f(z)z^{-(k+1)} - p_{n-k-1}(z)\| \leq \varepsilon, \text{ for } n \text{ large.}$$

Hence

$$\|f(z) - z^{k+1}p_{n-k-1}(z)\| = \left\|z^{k+1}\left(f(z)z^{-(k+1)} - p_{n-k-1}(z)\right)\right\| \leq \|z^{k+1}\|\varepsilon.$$

But $z^{k+1}p_{n-k-1} \in B_n(A)$; thus (3.1) follows. ∎

The case when zero lies interior to E is also easy to handle.

THEOREM 3.2. *Assume $0 \in E^o$, the interior of E, and $\overline{\mathbf{C}} \setminus E$ is connected. If $f \in \mathcal{A}(E)$, then*

$$\lim_{n\to\infty} E_n(A,f) = 0 \textit{ if and only if } A\underline{f} = \underline{0}.$$

PROOF. First assume that $\lim_{n\to\infty} E_n(A,f) = 0$ and $0 \in E^o$. Then $\lim_{n\to\infty} p_{n,A}^{*(j)}(0) = f^{(j)}(0), j = 0, \ldots, k$. Also note that $A\underline{p}_{n,A}^* = \underline{0}$ and so letting $n \longrightarrow \infty$ we get $A\underline{f} = \underline{0}$.

Next assume that $A\underline{f} = \underline{0}$ and set

$$v_n(z) := \sum_{i=0}^{k} \frac{\left(p_n^{*(i)}(0) - f^{(i)}(0)\right)}{i!} z^i.$$

Since $A\underline{f} = \underline{0}$, we have $A\underline{v}_n = A(\underline{p}_n^* - \underline{f}) = A\underline{p}_n^*$. Thus

$$\alpha_{n,A}(p_n^*) \leq \|v_n\| \leq \sum_{i=0}^{k} \frac{|p_n^{*(i)}(0) - f^{(i)}(0)|}{i!}\|z^i\|, \quad n \geq k. \tag{3.2}$$

Now, by Mergelyan's theorem, $\lim_{n\to\infty} E_n(f) = 0$ and since $0 \in E^o$, we have $\lim_{n\to\infty} p_n^{*(j)}(0) = f^{(j)}(0), j = 0, \ldots, k$. Hence with (3.2) we get

$$\lim_{n\to\infty} \alpha_{n,A}(p_n^*) = 0$$

and the theorem follows from Lemma 2.2. ∎

It remains to consider the more interesting case when $0 \in \partial E$, the boundary of E. It can be seen from the results of Nersesyan [N] that the essential condition needed for convergence is that the constraint $A\underline{p} = \underline{0}$ does not imply that $p(0) = 0$. Here we provide a simple direct proof that utilizes the following result of [CHS, p. 68] stated in a slightly more general form.

LEMMA 3.3. *Let $0 \in \partial E$. For any $\varepsilon > 0$ and positive integer m there is a polynomial $q_0(z)$ such that*

$$\|z - z^{2m+1} q_0(z)^{2m}\| < \varepsilon.$$

Now we can state

THEOREM 3.4. *Assume $\overline{\mathbf{C}} \setminus E$ is connected and $0 \in \partial E$. Then the following conditions are equivalent:*

(i) $\lim_{n\to\infty} E_n(A,f) = 0$ *for all* $f \in \mathcal{A}(E)$;

(ii) $B_k(A) \setminus \mathcal{P}_{k,0} \neq \emptyset$, *where* $\mathcal{P}_{k,0} := z\mathcal{P}_{k-1}$ *and* $\mathcal{P}_{0,0} := \{0\}$; *that is, there exists a polynomial* $p \in B_k(A)$ *such that* $p(0) \neq 0$;

(iii) A *has* 0 *as an eigenvalue (i.e.* $\det A = 0$*) and has an associated eigenvector with first component equal to* 1.

PROOF. First observe that (ii) $\Leftrightarrow$ (iii) is trivial.

We now show that (iii) $\Rightarrow$ (i). For the linear system $A\underline{x} = \underline{0}$, where $\underline{x} := (x_0, \ldots, x_k)^T$, assertion (iii) states that there is a solution with first component not equal to zero. So it is easy to see that there is a submatrix $A_{i_1,\ldots,i_l;1,j_1,\ldots,j_{l-1}}$ whose determinant is nonzero, where $l := k + 1 - \operatorname{rank}(A)$, and $A_{i_1,\ldots,i_l;1,j_1,\ldots,j_{l-1}}$ denotes the submatrix obtained by deleting the i_1th, ..., i_lth rows and 1st, j_1th, ..., j_{l-1}th columns from A. (We remark that $l \leq k$ since $A \neq 0$.) Without loss of generality we can assume

$$\det \begin{pmatrix} a_{0,1} & a_{0,2} & \cdots & a_{0,k+1-l} \\ a_{1,1} & a_{1,2} & \cdots & a_{1,k+1-l} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k-l,1} & a_{k-l,2} & \cdots & a_{k-l,k+1-l} \end{pmatrix} \neq 0.$$

Hence there exist constants $b_{i,j}$ and c_i such that for $n \geq k$

$$B_n(A) = \left\{ p \in \mathcal{P}_n : p^{(i)}(0) = c_i p(0) + \sum_{j=k-l+2}^{k} b_{i,j} p^{(j)}(0),\ i = 1, \ldots, k+1-l \right\}$$

For any $0 < \varepsilon < 1$, choose a polynomial p_0, using Mergelyan's theorem, such that $\|f - p_0\| < \varepsilon$. Assuming $p_0 \in \mathcal{P}_n$ with $n \geq k$, set

$$d_i := p_0^{(i)}(0) - c_i p_0(0) - \sum_{j=k-l+2}^{k} b_{i,j} p_0^{(j)}(0), \quad i = 1, \ldots, k+1-l$$

and $d := \max_{1 \leq i \leq k+1-l}\{|d_i|\}$. Also define

$$\varepsilon_1 := \begin{cases} \varepsilon & \text{if } d \leq \varepsilon \\ \varepsilon/d & \text{otherwise.} \end{cases}$$

Let m be a fixed positive integer with $2m + 2 > k$. From Lemma 3.3, we know that there exists a polynomial q_0 such that

$$\|z - z^{2m+1} q_0(z)^{2m}\| < \varepsilon_1.$$

Then we have

$$[z - z^{2m+1}q_0(z)^{2m}]^i = z^i - z^{2m+2}p_i(z) =: Q_i(z), \quad i = 1, \ldots, k+1-l,$$

where the p_i's are polynomials. Also note that $\varepsilon_1 < 1$ so that

$$\|Q_i\| < \varepsilon_1, \quad i = 1, \ldots, k+1-l. \tag{3.3}$$

Consider

$$r(z) := p_0(z) - \sum_{j=1}^{k+1-l} d_j Q_j(z)/j!$$

Then

$$\begin{aligned} r(0) &= p_0(0), \\ r^{(i)}(0) &= p_0^{(i)}(0) - d_i, \quad i = 1, \ldots, k+1-l, \\ r^{(i)}(0) &= p_0^{(i)}(0), \quad i = k+2-l, \ldots, k. \end{aligned}$$

Thus we have for $i = 1, \ldots, k+1-l$:

$$\begin{aligned} r^{(i)}(0) &= p_0^{(i)}(0) - p_0^{(i)}(0) + c_i p_0(0) + \sum_{j=k-l+2}^{k} b_{i,j} p_0^{(j)}(0) \\ &= c_i p_0(0) + \sum_{j=k-l+2}^{k} b_{i,j} p_0^{(j)}(0) \\ &= c_i r(0) + \sum_{j=k-l+2}^{k} b_{i,j} r^{(j)}(0), \end{aligned}$$

and so $r(z) \in B_t(A)$ for some positive integer t. From (3.3) and the definition of ε_1, we have $\|d_i Q_i\| \leq \varepsilon, i = 1, \ldots, k+1-l$, and so

$$\begin{aligned} \|f - r\| &= \left\| f - p_0 + \sum_{j=1}^{k+1-l} d_j Q_j / j! \right\| \\ &\leq \|f - p_0\| + \sum_{j=1}^{k+1-l} \|d_j Q_j / j!\| \\ &\leq \varepsilon + (k+1-l)\varepsilon. \end{aligned}$$

Thus $\lim_{n\to\infty} E_n(A, f) = 0$.

Finally, to show that (i) implies (iii), assume that (iii) is not true. Then $p(0) = 0$ for all $p \in B_k(A)$ and hence $p(0) = 0$ for all $p \in B_n(A)$, $n = 0, 1, 2, \ldots$. Thus for $f \equiv 1$, $E_n(A, f)$ does not tend to zero, which contradicts (i). ∎

4. **Distribution of zeros of $p^*_{n,A}$.** To state our results, we need to introduce some terminology from potential theory. We denote the logarithmic capacity (transfinite diameter) of the set E by cap(E) (*cf.* [T]). If cap(E) > 0, let μ_E be the unique positive unit measure with supp(μ_E) $\subset E$ that minimizes the energy integral

$$I[\mu] := \iint_E \log|z-t|^{-1}\, d\mu(t)\, d\mu(z)$$

over all unit measures supported on E. The extremal measure μ_E is called the *equilibrium distribution* for E and

$$U(\mu_E; z) := \int \log|z-t|^{-1}\, d\mu_E(t)$$

is the *conductor potential* of E. The minimum energy $I[\mu_E]$ is related to the capacity of E via

$$\mathrm{cap}(E) = \exp(-I[\mu_E]).$$

The Green function $g_K(z,\infty)$ with pole at infinity for K, the unbounded component of $\overline{\mathbf{C}} \setminus E$, is given by (*cf.* [T, p. 82])

$$g_K(z,\infty) = -\{\log[\mathrm{cap}(E)] + U(\mu_E; z)\}, \tag{4.1}$$

and is positive and harmonic in $K \setminus \{\infty\}$. We define for each $\sigma > 1$, the closed region

$$E_\sigma := E \bigcup \{z \in K : 0 < g_K(z,\infty) \le \log\sigma\},$$

which has boundary

$$\Gamma_\sigma := \{z \in K : g_K(z,\infty) = \log\sigma\}.$$

Note that if we define $K_\sigma := \overline{\mathbf{C}} \setminus E_\sigma$, then

$$g_{K_\sigma}(z,\infty) = g_K(z,\infty) - \log\sigma$$

and from (4.1) it is easy to see that

$$\mathrm{cap}(E_\sigma) = \mathrm{cap}(E)\sigma. \tag{4.2}$$

In this section, we will examine the geometric rate of convergence of $E_n(A,f)$ and the limiting distribution of the zeros of the polynomials $p^*_{n,A}$. For a polynomial p_n of precise degree n, we denote by $\nu_n = \nu(p_n)$ the discrete unit measure (defined on the Borel sets in $\mathbf{C}$) having mass $1/n$ at each zero of p_n, with the obvious modification in this definition for the case when p_n has multiple zeros. We say that ν_n converges in the weak-star topology to the measure μ as $n \to \infty$ and write $\nu_n \xrightarrow{*} \mu$ if

$$\lim_{n\to\infty} \int \phi\, d\nu_n = \int \phi\, d\mu,$$

for every continuous function ϕ on $\mathbf{C}$ having compact support.

Before we state our main results, we need the following lemma of Blatt, Saff and Simkani.

LEMMA 4.1 ([BSS]). *Let E be a compact set with* $\operatorname{cap}(E) > 0$ *and set* $E^* := \operatorname{supp}(\mu_E)$. *Let Λ be an infinite subset of positive integers and $\{p_n\}_{n\in\Lambda}$ be a sequence of monic polynomials of respective degrees precisely n. Then $\nu_n = \nu(p_n)$ converges in the weak-star topology to μ_E as $n \longrightarrow \infty$, $n \in \Lambda$, if conditions* (i) *and* (ii) *below are satisfied.*

(i) $\limsup_{n\to\infty} \|p_n\|_{E^*}^{1/n} \leq \operatorname{cap}(E)$, $n \in \Lambda$;

(ii) $\lim_{n\to\infty} \nu_n(B) = 0$, $n \in \Lambda$, *for every closed set B contained in the union of the bounded (open) components of $\overline{\mathbf{C}} \setminus E^*$.*

We first consider the case when $0 \in E^o$.

THEOREM 4.2. *Suppose $\overline{\mathbf{C}} \setminus E$ is connected and regular, $0 \in E^o$ and* $\operatorname{cap}(E) > 0$. *Assume $f \in \mathcal{A}(E)$, but f is not analytic on E and f does not vanish identically on any component of E^o. If $A\underline{f} = \underline{0}$, then*

(i) $\limsup_{n\to\infty} E_n^{1/n}(A,f) = 1$;

(ii) $\nu(p^*_{nA}) \xrightarrow{*} \mu_E$ *as $n \longrightarrow \infty$, $n \in \Lambda$, where $\Lambda \subseteq \mathbf{N}$ is a sequence that depends on f.*

PROOF. Clearly $E_n(A,f) \leq \|f\|$ and so

$$\limsup_{n\to\infty} E_n^{1/n}(A,f) \leq 1.$$

Since f is not analytic on E, we also have $\limsup_{n\to\infty} E_n^{1/n}(f) = 1$ (*cf.* [W, §4.7]). Hence

$$1 = \limsup_{n\to\infty} E_n^{1/n}(f) \leq \limsup_{n\to\infty} E_n^{1/n}(A,f) \leq 1,$$

which yields (i).

Write $p^*_{nA}(z) = a^*_{nA}z^n + \cdots$ and for $n > k$ choose

$$T_{nA}(z) \in B_n(A), \quad T_{nA}(z) = z^n + \cdots$$

such that

$$\|T_{nA}\| = \inf\{\|p\| : p \in B_n(A) \text{ and } p = z^n + \cdots\}.$$

Then, for $n > k$,

$$E_{n-1}(A,f) \leq \|f - p^*_{nA} + a^*_{nA}T_{nA}\| \leq E_n(A,f) + |a^*_{nA}|\|T_{nA}\|. \tag{4.3}$$

Let $T_n(z) = z^n + \cdots$ denote the (unconstrained) Chebyshev polynomials for E; that is

$$\|T_n\| = \inf\{\|p\| : p \in \mathcal{P}_n \text{ and } p(z) = z^n + \cdots\}.$$

It is well-known (*cf.* [T]) that $\lim_{n\to\infty} \|T_n\|^{1/n} = \operatorname{cap}(E)$. Note that

$$\|T_n\| \leq \|T_{nA}\| \leq \|z^{k+1}T_{n-k-1}\| \leq \|z^{k+1}\|\|T_{n-k-1}\|,$$

so we have

$$\lim_{n\to\infty} \|T_{nA}\|^{1/n} = \operatorname{cap}(E). \tag{4.4}$$

From (4.3) it follows that

$$E_{n-1}(A,f) - E_n(A,f) \leq |a^*_{nA}| \|T_{nA}\|.$$

Next observe from Theorem 3.2 that $E_n(A,f) \to 0$ as $n \to \infty$, and hence from (i) it follows that

$$\limsup_{n\to\infty} [E_{n-1}(A,f) - E_n(A,f)]^{1/n} = 1. \tag{4.6}$$

From (4.4), (4.5) and (4.6), it is easy to see that there is a subsequence $\Lambda \subseteq \mathbf{N}$ such that

$$\liminf_{n\to\infty} |a^*_{nA}|^{1/n} \geq 1/\operatorname{cap}(E), \quad n \in \Lambda.$$

Since the p^*_{nA} are uniformly bounded on E, the monic polynomials $p_n(z) := p^*_{nA}(z)/a^*_{nA}$, $n \in \Lambda$, satisfy condition (i) of Lemma 4.1. Finally the assumption that f does not identically vanish in any component of E^o together with Hurwitz's theorem imply that condition (ii) of Lemma 4.1 also holds for the sequence $\{p_n\}_{n\in\Lambda}$. Hence $\nu(p^*_{nA}) = \nu(p_n) \xrightarrow{*} \mu_E$, as $n \to \infty, n \in \Lambda$, by Lemma 4.1. ∎

REMARK. As can be seen from the proof, conclusion (ii) of Theorem 4.2 holds for any sequence $\Lambda \subseteq \mathbf{N}$ such that

$$\lim_{\substack{n\to\infty\\ n\in\Lambda}} [E_{n-1}(A,f) - E_n(A,f)]^{1/n} = 1$$

THEOREM 4.3. *Assume E is compact, $0 \in \partial E$, $\operatorname{cap}(E) > 0$, and $K = \overline{\mathbf{C}} \setminus E$ is connected and regular. Suppose f is analytic on E and f does not vanish identically on any component of E^o. Furthermore, assume $B_k(A) \setminus \mathcal{P}_{k,0} \neq \emptyset$ and $A\underline{f} \neq \underline{0}$. Then*

(i) $\limsup_{n\to\infty} E_n^{1/n}(A,f) = 1$;

(ii) $\nu(p^*_{nA}) \xrightarrow{*} \mu_E$, as $n \to \infty$, $n \in \Lambda$, where $\Lambda \subseteq \mathbf{N}$ is a sequence that depends on f.

PROOF. From Lemma 2.1 we know that

$$\alpha_{nA}(p^*_n) \leq E_n(f) + E_n(A,f) \leq 2E_n(A,f).$$

Together with (2.6), (2.10) and (2.11), for n large we have (with the same i_0 as in (2.9))

$$2E_n(A,f) \geq \alpha_{nA}(p^*_n) \geq \frac{1}{2}\left|\sum_{j=0}^{k} a_{i_0,j} f^{(j)}(0)\right| \beta^{-1}_{nA}.$$

Thus (2.8) and (4.8) imply that

$$\limsup_{n\to\infty} E_n^{1/n}(A,f) \geq \limsup_{n\to\infty} 1/\beta^{1/n}_{nA} \geq 1.$$

Since $1 \geq \limsup_{n\to\infty} E_n^{1/n}(A,f)$, we see that (i) holds.

The proof of (ii) is now the same as that of (ii) in Theorem 4.2. ∎

We next consider the case when 0 is outside E.

THEOREM 4.4. *Suppose E is compact, $K = \overline{\mathbb{C}} \setminus E$ is connected and regular, $0 \notin E$ and $g_K(0, \infty) = \log \sigma (\sigma > 1)$. Assume $f(z)$ is analytic on E_σ and does not vanish identically on any component of E_σ°. If $A\underline{f} \neq \underline{0}$, then*

(i) $\limsup_{n\to\infty} E_n^{1/n}(A, f) = 1/\sigma$;

(ii) $\nu(p_{n,k}^*) \xrightarrow{*} \mu_{E_\sigma}$, *as* $n \to \infty$, $n \in \Lambda$, *where* $\Lambda \subseteq \mathbb{N}$ *is a sequence that depends on* f.

REMARK. If $f \in \mathcal{A}(E_\sigma)$, but f is not analytic on E_σ, then (i) holds because $\limsup_{n\to\infty} E_n^{1/n}(f) = 1/\sigma$. If f is analytic on E_σ, then $\limsup_{n\to\infty} E_n^{1/n}(f) < 1/\sigma$; however Theorem 4.4 asserts that (i) holds provided $A\underline{f} \neq \underline{0}$.

PROOF OF THEOREM 4.4. We know that (*cf.* [W, §4.7]) since f is analytic on E_σ

$$\limsup_{n\to\infty} E_n^{1/n}(f) < 1/\sigma \tag{4.9}$$

and $\{p_n^*\}_{n=0}^\infty$ converges uniformly to f on some open set containing E_σ. Consequently,

$$\lim_{n\to\infty} p_n^{*(j)}(0) = f^{(j)}(0), \quad j = 0, \quad , k. \tag{4.10}$$

Let $\delta \in (\sigma, \infty)$. As in the proof of Theorem 2.5, for $p \in \mathcal{P}_n$, we have

$$|p^{(j)}(0)| \leq \frac{j!}{2\pi} \delta^n \frac{\text{length}(\Gamma_\delta)}{\text{dist}(0, \Gamma_\delta)^{j+1}}.$$

According to the definition of $\beta_{n,A}$ in (2.7) we get $\limsup_{n\to\infty} \beta_{n,A}^{1/n} \leq \delta$ and letting $\delta \to \sigma^+$ yields

$$\limsup_{n\to\infty} \beta_{n,A}^{1/n} \leq \sigma. \tag{4.11}$$

From (4.7), (2.10) and (4.10), we again deduce (4.8). Combining this with (4.11) we obtain

$$\limsup_{n\to\infty} E_n^{1/n}(A, f) \geq 1/\sigma. \tag{4.12}$$

Note that $f(z)/z^{k+1}$ has a pole at $z = 0$ (since $A\underline{f} \neq \underline{0}$) and so

$$\limsup_{n\to\infty} E_n^{1/n}\left(f(z)/z^{k+1}\right) = 1/\sigma.$$

Hence for $\varepsilon > 0$ there is a polynomial $q_{n-k-1} \in \mathcal{P}_{n-k-1}$ such that, for n large,

$$\|f(z)/z^{k+1} - q_{n-k-1}(z)\| \leq [(1+\varepsilon)/\sigma]^{n-k-}$$

and so

$$\|f(z) - z^{k+1} q_{n-k-1}(z)\| \leq \|z^{k+1}\| [(1+\varepsilon)/\sigma]^{n-k-1}.$$

Note that $z^{k+1} q_{n-k-1}(z) \in B_n(A)$ so we have

$$\limsup_{n\to\infty} E_n^{1/n}(A, f) \leq \limsup_{n\to\infty} \|f(z) - z^{k+1} q_{n-k-1}(z)\|^{1/n} \leq (1+\varepsilon)/\sigma.$$

As $\varepsilon > 0$ is arbitrary, we get $\limsup_{n\to\infty} E_n^{1/n}(A,f) \le 1/\sigma$. Together with (4.12), this yields (i).

Now we prove (ii). It suffices to check that the conditions in Lemma 4.1 are satisfied for $p_n(z) := p^*_{nA}(z)/a^*_{nA}$ and E replaced by E_σ. Since $\{p^*_{nA}\}_{n=0}^{\infty}$ converges uniformly to f on every closed set $D \subset E^o_\sigma$, the condition (ii) in Lemma 4.1 is satisfied for the sequence $\{p_n\}_{n\in\mathbf{N}}$.

From (4.3) we have

$$E_{n-1}(A,f) - E_n(A,f) \le |a^*_{nA}|\,\|T_{nA}\|.$$

Also from (i) and the fact that $\lim_{n\to\infty} \|T_{nA}\|^{1/n} = \operatorname{cap}(E)$, we have for a suitable
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subsequence Λ

$$\liminf_{n\to\infty} |a^*_{nA}|^{1/n} \ge \frac{1}{\operatorname{cap}(E)\sigma}, \quad n \in \Lambda \subseteq \mathbf{N}. \tag{4.13}$$

Note that by (i) for any $\rho < \sigma$ we have (*cf.* [W, §4.7])

$$\|p^*_{nA} - f\|_{E_\rho} \to 0 \text{ as } n \to \infty,$$

where $\|\cdot\|_{E_\rho}$ denotes the uniform norm on E_ρ. Thus the sequence $\{\|p^*_{nA}\|_{E_\rho}\}_{n=0}^{\infty}$ is bounded and using Lemma 2.4 we have

$$\limsup_{n\to\infty} \|p^*_{nA}\|_{E_\sigma}^{1/n} \le \sigma/\rho.$$

Letting $\rho \to \sigma^-$ we obtain

$$\limsup_{n\to\infty} \|p^*_{nA}\|_{E_\sigma}^{1/n} \le 1. \tag{4.14}$$

For the monic polynomials p_n, by (4.13) and (4.14) we therefore have

$$\limsup_{n\to\infty} \|p_n\|_{E_\sigma}^{1/n} \le \operatorname{cap}(E)\sigma = \operatorname{cap}(E_\sigma), \quad n \in \Lambda.$$

This yields condition (i) in Lemma 4.1 and completes the proof. ∎

5. Comparison of rates of convergence. In this section we will prove that when $0 \notin E^o$ there are " relatively few " functions $f \in \mathcal{A}(E)$ (in the sense of category) with rate of the convergence of $E_n(f)$ faster than that of $E_n(A,f)$.

For the case when $0 \in E^o$ the following result is straightforward to establish (*cf.* the proof of Theorem 3.2).

THEOREM 5.1. *Let E be a compact set, $\overline{\mathbf{C}}\backslash E$ be connected, and* $0 \in E^o$. *If $f \in \mathcal{A}(E)$ and $A\underline{f} = \underline{0}$, then*

$$E_n(A,f) = O\big(E_n(f)\big).$$

In the proof of the main result of this section we follow an argument of Saff and Totik which utilizes the following.

LEMMA 5.2 ([ST, PROOF OF THEOREM 1]). *For any integer $n_0 > k$, there is an $f \in \mathcal{A}(E)$ such that $\|f\| = 1$ and $p^*_{n_0}(f) \equiv 0$. In particular, $E_{n_0}(f) = E_{n_0}(A, f) = 1$.*

We can now state our main result.

THEOREM 5.3. *Let E be compact with $K = \overline{\mathbb{C}} \setminus E$ connected and $0 \notin E^o$. If $B_k(A) \setminus \mathcal{P}_{k.0} \neq \emptyset$, then the set S of functions $f \in \mathcal{A}(E)$ for which*

$$\limsup_{n \to \infty} \frac{E_n(f)}{E_n(A, f)} < 1 \tag{5.1}$$

is of the first category in the complete metric space $\mathcal{A}(E)$.

So that (5.1) is meaningful for all $f \in \mathcal{A}(E)$ we set $E_n(f)/E_n(A, f) = 0$ whenever $f \in \mathcal{P}_n$.

PROOF. Let

$$S_{m.l} := \left\{ f \in \mathcal{A}(E) : \frac{E_n(f)}{E_n(A, f)} \leq 1 - 1/m \text{ for all } n \geq l \right\}$$

$$S = \bigcup_{m=1}^{\infty} \bigcup_{l=1}^{\infty} S_{m.l}.$$

Assume to the contrary that S is not of the first category. Then for some m and l the set $S_{m.l}$ is not nowhere dense in $\mathcal{A}(E)$. We claim that $S_{m.l}$ is closed. In fact, if $\{f_\nu\}_{\nu=1}^{\infty} \subseteq S_{m.l}$ and f_ν converges to f uniformly on E, then $E_n(f_\nu) \to E_n(f)$ and $E_n(A, f_\nu) \to E_n(A, f)$ as $\nu \to \infty$ for fixed $n \geq l$, and so $E_n(f)/E_n(A, f) \leq 1 - 1/m$; that is, $f \in S_{m.l}$.

Since $S_{m.l}$ is closed and not nowhere dense in $\mathcal{A}(E)$, there is an $f_0 \in \mathcal{A}(E)$ and a $\delta_0 > 0$ such that the δ_0-neighborhood of f_0 in $\mathcal{A}(E)$ is contained in $S_{m.l}$. Choose a polynomial $p_0 \in B_{\deg p_0}(A)$ with $\|f_0 - p_0\| < \delta_0/2$ (this can be done by Theorems 3.1 and 3.4) and set $n_0 := \max\{l, \deg p_0\}$. If $f (\|f\| \neq 0)$ is any function in $\mathcal{A}(E)$, then the function

$$f^*(z) := p_0(z) + \frac{1}{2}\delta_0 \|f\|^{-1} f(z)$$

belongs to the δ_0-neighborhood of f_0. Hence

$$\frac{E_{n_0}(f^*)}{E_{n_0}(A, f^*)} \leq 1 - 1/m.$$

But note that since $p_0 \in B_{\deg p_0}(A)$ we have

$$E_{n_0}(f^*) = \delta_0 \|f\|^{-1} E_n(f)/2,$$

and

$$E_{n_0}(A, f^*) = \delta_0 \|f\|^{-1} E_n(A, f)/2.$$

Thus we can conclude that for every function $f \in \mathcal{A}(E) \setminus B_{n_0}(A)$,

$$\frac{E_{n_0}(f)}{E_{n_0}(A, f)} \leq 1 - 1/m,$$

which is impossible by Lemma 5.2. ∎

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