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Interpolatory properties of best L_2-approximants

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ABSTRACT

Let f be a continuous function and s_n be the polynomial of degree at most n of best $L_2(\mu)$-approximation to f on $[-1,1]$. Let $Z_n(f) := \{x \in [-1,1] : f(x) - s_n(x) = 0\}$. Under mild conditions on the measure μ, we prove that $\cup Z_n(f)$ is dense in $[-1,1]$. This answers a question posed independently by A. Kroó and V. Tikhomiroff. It also provides an analogue of the results of Kadec and Tashev (for L_∞) and Kroó and Peherstorfer (for L_1) for least squares approximation.

1. INTRODUCTION

Let f be a continuous real-valued function on $[-1,1]$. Let $\mathscr{P}_n$ denote the set of algebraic polynomials of degree at most n and let $s_{n,p}$ be the polynomial of best approximation to f from $\mathscr{P}_n$ in the norm of the space $L_p(\mu)$. It is well-known (cf. [Si]) that this polynomial satisfies the orthogonality condition:

$$(1.1) \qquad \int_{-1}^{1} |f(x) - s_{n,p}(x)|^{p-1} \operatorname{sign}(f(x) - s_{n,p}(x))\, q(x)\, d\mu = 0$$

for all $q \in \mathscr{P}_n$. As an immediate consequence of (1.1) we see that if the support of μ contains infinitely many points, then for each $n = 0, 1, \ldots$ the error $f(x) - s_{n,p}(x)$ vanishes in at least $n+1$ points of the interval $[-1,1]$; i.e., $s_{n,p}(x)$ interpolates $f(x)$ at $n+1$ points of $[-1,1]$.

This property raises the natural question of the denseness of such points. For $p = \infty$, it was proved by Kadec [Ka] that for a subsequence of integers n_j, these

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interpolation points are dense in the interval $[-1, 1]$ (in fact, he showed they have the arcsine distribution). On the other hand, Lorentz [L] gave an example showing that there exists a function f and a subsequence $\{n_k\}$ such that the zeros of $f(x) - s_{n_k, \infty}(x)$ avoid a subinterval of $[-1, 1]$. Further investigations of the case $p = \infty$ can be found in [BL], [BKGS], [KS], [BST], [Ta].

For $p = 1$ this problem was addressed by Kroó and Peherstorfer [KP1], [KP2]. They observed that in this case the zeros of $f(x) - s_{n_k, 1}(x)$ are dense on $[-1, 1]$ for any subsequence n_k. Moreover, this result holds if the subspaces $\mathscr{P}_n$ are replaced by an arbitrary sequence of subspaces $U_n \subset L_1(\mu)$ such that every function $f \in L_1(\mu)$ is a limit of a sequence $\{u_n\}$, $u_n \in U_n$.

The analogous question for $L_2(\mu)$ was posed independently by A. Kroó and V. Tikhomiroff (cf. [Kr], [Ti]). In this paper we show that, under mild conditions on the measure μ, the zeros of $\{f(x) - s_{n,2}(x)\}_{n=1}^{\infty}$ are indeed dense in $[-1, 1]$. Moreover we prove that, as in the case $p = \infty$, this result cannot be extended to an arbitrary subsequence $\{n_k\}$. Furthermore, we give an example of a complete orthonormal sequence $u_n \in L_2(\mu)$ and a function $\sum_{n=0}^{\infty} a_n u_n \in L_2(\mu)$ such that the remainders $\sum_{k=n}^{\infty} a_k u_k, n = 1, 2, \ldots,$ are all positive on a fixed subinterval of $[-1, 1]$.

2. MAIN THEOREM

Throughout this section μ denotes a finite, positive Borel measure on $[-1, 1]$ such that $\mu' > 0$ a.e. (with respect to Lebesgue measure) on $[-1, 1]$. Let $p_n = p_n(x, \mu)$ be the sequence of orthonormal polynomials with respect to the measure μ. For $f \in L_2(\mu)$ we let

$$s_n = s_{n,2}(f) := \sum_{k=0}^{n} a_k p_k, \quad a_k = a_k(f) := \int_{-1}^{1} f p_k \, d\mu,$$

denote the n-th partial sum of the Fourier expansion $\sum_0^{\infty} a_k p_k$ of f. As is well known,

$$\int_{-1}^{1} |f - s_n|^2 \, d\mu \leq \int_{-1}^{1} |f - P|^2 \, d\mu, \text{ for all } P \in \mathscr{P}_n.$$

THEOREM 2.1. *Let $f \in L_2(\mu)$, with f not $d\mu$-a.e. equal to a polynomial. Let $[a, b]$ be an arbitrary subinterval of $[-1, 1]$ having positive length. Then there exists a subsequence of integers $\{n_j\}_{j=1}^{\infty}$, $n_1 < n_2 < \ldots$, such that $f(x) - s_{n_j}(x)$ changes sign on the interval $[a, b]$.*

The proof requires several lemmas.

LEMMA 2.2. [MNT] *For each subinterval $[a, b] \subset [-1, 1]$ $(a \neq b)$, there exists a constant $\tau > 0$, depending only on $b - a$, such that the orthonormal polynomials p_n satisfy*

$$\int_a^b |p_n|^2 \, d\mu \geq \tau, \quad n = 0, 1, 2, \ldots.$$

LEMMA 2.3. *Let* $[a,b]\subset[-1,1] (a\neq b)$. *Then*

$$\limsup_{n\to\infty}\left[\frac{(\int_a^b |p_n|^2 d\mu)^{1/2}}{\int_a^b |p_n| d\mu}\right]^{1/n}\leq 1.$$

PROOF. Since $\mu'>0$ a.e. on $[a,b]$, the restricted measure $\mu|_{[a,b]}$ is a regular measure on $[a,b]$. Hence by the result of [ST, Lemma 2.1], we have

$$\limsup_{n\to\infty}\left[\frac{\max\{|p_n(x)|:x\in[a,b]\}}{\int_a^b |p_n| d\mu}\right]^{1/n}\leq 1.$$

From this, the lemma immediately follows. ■

Combining Lemmas 2.2 and 2.3 we get

COROLLARY 2.4. *Let* $[a,b]\subset[-1,1]$ $(a\neq b)$. *Then*

$$\liminf_{n\to\infty}\left(\int_a^b |p_n| d\mu\right)^{1/n}\geq 1.$$

The next lemma is in the spirit of the "contamination principle" introduced by Saff (cf. [S], [LSS]). For a fixed function f and for $-1\leq a'<b'\leq 1$, let

$$E_n=E_n(f):=\|f-s_n\|_{L_2(\mu)}=\left(\int_{-1}^{1} |f-s_n|^2 d\mu\right)^{1/2},$$

and

$$r_n=r_n(f,a',b'):=\frac{1}{E_n}\int_{a'}^{b'} |f-s_n| d\mu.$$

LEMMA 2.5. *For any* $f\in L_2(\mu)$, *with* f *not* $d\mu$*-a.e. a polynomial, the exists a subsequence* $\{n_j\}\subset\mathbb{N}$ *such that for any* $-1\leq a'<b'\leq 1$,

$$\max\{r_{n_j}, r_{n_j-1}\}\geq\frac{1}{n_j}\int_{a'}^{b'} |p_{n_j}| d\mu. \tag{2.1}$$

PROOF. We have

$$\int_{a'}^{b'} |a_n p_n| d\mu=\int_{a'}^{b'} |s_n-s_{n-1}| d\mu\leq\int_{a'}^{b'} |f-s_n| d\mu+\int_{a'}^{b'} |f-s_{n-1}| d\mu$$

$$=r_nE_n+r_{n-1}E_{n-1}\leq\max(r_n,r_{n-1})\cdot[E_n+E_{n-1}].$$

On the other hand,

$$\int_{a'}^{b'} |a_n p_n| d\mu=|a_n|\int_{a'}^{b'} |p_n| d\mu=\sqrt{E_{n-1}^2-E_n^2}\int_{a'}^{b'} |p_n| d\mu.$$

Hence

$$\max(r_n,r_{n-1})\geq\left(\frac{E_{n-1}-E_n}{E_{n-1}+E_n}\right)^{1/2}\cdot\int_{a'}^{b'} |p_n| d\mu. \tag{2.2}$$

Now observe that the positive sequence $\{E_n\}$ decreases to zero as $n \to \infty$; thus from elementary properties of series, it follows that

$$\sum_{n=1}^{\infty} \frac{E_{n-1}-E_n}{E_{n-1}+E_n} = \infty.$$

Consequently there exists a subsequence $\{n_j\} \subset \mathbb{N}$ such that

$$\frac{E_{n_j-1}-E_{n_j}}{E_{n_j-1}+E_{n_j}} \geq \frac{1}{n_j^2}, \quad j=1,2,\dots.$$

Combining this with (2.2) we get the desired result. ■

The next lemma is a slight modification of [BKGS, Lemma 2.1].

LEMMA 2.6. *Given* $-1 \leq a < b \leq 1$ *and* $n \in \mathbb{N}$, *let* $\eta := (b-a)/4$ *and set* $a' := a+\eta$, $b' := b-\eta$. *Then there exists a* $q_n \in \mathcal{P}_n$ *such that*

(2.3) $\quad \max\{|q_n(x)| : x \in [-1,a] \cup [b,1]\} \leq 1,$

(2.4) $\quad q_n(x) \geq 0$ *on* $[a,b]$,

(2.5) $\quad q_n(x) \geq \frac{1}{4}(1+\sqrt{2\eta})^{n/2} \geq c_1 e^{c_2 n}$ *on* $[a', b']$,

where $c_1, c_2 > 0$ *are constants independent of* n (*but dependent on* a *and* b).

PROOF. Let

$$T_m(x) := \cos(m \arccos x)$$

denote the Chebyshev polynomial of degree m. For $m := [n/2]$, $\sigma := 2\eta = (b-a)/2$, set

$$q_n(x) := \frac{1}{2} T_m\left(1 + \frac{\sigma^2}{2} - \left(2 + \frac{\sigma^2}{2}\right)\frac{y^2}{4}\right),$$

where $y = x - (b+a)/2$. It is easy to see that, for $x \in [-1,a] \cup [b,1]$;

$$1 + \frac{\sigma^2}{2} - \left(2 + \frac{\sigma^2}{2}\right)\frac{y^2}{4} \in [-1,1]$$

and so $|q_n(x)| \leq 1$. On the other hand, on the interval $[a', b']$, the polynomial q_n attains its minimum at one of the endpoints $x = a+\eta$ or $x = b-\eta$. Substituting these values of x we get

$$q_n(x) = \frac{1}{2} T_m\left(1 + \frac{3}{2}\eta^2 - \frac{\eta^4}{2}\right) \geq \frac{1}{2} T_m(1+\eta^2) \geq \frac{1}{4}(1+\sqrt{2\eta})^m. \quad ■$$

PROOF OF THE THEOREM 2.1. Let $-1 \leq a < b \leq 1$ and suppose to the contrary that for all n sufficiently large the functions

$$f(x) - s_n(x)$$

do not change sign on $[a,b]$. Let a', b' and q_n be as in Lemma 2.6. In what follows it suffices to assume that $f(x)-s_n(x)\geq 0$ $d\mu$-a.e. on $[a,b]$ (otherwise we can use $-q_n(x)$). By the orthogonality condition (1.1) we have

$$0=\int_{-1}^{1} q_n(f-s_n)d\mu=\int_{a}^{b} q_n(f-s_n)d\mu+\int_{[-1,1]\setminus[a,b]} q_n(f-s_n)d\mu$$
$$\geq \int_{a'}^{b'} q_n(f-s_n)d\mu - M\Big(\int_{-1}^{1} |f-s_n|^2 d\mu\Big)^{1/2},$$

where $M:=\sqrt{\mu([-1,1])}$. Hence

$$M\Big(\int_{-1}^{1} |f-s_n|^2 d\mu\Big)^{1/2}\geq \int_{a'}^{b'} q_n|f-s_n|d\mu$$
$$\geq c_1 e^{c_2 n}\int_{a'}^{b'} |f-s_n|d\mu,$$

and so for all n large,

$$\text{(2.6)}\qquad \frac{M}{c_1 e^{c_2 n}}\geq r_n(f,a',b'):=\frac{1}{E_n}\int_{a'}^{b'}|f-s_n|d\mu.$$

Observe that from Lemma 2.5 we can find a sequence $\{n_j\}_1^\infty$ such that

$$[\max(r_{n_j}, r_{n_j-1})]^{1/n_j}\geq\left(\frac{1}{n_j}\right)^{1/n_j}\cdot\Big(\int_{a'}^{b'}|p_{n_j}|d\mu\Big)^{1/n_j}.$$

From this and Corollary 2.4 we see that there exists a (possibly different) subsequence $\{n_j\}$ such that

$$\text{(2.7)}\qquad \liminf_{j\to\infty}(r_{n_j})^{1/n_j}\geq 1.$$

Combining (2.7) with (2.6) we obtain

$$1>\frac{1}{e^{c_2}}=\limsup_{j\to\infty}\left(\frac{M}{c_1 e^{c_2 n_j}}\right)^{1/n_j}\geq\liminf_{j\to\infty}(r_{n_j})^{1/n_j}=1,$$

which yields the desired contradiction. ■

REMARK. Corollary 2.4 and hence Theorem 2.1 hold, more generally, for regular measures μ; that is, measures for which the leading coefficients γ_n of $p_n(x,\mu)$ satisfy $\lim_{n\to\infty}\gamma_n^{1/n}=2$.

3. SHARPNESS

In this section we analyze the sharpness of Theorem 2.1. Our first result shows that the conclusion of Theorem 2.1 does not, in general, hold for *all* sufficiently large integers n.

THEOREM 3.1. *Given $\varepsilon>0$ there exists an entire function g and a subsequence $\{n_j\}\subset\mathbb{N}$ such that for the Legendre expansion of g on $[-1,1]$ all the zeros of $\{g-s_{n_j}(g)\}_{j=1}^\infty$ lie outside of the interval $(-1,1-\varepsilon)$.*

PROOF. Choose $\beta>0$ such that

$$(3.1) \qquad 1-\frac{\varepsilon}{2}<\frac{2\beta^2}{(2+\beta)^2}-1,$$

and for $k=1,2,\ldots$, select a sequence of positive integers $m_1<m_2<\cdots$ such that

$$(3.2) \qquad \lim_{k\to\infty}\frac{m_k}{k}=\beta.$$

Consider the Jacobi polynomials $P_k^{(0,m_k)}(x)=:Q_k$ of degree k corresponding to the weight $(x+1)^{m_k}$. From (3.1) and (3.2) it is known (cf. [MSV], Corollary 1) that all the zeros of Q_k lie in the interval $(1-\varepsilon,1)$, for all sufficiently large k, say $k\geq k_0$. Hereafter we consider only even indices $k\geq k_0$ so that $Q_k>0$ for $x\in(-1,1-\varepsilon)$.

Let P_n denote the Legendre polynomial of degree n. Then

$$(x+1)^{m_k}Q_k(x)=\sum_{n=0}^{m_k+k}c_nP_n(x),$$

where

$$c_n:=\frac{2n+1}{2}\int_{-1}^{1}(x+1)^{m_k}Q_k(x)P_n(x)dx.$$

Notice that from the definition of Q_k we have $c_n=0$ for $n<k$. Hence

$$(3.3) \qquad (x+1)^{m_k}Q_k(x)=\sum_{n=k}^{m_k+k}c_nP_n(x).$$

Choose integers k_j so that $m_{k_j}+k_j<k_{j+1}$ and set

$$R_j(x):=(x+1)^{m_{k_j}}Q_{k_j}(x).$$

Next choose positive constants a_j so small that the function

$$g(x):=\sum_{j=1}^{\infty}a_jR_j(x)$$

is entire. From (3.3) it follows that the $(k_j+m_{k_j})$-th partial sum of the Legendre expansion of g is given by

$$s_{k_j+m_{k_j}}(x)=\sum_{i=1}^{j}a_iR_i(x)$$

and so

$$g(x)-s_{k_j+m_{k_j}}(x)=\sum_{i=j+1}^{\infty}a_iR_i(x).$$

Since, for all i, we have $a_i>0$ and $R_i(x)>0$ for $x\in(-1,1-\varepsilon)$, we see that the sequence $n_j:=k_j+m_{k_j}$ verifies the theorem. ■

REMARK. Similar examples can be constructed using the approach in [KP2].

Next we give an example of a complete orthonormal sequence of functions in $L_2[-1,1]$ for which the appropriate differences $g-s_n$ do not change sign on

$[-1,0]$. Hence (unlike the L_1 case) the polynomial spaces $\mathscr{P}_n$ cannot be replaced by an arbitrary sequence of subspaces U_n with $\overline{\cup U_n}=L_2$.

THEOREM 3.2. *There exists a complete orthonormal sequence $u_n \in L_2[-1,1]$ and a function*

$$g=\sum_{k=0}^{\infty} a_k u_k$$

such that $\sum_{k=n}^{\infty} a_k u_k$ is positive on $[-1,0]$ for all n.

PROOF. Define a sequence of functions $\{\varphi_k\}$ by $\varphi_0 \equiv 1$,

$$\varphi_n(x):=\begin{cases} 1, & \text{if } x\in\left[-1,\dfrac{1}{2^n}\right] \\ b_n, & \text{if } x\in\left(\dfrac{1}{2^n},\dfrac{1}{2^{n-1}}\right] \\ 0, & \text{if } x\in\left(\dfrac{1}{2^{n-1}},1\right] \end{cases}$$

where $b_n:=-(2^n+1), n=1,2,\dots$.

It is easy to see that φ_n is an orthogonal sequence of functions in $L_2[-1,1]$ and that $\varphi_n(x)>0$ for all $x\in[-1,0], n=0,1,\dots$. Let $\{\Psi_k\}$ be an orthonormal basis in $[\operatorname{span}\{\varphi_k\}]^{\perp}$.

Define

$$u_n:=\begin{cases} \varphi_k/\|\varphi_k\|_{L_2}, & \text{if } n=2k \\ \Psi_k, & \text{if } n=2k+1. \end{cases}$$

Then u_n is an orthonormal basis in $L_2[-1,1]$. Set

$$g:=\sum_{k=0}^{\infty} c_k u_{2k};\ c_k>0;\ \sum_{k=0}^{\infty} c_k^2<\infty.$$

Then $g\in L_2[-1,1]$ and verifies the theorem. ■

REMARK. For most of the standard approximating families such as trigonometric polynomials, Walsh functions and various spline functions the analog of the Theorem 2.1 holds. The trigonometric case follows from the algebraic case by the standard change of variable. The other two cases follow from the existence of locally supported functions.

REMARK. It is interesting to compare Theorems 3.1 and 3.2 with the Theorem 3 of [KP1].

4. TWO CONJECTURES

We conclude with the discussion of two bold conjectures.

CONJECTURE 1. *Let $f \in C[-1,1]$ and let $s_{n,p} \in \mathscr{P}_n$ be the best $L_p(\mu)$, $1<p<\infty$, approximation to f from $\mathscr{P}_n$. Let $\{t_j^{(n)}\}_{j=1}^n$ be an n-point Fekete subset of the zero set $Z(f-s_{n,p})$. Then there exists a subsequence n_k such that*

$$\frac{\#\{t_j^{(n_k)} : t_j^{(n_k)} \in (a,b)\}}{n_k} \to \frac{1}{\pi}(\arcsin b - \arcsin a)$$

for all intervals $(a,b) \subset [-1,1]$.

In support of this conjecture we present the following theorem for Jacobi expansions (cf. [Sz], [N]).

THEOREM 4.1. *Let $d\mu = (1-x)^\alpha (1+x)^\beta dx$, $\alpha > -1, \beta > -1$ and let $f \in L_2(\mu)$, with f not $d\mu$-a.e. equal to a polynomial. If $s_n(x)$ is the best $L_2(\mu)$ approximation to f from $\mathscr{P}_n$, then there exists a subsequence n_j and a positive constant $c>0$ such that*

(a) *Either $f(x)-s_{n_j}(x)$ or $f(x)-s_{n_j-1}(x)$ changes sign on every interval (a,b) with*

$$b-a \geq c \cdot \frac{\log n_j}{n_j}.$$

(b) *If b is sufficiently close to 1, then either $f(x)-s_{n_j}(x)$ or $f(x)-s_{n_j-1}(x)$ changes sign on the interval (a,b) if*

$$b-a \geq c\left(\frac{\log n_j}{n_j}\right)^2.$$

PROOF. Using the asymptotic properties of Jacobi polynomials ([Sz, p. 167–169]) it is easy to conclude that

$$\int_a^b |p_n| d\mu \geq \frac{k}{n^\sigma}(b-a)$$

for $b-a \geq k_1 \log n/n$ and for $b-a \geq k_2(\log n/n)^2$ if b is sufficiently close to 1. (Here k, k_1, k_2 and σ are positive constants that depend on α, β.) The estimates then follow from (2.6), (2.1) and (2.5) by straightforward computations. ■

REMARK. At present we are unable to prove the analog of the Theorem 2.1 for $p \neq 1, 2, \infty$. The best we can do is

PROPOSITION 4.2. *Let $f \in L_p(\mu)$ and assume that f is not $d\mu$-a.e. equal to a function analytic in a 2-dimensional neighborhood of $[-1,1]$. Then for every interval $(a,b) \subset [-1,1]$, there exists a subsequence $\{n_j\} \subset \mathbb{N}$ such that $f(x)-s_{n_j,p}(x)$ changes sign on (a,b).*

PROOF. Suppose that for all n large $f(x)-s_{n,p}(x)$ does not change sign on $[a,b]$. Then using the equation (1.1) with q_n from Lemma 2.6 we obtain (as in the proof of the Theorem 2.1)

$$\int_{a'}^{b'} |f-s_{n,p}|^{p-1}\,d\mu \le \frac{c_1}{e^{c_2 n}}\,\|f-s_{n,p}\|^p_{L_p(\mu)}.$$

Since $\|f-s_{n,p}\|^p_{L_p(\mu)} \to 0$, we see that the sequence $s_{n,p}$ converges to f geometrically fast on $[a',b']$ in the $L_{p-1}(\mu)$-norm. Hence (cf. [ST]) the function f is analytic on $[a',b']$ and the sequence $s_{n,p}(z)$ converges uniformly to $f(z)$ in some ellipse containing $[a',b']$. However, since f is not analytic on $[-1,1]$, it is known (cf. [LSS]) that the sequence $\{s_{n,p}\}_{n=1}^{\infty}$ does not converge uniformly in any open disk containing a point of $[-1,1]$. This yields the desired contradiction. ■

CONJECTURE 2. *Let* $1 \le p \le \infty, f \in C[-1,1]$, *and* $s_{n,p} \in \mathscr{P}_n$ *be its best* $L_p(\mu)$ *approximation. If there exists a subsequence* $\{n_k\}$ *such that* $f-s_{n_k,p}$ *is zero-free on some interval* $(a,b) \subset [-1,1]$, *then* $\{n_k\}$ *is in some sense lacunary.*

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