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Behavior of Best L_p Polynomial Approximants on the Unit Interval and on the Unit Circle

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For function f defined on the interval $I := [-1, 1]$, let $p^*_{n,2}(f)$ be its best approximant out of $\mathscr{P}_n$ under the L_2 norm

$$\|g\|_{L_2(d\alpha)} := \left(\int_I |g(x)|^2 \, d\alpha\right)^{1/2},$$

where $d\alpha$ is a finite Borel measure on I. We compare the L_2 norm of the error function $f - p^*_{n,2}(f)$ on subintervals vs that on the whole interval I. Then we consider the distribution of the zeros of the best L_p approximants. Corresponding results are also obtained for approximation on the unit circle $\{z \in \mathbf{C} : |z| = 1\}$.

1. INTRODUCTION

Let $\mathbf{C}$ be the complex plane, $\Delta := \{z \in \mathbf{C} : |z| \leqslant 1\}$ the closed unit disk, and $I := [-1, 1]$ the closed unit interval. Throughout this chapter, we use

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$d\alpha$ to denote a finite positive Borel measure on I with supp$(d\alpha)$ an infinite set, and $d\mu$ to denote a finite positive Borel measure on $\partial\Delta := \{z \in \mathbf{C} : |z| = 1\}$ with supp$(d\mu)$ an infinite set. Given $p > 0$, for a Borel set $E \subset I$, define

$$\|f\|_{L_p(d\alpha, E)} := \left(\int_E |f(x)|^p \, d\alpha \right)^{1/p},$$

while for a Borel set $F \subset \partial\Delta$, define

$$\|f\|_{L_p(d\mu, F)} := \left(\int_F |f(e^{i\theta})|^p \, d\mu \right)^{1/p}.$$

Let $L_p(d\alpha)$ (resp. $L_p(d\mu)$) be the space of Borel measurable functions f on I (resp. $\partial\Delta$) with $\|f\|_{L_p(d\alpha)} := \|f\|_{L_p(d\alpha, I)} < \infty$ (resp. $\|f\|_{L_p(d\mu)} := \|f\|_{L_p(d\mu, \partial\Delta)} < \infty$).

For a given $f \in C(I)$ (we use $C(K)$ to denote the space of continuous functions defined on $K \subset \mathbf{C}$), we denote by $p^*_{n,\infty}(f)$ its best uniform approximant out of $\mathscr{P}_n$, the set of all algebraic polynomials of degree at most n, i.e.,

$$\|f - p^*_{n,\infty}(f)\|_I := \inf_{p_n \in \mathscr{P}_n} \|f - p_n\|_I,$$

where $\|\cdot\|_K$ means the uniform norm on $K \subset \mathbf{C}$. Similarly, define $s^*_{n,\infty}(f)$ (for $f \in C(\partial\Delta)$), $p^*_{n,p}(f)$ (for $f \in L_p(d\alpha)$) and $s^*_{n,p}(f)$ (for $f \in L_p(d\mu)$) in $\mathscr{P}_n$ as follows:

$$\|f - s^*_{n,\infty}(f)\|_{\partial\Delta} := \inf_{p_n \in \mathscr{P}_n} \|f - p_n\|_{\partial\Delta},$$

$$\|f - p^*_{n,p}(f)\|_{L_p(d\alpha)} := \inf_{p_n \in \mathscr{P}_n} \|f - p_n\|_{L_p(d\alpha)},$$

and

$$\|f - s^*_{n,p}(f)\|_{L_p(d\mu)} := \inf_{p_n \in \mathscr{P}_n} \|f - p_n\|_{L_p(d\mu)}.$$

Kadec [6] proved that for *real-valued* $f \in C(I)$, there are $(n+2)$-point subsets of the extremal point sets $A_n := \{x \in I : |f(x) - p^*_{n,\infty}(f, x)| = \|f - p^*_{n,\infty}(f)\|_I\}$ that, for a suitable subsequence of integers n, are distributed like the extrema of Chebyshev polynomials $T_n(x) := (1/2^{n-1}) \cos(n \arccos x)$. So, by the denseness of such extrema, there is an increasing subsequence of the positive integers, say $\Lambda(f) \subset \mathbf{N}$, such that for any subinterval $[a, b] \subset I$ $(a \neq b)$,

$$\frac{\|f - p^*_{n,\infty}(f)\|_{[a,b]}}{\|f - p^*_{n,\infty}(f)\|_I} = 1, \qquad n \in \Lambda(f), \quad n \geq n_{[a,b]}. \tag{1}$$

Essentially, (1) tells us that $\{p^*_{n,\infty}(f)\}_{n=0}^{\infty}$ does not approximate f better on any subinterval of I than it does on the whole interval I, which illustrates the *principle of contamination* introduced by Saff [13]. Recently, Kroó and Saff [7] proved a result which implies that (1) also holds for *complex-valued* $f \in C(I)$ and also for the analogous case of uniform approximation on the unit circle $\partial\Delta$. More precisely, if $f \in A(\Delta) := \{f \in C(\Delta) \colon f \text{ analytic in } \Delta^\circ\}$, where $\Delta^\circ := \{z \in \mathbf{C} : |z| < 1\}$, then there is a subsequence of $\mathbf{N}$, say $\Lambda(f)$, such that

$$\frac{\|f - s^*_{n,\infty}(f)\|_\Gamma}{\|f - s^*_{n,\infty}(f)\|_{\partial\Delta}} = 1, \qquad n \in \Lambda(f), \quad n \geqslant n_\Gamma, \tag{2}$$

for any subarc Γ (not a single point) of $\partial\Delta$.

In this paper, we first prove the analogues of (1) and (2) for general L_2 best approximation on I and $\partial\Delta$, which illustrate an L_2 version of the principle of contamination (this is done in Section 2). Then we treat the problem of the distribution of zeros of the L_p $(p > 0)$ best approximants $p^*_{n,p}$ and $s^*_{n,p}$, and so generalize the Jentzsch–Szegö-type theorem in [1]. This is done in Section 4. In the proof of the Jentzsch–Szegö-type theorem for the unit circle case, the regularity of the measure plays a very important role (cf. Definition 3.1). It turns out that the regularity of a measure is equivalent to the regular nth root asymptotic behavior of the corresponding orthonormal polynomials (cf. Theorem 3.3). Because of its own interest, we state and prove this fact in Section 3.

2. Norm Comparisons in L_2 Approximation

Set

$$\alpha(x) := d\alpha([-1, x)), \qquad x \in I,$$

and

$$\mu(\theta) := d\mu(\{z = e^{it} \colon t \in [0, \theta)\}), \qquad \theta \in [0, 2\pi].$$

Then α' and μ' exist a.e. on I and $[0, 2\pi]$, respectively.

Theorem 2.1. *Suppose that $\alpha' > 0$ a.e. on I. Let $f \in L_2(d\alpha)$, f not a polynomial, and $\delta \in (0, 2]$. Then*

$$\sum_{n=0}^{\infty} \left(\frac{\|f - p^*_{n,2}(f)\|_{L_2(d\alpha,\,[a,b])}}{\|f - p^*_{n,2}(f)\|_{L_2(d\alpha)}} \right)^2 = \infty, \tag{3}$$

uniformly for $[a, b] \subset [-1, 1]$ with $b - a \geqslant \delta$.

Before proceeding with the proof of Theorem 2.1 we state a needed lemma.

Let $\{p_n\}_{n=0}^{\infty}$ be the unique system of polynomials orthonormal with respect to $d\alpha$, i.e., polynomials

$$p_n(x) := p_n(d\alpha, x) = \gamma_n x^n + \cdots \qquad (\gamma_n = \gamma_n(d\alpha) > 0) \tag{4}$$

such that

$$\int_I p_m(x)\, p_n(x)\, d\alpha = \delta_{mn},$$

where $\delta_{mn} = 1$ if $m = n$ and $\delta_{mn} = 0$ otherwise. Then we have the following result of Máté, Nevai and Totik:

LEMMA 2.2 (Theorem 13.3 in [9]). *Assume $\alpha' > 0$ a.e. on I. Then for each $[a, b] \subset I$ $(a \neq b)$, there is a constant $\tau > 0$, depending only on $b - a$, such that*

$$\int_a^b |p_n(d\alpha, x)|^2\, dx \geqslant \tau, \qquad n \geqslant 0.$$

Proof of Theorem 2.1. Set $a_n := \int_I f(x)\, p_n(d\alpha, x)\, d\alpha$, $n = 0, 1, 2, \ldots$. Then

$$p^*_{n,2}(x) := p^*_{n,2}(f, x) = \sum_{k=0}^{n} a_k\, p_k(d\alpha, x), \qquad n = 0, 1, 2, \ldots$$

and

$$E_n(f) := \|f - p^*_{n,2}\|_{L_2(d\alpha)} = \left(\sum_{k=n+1}^{\infty} |a_k|^2 \right)^{1/2}, \qquad n = 0, 1, 2, \ldots.$$

Letting

$$r_n := \frac{\|f - p^*_{n,2}(f)\|_{L_2(d\alpha, [a,b])}}{E_n(f)}, \qquad n = 0, 1, 2, \ldots,$$

we have

$$\begin{aligned}
&\|a_n\, p_n(d\alpha, \cdot)\|_{L_2(dx, [a,b])} \\
&\quad = \|p^*_{n,2} - p^*_{n-1,2}\|_{L_2(dx, [a,b])} \\
&\quad \leqslant \|f - p^*_{n,2}\|_{L_2(dx, [a,b])} + \|f - p^*_{n-1,2}\|_{L_2(dx, [a,b])} \\
&\quad \leqslant \max\{r_n, r_{n-1}\}(E_n(f) + E_{n-1}(f)).
\end{aligned} \tag{5}$$

On the other hand, by Lemma 2.2,

$$\|a_n p_n(d\alpha, \cdot)\|_{L_2(d\alpha, [a,b])} = |a_n| \, \|p_n(d\alpha, \cdot)\|_{L_2(d\alpha, [a,b])} \geqslant c\,|a_n|, \qquad n = 0, 1, 2, ..., \tag{6}$$

for some constant $c > 0$. But

$$|a_n|^2 = \sum_{k=n}^{\infty} |a_k|^2 - \sum_{k=n+1}^{\infty} |a_k|^2 = E_{n-1}(f)^2 - E_n(f)^2,$$

and so, combining (5) and (6), it follows that

$$c^2(E_{n-1}(f)^2 - E_n(f)^2) \leqslant \max\{r_n^2, r_{n-1}^2\}(E_{n-1}(f) + E_n(f))^2, \qquad n = 1, 2, 3, ...$$

Thus

$$c^2 \frac{E_{n-1}(f) - E_n(f)}{E_{n-1}(f) + E_n(f)} \leqslant \max\{r_n^2, r_{n-1}^2\}, \qquad n = 1, 2, 3, ... \tag{7}$$

Next we note that since $E_n(f)$ decreases to zero as $n \to \infty$, it follows from elementary properties of series that

$$\sum_{n=1}^{\infty} \frac{E_{n-1}(f) - E_n(f)}{E_{n-1}(f) + E_n(f)} = \infty. \tag{8}$$

Therefore (7) implies that $\sum_{n=1}^{\infty} \max(r_n^2, r_{n-1}^2) = \infty$, which is equivalent to (3). ∎

For the unit circle, we have the following companion of Theorem 2.1.

THEOREM 2.3. *Suppose that* $\mu' > 0$ *a.e. on* $[0, 2\pi]$. *Let* $f \in L_2(d\mu)$, f *not a polynomial, and* $\delta \in (0, 2\pi]$. *Then*

$$\sum_{n=0}^{\infty} \left(\frac{\|f - s_{n,2}^*(f)\|_{L_2(d\mu, F)}}{\|f - s_{n,2}^*(f)\|_{L_2(d\mu)}} \right)^2 = \infty, \tag{9}$$

uniformly for Borel sets $F \subset \partial\Delta$ *with (linear) Lebesgue measure* $\geqslant \delta$.

Proof. We first introduce the orthonormal polynomials with respect to $d\mu$; that is,

$$\varphi_n(z) := \varphi_n(d\mu, z) = \kappa_n z^n + \cdots \qquad (\kappa_n := \kappa_n(d\mu) > 0), \tag{10}$$

satisfying

$$\frac{1}{2\pi}\int_{\partial\Delta} \varphi_m(z)\,\overline{\varphi_n(z)}\,d\mu = \delta_{mn}.$$

Then we proceed exactly as in the proof of Theorem 2.1, using the following result of Máté, Nevai, and Totik instead of Lemma 2.2.

LEMMA 2.4 (Corollary 7.5 in [9]). *Assume $\mu' > 0$ a.e. on $[0, 2\pi]$. Then, for each $\delta > 0$ there is a constant $\tau > 0$ such that*

$$\int_F |\varphi_n(d\mu, z)|^2\,d\mu \geqslant \tau, \qquad n \geqslant 0,$$

for every Borel subset F of $\partial\Delta$ with $|F| \geqslant \delta$, where $|\cdot|$ denotes the Lebesgue measure on $\partial\Delta$.

Remark. The inequalities in Lemmas 2.2 and 2.4 are the so-called Turán-type inequalities, see [9].

COROLLARY 2.5. (i) *With the assumptions of Theorem 2.1, if $f \in L_2(d\alpha)$, $\varepsilon > 0$, and $-1 \leqslant a < b \leqslant 1$, then there is a subsequence $\Lambda \subset \mathbf{N}$ such that*

$$\|f - p^*_{n,2}(f)\|_{L_2(d\alpha,[a,b])} \geqslant \frac{C}{n^{1/2+\varepsilon}}\,\|f - p^*_{n,2}(f)\|_{L_2(d\alpha)}, \qquad n \in \Lambda, \tag{11}$$

where C is a positive constant depending only on $b - a$.

(ii) *With the assumptions of Theorem 2.3, if $f \in L_2(d\mu)$, $\varepsilon > 0$, and $F \subset \partial\Delta$ is any Borel set with $|F| > 0$, then there is a subsequence $\Lambda \subset \mathbf{N}$ such that*

$$\|f - s^*_{n,2}(f)\|_{L_2(d\mu,F)} \geqslant \frac{C}{n^{1/2+\varepsilon}}\,\|f - s^*_{n,2}(f)\|_{L_2(d\mu)}, \qquad n \in \Lambda, \tag{12}$$

where C is a positive constant depending only on $|F|$.

Proof. By (8), for any $\delta > 0$, there is a subsequence of positive integers, $\Lambda_0 \subset \mathbf{N}$, depending only on f and δ, such that

$$\frac{1}{n^{1+\delta}} < \frac{E_{n-1}(f) - E_n(f)}{E_{n-1}(f) + E_n(f)}, \qquad n \in \Lambda_0.$$

Together with (7), this gives

$$\frac{c}{n^{1/2+\delta/2}} \leqslant \max\{r_n, r_{n-1}\}, \qquad n \in \Lambda_0,$$

which implies (11). The proof of (12) is identical. ∎

Our next result shows that Theorem 2.1 is best possible in the sense that the exponent 2 appearing in (3) cannot be replaced by any larger value.

PROPOSITION 2.6. *Let* $d\alpha(x) = (2/\pi(1-x^2)^{1/2})\, dx$, $x \in (-1, 1)$. *Then for each* $r > 1$,

$$f_r(x) := \sum_{k=1}^{\infty} \frac{1}{k^r} \cos(k \arccos x)$$

satisfies

$$\sum_{n=0}^{\infty} \left(\frac{\|f_r - p^*_{n,2}(f_r)\|_{L_2(d\alpha, [-1,b])}}{\|f_r - p^*_{n,2}(f_r)\|_{L_2(d\alpha)}} \right)^{2+\delta} < \infty, \tag{13}$$

for every $b \in (-1, 1)$ *and* $\delta > 0$.

Remark. It is easy to see that, by a modification of Proposition 2.6, we can show that (9) is also best possible.

Proof of Proposition 2.6. We use $C_1, C_2, ...,$ to denote absolute constants. Note that for the given $d\alpha(x)$,

$$p_n(d\alpha, x) = \cos(n \arccos x) =: t_n(x),$$

$n = 1, 2, 3, ...,$ and $p_0(d\alpha, x) = 1/\sqrt{2}$. So

$$p^*_{n,2}(f_r, x) = \sum_{k=1}^{n} \frac{1}{k^r} t_k(x), \qquad n = 1, 2, 3, ...$$

and $p^*_{0,2}(f_r, x) \equiv 0$.

Set

$$D_k(\theta) := \frac{1}{2} + \sum_{j=1}^{k} \cos j\theta, \qquad k = 1, 2, 3, ...$$

and $\theta := \arccos x \in [0, \pi]$. Then

$$R_n(x) := \sum_{k=n}^{\infty} \frac{1}{k^r} t_k(x) = \sum_{k=n}^{\infty} \frac{1}{k^r} (D_k(\theta) - D_{k-1}(\theta))$$

$$= \sum_{k=n}^{\infty} \left(\frac{1}{k^r} - \frac{1}{(k+1)^r} \right) D_k(\theta) - \frac{D_{n-1}(\theta)}{n^r}.$$

Thus, for $x \in [-1, 1]$,

$$|R_n(x)| \leqslant C_1 \left(\sum_{k=n}^{\infty} \frac{1}{k^{r+1}} |D_k(\theta)| + \frac{|D_{n-1}(\theta)|}{n^r} \right), \qquad n = 1, 2, 3, \ldots . \quad (14)$$

Since

$$D_k(\theta) = \frac{\sin(k + 1/2)\theta}{2 \sin \theta/2}, \qquad k = 1, 2, 3, \ldots,$$

we have

$$|D_k(\theta)| \leqslant \frac{1}{2 \sin \tau/2}, \qquad \text{for} \quad 0 < \tau \leqslant \theta \leqslant \pi,$$

$k = 1, 2, 3, \ldots$. Thus, with $|\sin \theta/2| = \sqrt{(1-x)/2}$, it follows from (14) that

$$|R_n(x)| \leqslant C_2 \frac{1}{n^r}, \qquad \text{for } -1 \leqslant x \leqslant b < 1,$$

and so

$$\left(\int_{-1}^{b} |R_n(x)|^2 \, d\alpha \right)^{1/2} \leqslant C_3 \frac{1}{n^r}, \qquad n = 1, 2, 3, \ldots . \quad (15)$$

But, for $n = 1, 2, 3, \ldots$,

$$\int_{-1}^{1} |R_n(x)|^2 \, d\alpha = \sum_{k=n}^{\infty} \frac{1}{k^{2r}} \geqslant C_4 \frac{1}{n^{2r-1}};$$

hence, from (15) we get

$$\left(\frac{\|f_r - p_{n,2}^*(f_r)\|_{L_2(d\alpha, [-1,b])}}{\|f_r - p_{n,2}^*(f_r)\|_{L_2(d\alpha)}} \right)^{2+\delta} \leqslant \frac{C_5}{n^{1+\delta/2}}, \qquad n = 1, 2, 3, \ldots,$$

which implies that the series in (13) is convergent. ∎

The generalizations of Theorems 2.1 and 2.3 for best L_p polynomial approximants remain open problems. In light of the Kadec result (1) for the case $p = \infty$, it is tempting to make the following

Conjecture. If $\alpha' > 0$ a.e. on I, f not a polynomial, then

$$\sum_{n=0}^{\infty} \left(\frac{\|f - p_{n,p}^*(f)\|_{L_p(d\alpha, [a,b])}}{\|f - p_{n,p}^*(f)\|_{L_p(d\alpha)}} \right)^p = \infty.$$

3. Regularity of Measure

In Section 2, we used $\alpha' > 0$ a.e. or $\mu' > 0$ a.e. in our assumptions. By a theorem of Rahmanov (cf. [12, 10]), we know that these assumptions imply that $\lim_{n\to\infty} \gamma_n^{1/n} = 2$ and $\lim_{n\to\infty} \kappa_n^{1/n} = 1$, respectively (cf. (4), (10)). When we consider the distribution of zeros of the best L_p $(p>0)$ approximants, these limit conditions suffice for our purpose.

Definition 3.1. We call $d\alpha$ (resp. $d\mu$) a *regular measure* with respect to I (resp. $\partial\Delta$) if $\lim_{n\to\infty} \gamma_n^{1/n} = 2$ (resp. $\lim_{n\to\infty} \kappa_n^{1/n} = 1$).[1]

For measures on I, we have the following result of Erdős and Turán.

Theorem 3.2 [3]. *The measure $d\alpha$ is regular with respect to I if and only if*

$$\lim_{n\to\infty} |p_n(d\alpha, z)|^{1/n} = |z + \sqrt{z^2 - 1}|, \qquad z \in \mathbf{C}\setminus I, \tag{16}$$

where the convergence in (16) *is locally uniform in* $\mathbf{C}\setminus I$.

In (16), the branch of the square root is taken so that $\sqrt{z^2-1}$ behaves like z near infinity.

The main result in this section is

Theorem 3.3. *A measure $d\mu$ on $\partial\Delta$ is regular with respect to $\partial\Delta$ if and only if*

$$\lim_{n\to\infty} |\varphi_n(d\mu, z)|^{1/n} = |z|, \qquad |z| > 1, \tag{17}$$

where the convergence in (17) *is locally uniform in* $|z| > 1$.

Before giving the proof of Theorem 3.3 we need to recall some properties of the orthogonal polynomials on the unit circle. Let

$$\Phi_n(z) = \Phi_n(d\mu, z) := \frac{1}{\kappa_n} \varphi_n(d\mu, z) = z^n + \cdots, \qquad n = 0, 1, 2, \ldots.$$

Then the monic polynomials Φ_n satisfy the following recursive relation (cf. [17, p. 293; 5, p. 132]),

$$\Phi_{n+1}^*(z) = \Phi_n^*(z) - a_n z \Phi_n(z), \tag{18}$$

[1] Regularity of general measures (with arbitrary compact support) is treated in [16]. Simultaneously, yet independently, results corresponding to Theorems 3.2, 3.3, and 4.1, (for $p = 2$), and Corollaries 3.4, 3.5, and 3.6 for the general case have been partially announced in [16].

where

$$\Phi_n^*(z) := z^n \overline{\Phi_n(1/\bar{z})}$$

and

$$a_n := -\overline{\Phi_{n+1}(0)} = -\frac{\overline{\varphi_{n+1}(0)}}{\kappa_{n+1}}, \qquad n = 0, 1, 2, \ldots. \tag{19}$$

Also we have (cf. [5, p. 2])

$$\kappa_{n+1}^2 - \kappa_n^2 = |\varphi_{n+1}(0)|^2, \qquad n = 0, 1, 2, \ldots. \tag{20}$$

Proof of Theorem 3.3. Note that by the maximum principle,

$$\|\Phi_n(z)\|_{\partial\Delta} \geqslant 1, \tag{21}$$

for $n = 1, 2, 3, \ldots$, and hence

$$\kappa_n^{1/n} \leqslant \|\varphi_n(d\mu, \cdot)\|_{\partial\Delta}^{1/n}, \qquad n = 1, 2, 3, \ldots. \tag{22}$$

If (17) is true, then

$$\limsup_{n \to \infty} \|\varphi_n(d\mu, \cdot)\|_{\partial\Delta}^{1/n} \leqslant \lim_{n \to \infty} \|\varphi_n(d\mu, \cdot)\|_{\{z: |z| = 1+\rho\}}^{1/n} = 1 + \rho, \qquad \text{for} \quad \rho > 0.$$

With (22), this yields

$$\limsup_{n \to \infty} \kappa_n^{1/n} \leqslant 1 + \rho,$$

and, since $\rho > 0$ is arbitrary, we get

$$\limsup_{n \to \infty} \kappa_n^{1/n} \leqslant 1.$$

On the other hand, by the monotonicity of κ_n (cf. (20)), we have

$$0 < \kappa_0 \leqslant \kappa_n, \qquad n = 0, 1, 2, \ldots,$$

and so

$$\liminf_{n \to \infty} \kappa_n^{1/n} \geqslant 1. \tag{23}$$

Thus

$$\lim_{n \to \infty} \kappa_n^{1/n} = 1,$$

i.e., the measure $d\mu$ is regular when (17) is satisfied.

Now let us assume that the measure $d\mu$ is regular with respect to $\partial\Delta$. We make use of the formula

$$\Phi_n^*(z) = \prod_{k=0}^{n-1} \left\{1 - a_k z \frac{\Phi_k(z)}{\Phi_k^*(z)}\right\}, \qquad n = 1, 2, 3, ...,$$

which follows from (18). Since

$$\left|\frac{\Phi_k(z)}{\Phi_k^*(z)}\right| = \begin{cases} \leqslant 1, & |z| < 1, \\ = 1, & |z| = 1, \\ \geqslant 1, & |z| > 1, \end{cases}$$

we have, for $|z| \leqslant 1$,

$$|\Phi_n^*(z)| \leqslant \prod_{k=0}^{n-1} \{1 + |a_k|\}, \qquad n = 1, 2, 3, ... \tag{24}$$

Also note that, from (19) and (20),

$$\left(\frac{\kappa_n}{\kappa_{n+1}}\right)^2 = 1 - |a_n|^2, \qquad n = 0, 1, 2, ... \tag{25}$$

Now we claim: *if $d\mu$ is regular, then for every $\delta > 0$, we have*

$$\lim_{n \to \infty} \frac{j_n(\delta)}{n} = 0, \tag{26}$$

where $j_n(\delta)$ is the cardinality of the set

$$I_n(\delta) := \{j : 0 \leqslant j \leqslant n, |a_j| > \delta\}.$$

In fact, for $j \in I_n(\delta)$ $(0 < \delta < 1)$,

$$0 < 1 - |a_j|^2 < 1 - \delta^2$$

(the left-hand inequality follows from the fact that $|a_j| = |\Phi_{j+1}(0)| < 1$), and so

$$\begin{aligned} \left(\frac{\kappa_0}{\kappa_{n+1}}\right)^2 &= \prod_{j=0}^{n} \left(\frac{\kappa_j}{\kappa_{j+1}}\right)^2 \\ &= \prod_{j=0}^{n} (1 - |a_j|^2) \\ &= \prod_{j \in I_n(\delta)} (1 - |a_j|^2) \cdot \prod_{\substack{j \notin I_n(\delta) \\ 0 \leqslant j \leqslant n}} (1 - |a_j|^2) \\ &\leqslant \prod_{j \in I_n(\delta)} (1 - |a_j|^2) \\ &\leqslant (1 - \delta^2)^{j_n(\delta)}. \end{aligned} \tag{27}$$

Thus, by regularity of $d\mu$, we have

$$1 = \liminf_{n \to \infty} \left(\frac{\kappa_0}{\kappa_{n+1}} \right)^{2/n} \leq (1 - \delta^2)^{\limsup_{n \to \infty} j_n(\delta)/n},$$

and so

$$\limsup_{n \to \infty} \frac{j_n(\delta)}{n} = 0,$$

which proves our claim.

Now by (24), for any $\delta \in (0, 1)$ and $|z| \leq 1$,

$$|\Phi_{n+1}^*(z)|^{1/n} \leq \prod_{j \in I_n(\delta)} (1 + |a_j|)^{1/n} \cdot \prod_{\substack{j \notin I_n(\delta) \\ 0 \leq j \leq n}} (1 + |a_j|)^{1/n}$$
$$\leq 2^{j_n(\delta)/n} \cdot (1 + \delta)^{(n - j_n(\delta))/n}.$$

Hence

$$\limsup_{n \to \infty} \|\Phi_n^*\|_{\partial \Delta}^{1/n} \leq 1 + \delta,$$

and, by the arbitrariness of $\delta \in (0, 1)$, we obtain

$$\limsup_{n \to \infty} \|\Phi_n\|_{\partial \Delta}^{1/n} = \limsup_{n \to \infty} \|\Phi_n^*\|_{\partial \Delta}^{1/n} \leq 1.$$

With (21), it follows that

$$\lim_{n \to \infty} \|\Phi_n\|_{\partial \Delta}^{1/n} = 1. \tag{28}$$

But recall that all the zeros of Φ_n lie in $|z| < 1$ (cf. [17, p. 292]), and so (cf. [4, Chap. 2 Sect. 2.B]) (28) is equivalent to

$$\lim_{n \to \infty} |\Phi_n(z)|^{1/n} = |z|,$$

locally uniformly in $|z| > 1$. Thus

$$\lim_{n \to \infty} |\varphi_n(z)|^{1/n} = \lim_{n \to \infty} |\kappa_n \Phi_n(z)|^{1/n}$$
$$= \lim_{n \to \infty} |\kappa_n|^{1/n} \lim_{n \to \infty} |\Phi_n(z)|^{1/n}$$
$$= |z|,$$

locally uniformly in $|z| > 1$. ∎

From the proof we have the following

COROLLARY 3.4. *The following assertions are pairwise equivalent:*

(i) $\lim_{n\to\infty} \|\varphi_n\|_{\partial\Delta}^{1/n} = 1.$

(ii) $\lim_{n\to\infty} \kappa_n^{1/n} = 1.$

(iii) $\lim_{n\to\infty} (n+1)^{-1} \sum_{j=0}^{n} \ln(1-|a_j|^2) = 0.$

Proof. (i) $\Rightarrow$ (ii) The proof follows from (22) and (23).

(ii) $\Rightarrow$ (i) By (28),

$$\lim_{n\to\infty} \|\varphi_n\|_{\partial\Delta}^{1/n} = \lim_{n\to\infty} \|\kappa_n \Phi_n\|_{\partial\Delta}^{1/n} = \lim_{n\to\infty} \kappa_n^{1/n} \|\Phi_n\|_{\partial\Delta}^{1/n} = 1.$$

(ii) $\Leftrightarrow$ (iii) Note that by (27),

$$\frac{1}{n+1} \ln \kappa_{n+1} = \frac{1}{n+1} \ln \kappa_0 - \frac{1}{2(n+1)} \sum_{k=0}^{n} \ln(1-|a_j|^2). \quad \blacksquare$$

The following corollary illustrates the importance of the regularity of measures (cf. [15]).

COROLLARY 3.5. *For any $p>0$, if $d\alpha$ ($d\mu$) is regular with respect to I (resp. $\partial\Delta$), then for any $\varepsilon>0$, there is $N_{\varepsilon,p}>0$, depending only on ε and p, such that*

$$\|P_n\|_I \leqslant (1+\varepsilon)^n \|P_n\|_{L_p(d\alpha)} \tag{29}$$

(*respectively*,

$$\|P_n\|_{\partial\Delta} \leqslant (1+\varepsilon)^n \|P_n\|_{L_p(d\mu)}), \tag{30}$$

for $n>N_{\varepsilon,p}$ and all $P_n \in \mathscr{P}_n$.

Proof. Note that (29) is equivalent to

$$\limsup_{n\to\infty} \left\{ \sup_{\substack{P_n\in\mathscr{P}_n \\ P_n \not\equiv 0}} \frac{\|P_n\|_I}{\|P_n\|_{L_p(d\alpha)}} \right\}^{1/n} \leqslant 1. \tag{31}$$

Since $d\alpha$ is regular, Theorem 3.2 implies that

$$\lim_{n\to\infty} \|p_n(d\alpha,\cdot)\|_I^{1/n} = 1.$$

Then by expanding any $P_n \in \mathscr{P}_n$ in terms of $\{p_k(d\alpha,\cdot)\}_{k=0}^n$, we see that (31) is true for $p=2$. Then following Saff and Totik (cf. the proof of

Theorem 1.5(ii) in [15]), we know that (31) is true for all $p>0$. This proves (29).

Using Theorem 3.3 (or Corollary 3.4) instead of Theorem 3.2, we can prove (30) in a similar way. ∎

By Theorem 1.1 in [15], we know that for $d\alpha$ regular with respect to I, f is equal ($d\alpha$—a.e. on I) to a function that is analytic on I if and only if

$$\limsup_{n\to\infty} \|f-p_{n,p}^*(f)\|_{L_p(d\alpha)}^{1/n}<1. \tag{32}$$

As a consequence of Theorem 3.3, for the unit circle, we have

Corollary 3.6. *Assume $d\mu$ is regular with respect to $\partial\Delta$. Let $f\in L_p(d\mu)$ for some $p>0$. Then f is equal ($d\mu$—a.e. on $\partial\Delta$) to a function that is analytic on an open set containing Δ if and only if*

$$\limsup_{n\to\infty} \|f-s_{n,p}^*(f)\|_{L_p(d\mu)}^{1/n}<1. \tag{33}$$

Proof. We use the same method as in [18, Sect. 4.5, Theorem 5], and briefly describe the main steps.

First, if f is analytic on Δ, then (cf. [18, p. 76]) there exist polynomials $q_n\in\mathscr{P}_n$, $n=0, 1, 2, \ldots$, such that

$$\limsup_{n\to\infty} \|f-q_n\|_{\partial\Delta}^{1/n}<1,$$

and so

$$\limsup_{n\to\infty} \|f-s_{n,p}^*(f)\|_{L_p(d\mu)}^{1/n}\leqslant \limsup_{n\to\infty} \|f-q_n\|_{\partial\Delta}^{1/n}<1.$$

This proves the necessity of (33).

Next, if (33) holds, then

$$\limsup_{n\to\infty} \|s_{n,p}^*(f)-s_{n-1,p}^*(f)\|_{L_p(d\mu)}^{1/n}<1,$$

and so, by Corollary 3.5,

$$\limsup_{n\to\infty} \|s_{n,p}^*(f)-s_{n-1,p}^*(f)\|_{\partial\Delta}^{1/n}<1.$$

Hence $g(z):=\sum_{n=1}^{\infty}(s_{n,p}^*(f)-s_{n-1,p}^*(f))+s_{0,p}^*(f)$ is analytic on Δ and $f=g$ $d\mu$—a.e. on $\partial\Delta$. This gives the sufficiency of (33). ∎

4. Jentzsch–Szegö-Type Theorems in L_p Approximation

Let P_n be a polynomial of exact degree n, and let $z_1, z_2, ..., z_n$ be the zeros of P_n (counting multiplicity). Define the measure $\nu(P_n)$ as

$$\nu(P_n) := \frac{1}{n} \sum_{j=1}^{n} \delta_{z_j}, \tag{34}$$

where δ_z denotes the Dirac's measure for the point $z \in \mathbf{C}$.

The arcsine measure is the measure $dx/\pi\sqrt{1-x^2}$ on I. The uniform measure on $\partial \Delta$, denoted by μ^*, is $d\theta/2\pi$ ($z = e^{i\theta}$).

As a consequence of Corollary 3.5, we prove

THEOREM 4.1. *Let $p > 0$ and $d\alpha$ be regular with respect to I. Let $T_{n,p} \in \mathscr{P}_n$, $T_{n,p}(x) = x^n + \cdots$, satisfy*

$$\|T_{n,p}\|_{L_p(d\alpha)} = \inf_{\substack{P_n \in \mathscr{P}_n \\ P_n = x^n + \cdots}} \|P_n\|_{L_p(d\alpha)}, \qquad n = 0, 1, 2, \ldots .$$

Then $\nu(T_{n,p})$ converges in the weak-star topology to the arcsine measure as $n \to \infty$.

Proof. By Theorem 2.1 in [1], we only need show that

$$\limsup_{n \to \infty} \|T_{n,p}\|_I^{1/n} \leqslant \tfrac{1}{2}. \tag{35}$$

By Corollary 3.5, for $\varepsilon > 0$ and n large enough,

$$\begin{aligned} \|T_{n,p}\|_I &\leqslant (1+\varepsilon)^n \|T_{n,p}\|_{L_p(d\alpha)} \\ &\leqslant (1+\varepsilon)^n \|T_n\|_{L_p(d\alpha)} \\ &\leqslant (1+\varepsilon)^n \|T_n\|_I \left(\int_I d\alpha \right)^{1/p}, \end{aligned}$$

where $T_n(x) := (1/2^{n-1}) \cos(n \arccos x)$. Hence

$$\limsup_{n \to \infty} \|T_{n,p}\|_I^{1/n} \leqslant (1+\varepsilon)\tfrac{1}{2},$$

and so (35) follows by the arbitrariness of $\varepsilon > 0$. ∎

For the zero distribution of monic polynomials of minimal $L_p(d\mu)$ norm on the unit circle, we need to modify the measure $\nu(P_n)$ in (34). First, for $z \in \Delta^\circ$, define the positive unit measure

$$\hat{\delta}_z := \operatorname{Re}\left(\frac{t+z}{t-z}\right) \cdot \frac{|dt|}{2\pi}, \qquad t \in \partial \Delta.$$

Then $\hat{\delta}_z$ is the *harmonic measure* on $\partial\Delta$ for z (or, in the terminology of Landkof, the *Green measure* for the point z and the region Δ°, [8, p. 212]). Next, for a polynomial P_n of exact degree n with zeros $z_1, z_2, ..., z_n$ (counting multiplicity), define

$$\hat{v}(P_n) := \frac{1}{n}\left(\sum_{z_j \in \Delta^\circ} \hat{\delta}_{z_j} + \sum_{z_j \notin \Delta^\circ} \delta_{z_j}\right).$$

For a measure σ, we adopt the notations

$$\mathscr{U}(\sigma, z) := \int \log|z-t|^{-1}\, d\sigma(t)$$

and

$$I(\sigma) := \int \mathscr{U}(\sigma, z)\, d\sigma(z).$$

Then it is easy to see that, for $z \in \mathbf{C}\setminus\Delta$,

$$\mathscr{U}(v(P_n), z) = \mathscr{U}(\hat{v}(P_n), z). \tag{36}$$

Now we can state

THEOREM 4.2. *Let $p>0$ and $d\mu$ be regular with respect to $\partial\Delta$. Let $C_{n,p} \in \mathscr{P}_n$, $C_{n,p}(z) = z^n + \cdots$, satisfy*

$$\|C_{n,p}\|_{L_p(d\mu)} = \inf_{\substack{P_n \in \mathscr{P}_n \\ P_n = z^n + \cdots}} \|P_n\|_{L_p(d\mu)}, \qquad n = 0, 1, 2, ...$$

Then $\hat{v}(C_{n,p})$ converges in the weak-star topology to the uniform measure μ^* as $n \to \infty$.

Remark. From the definition of $C_{n,p}$ it is easy to show that all its zeros lie on Δ°.

Proof of Theorem 4.2. As in the proof of Theorem 4.1, by Corollary 3.5, for $\varepsilon > 0$ and n large enough,

$$\begin{aligned}\|C_{n,p}\|_{\partial\Delta} &\leqslant (1+\varepsilon)^n \|C_{n,p}\|_{L_p(d\mu)} \\ &\leqslant (1+\varepsilon)^n \|z^n\|_{L_p(d\mu)} \\ &\leqslant (1+\varepsilon)^n \left(\int_{\partial\Delta} d\mu\right)^{1/p}.\end{aligned}$$

Hence

$$\limsup_{n\to\infty} \|C_{n,p}\|_{\partial\Delta}^{1/n} = 1. \tag{37}$$

By the proof of Theorem 2.1 in [1], inequality (37) implies

$$\lim_{n \to \infty} \mathscr{U}(\nu(C_{n,p}), z) = \mathscr{U}(\mu^*, z), \qquad z \in \mathbf{C} \setminus \Delta.$$

So, by (36), we also have

$$\lim_{n \to \infty} \mathscr{U}(\hat{\nu}(C_{n,p}), z) = \mathscr{U}(\mu^*, z), \qquad z \in \mathbf{C} \setminus \Delta. \tag{38}$$

Now, if ν is any weak-star limit measure of the sequence $\{\hat{\nu}(C_{n,p})\}_{n=0}^{\infty}$, then, as in the proof of Theorem 2.1 in [1], we can obtain from (38) that

$$\mathscr{U}(\nu, z) \leqslant I[\mu^*], \qquad z \in \partial\Delta.$$

Since ν is supported on $\partial\Delta$ and $\nu(\partial\Delta) = 1$, integrating the last inequality yields $I[\nu] \leqslant I[\mu^*]$. Thus, by the uniqueness of the solution to the minimum energy problem (cf. [8, Chap. II]), we get $\nu = \mu^*$ and so the whole sequence $\{\hat{\nu}(C_{n,p})\}_{n=0}^{\infty}$ converges in the weak-star topology to μ^*. ∎

The following Jentzsch–Szegö-type theorems show that the L_p $(p > 0)$ best approximants also obey the principle of contamination.

THEOREM 4.3. *Let f be continuous but not analytic on I, $d\alpha$ a regular measure with respect to I, and $p > 0$. Then there is a subsequence $\Lambda(f) \subset \mathbf{N}$ such that $\nu(p_{n,p}^*(f))$ converges in the weak-star topology to the arcsine measure as $n \to \infty$, $n \in \Lambda(f)$.*

THEOREM 4.4. *Let f be analytic in Δ°, continuous on Δ, but not analytic on Δ, and let $d\mu$ be a regular measure with respect to $\partial\Delta$. Then, for each $p > 0$, there is a subsequence $\Lambda(f) \subset \mathbf{N}$ such that $\hat{\nu}(s_{n,p}^*(f))$ converges in the weak-star topology to μ^* as $n \to \infty$, $n \in \Lambda(f)$.*

Furthermore, in the special case that $\log \mu' \in L_1([0, 2\pi])$, *then $\nu(s_{n,p}^*(f))$ itself converges in the weak-star topology to μ^* as $n \to \infty$, $n \in \Lambda(f)$.*

Remarks. (i) For Jordan arcs or Jordan curves with length measure and weights w satisfying the condition that some negative power of w is integrable, results similar to Theorems 4.3 and 4.4 hold (cf. [1], [14]).

(ii) Theorem 4.4 is an L_p version of a recent result of Mhaskar and Saff [11].

Since the proof of Theorem 4.3 is similar to that of Theorem 4.4, we only give the latter.

Proof of Theorem 4.4. We first show that

$$\limsup_{n \to \infty} \|s_{n,p}^*(f)\|_{\partial\Delta}^{1/n} \leqslant 1 \tag{39}$$

and

$$\limsup_{n \to \infty} |a_{n,p}^*|^{1/n} \geqslant 1, \tag{40}$$

where $s_{n,p}^*(f, z) = a_{n,p}^* z^n + \cdots$, $n = 0, 1, 2, \ldots$.

Inequality (39) follows easily from Corollary 3.5:

$$\begin{aligned}\limsup_{n \to \infty} \|s_{n,p}^*(f)\|_{\partial\Delta}^{1/n} &\leqslant \limsup_{n \to \infty} \|s_{n,p}^*(f)\|_{L_p(d\mu)}^{1/n} \\ &\leqslant \limsup_{n \to \infty} (\max\{2^{1/p}, 2\} \|f\|_{L_p(d\mu)})^{1/n} \\ &\leqslant 1.\end{aligned}$$

For (40), note that for $p \geqslant 1$,

$$\begin{aligned}&\|f - s_{n,p}^*(f)\|_{L_p(d\mu)} - \|f - s_{n+1,p}^*(f)\|_{L_p(d\mu)} \\ &\quad \leqslant \|f - (s_{n+1,p}^*(f) - a_{n+1,p}^* z^{n+1})\|_{L_p(d\mu)} - \|f - s_{n+1,p}^*(f)\|_{L_p(d\mu)} \\ &\quad \leqslant |a_{n+1,p}^*| \left(\int_{\partial\Delta} d\mu \right)^{1/p}, \qquad n = 1, 2, 3, \ldots.\end{aligned} \tag{41}$$

For $0 < p < 1$, we similarly get

$$\|f - s_{n,p}^*(f)\|_{L_p(d\mu)}^p - \|f - s_{n+1,p}^*(f)\|_{L_p(d\mu)}^p \leqslant |a_{n+1,p}^*|^p \left(\int_{\partial\Delta} d\mu \right), \qquad n = 1, 2, 3, \ldots. \tag{42}$$

Now, since f is not analytic on Δ, Corollary 3.6 yields

$$\limsup_{n \to \infty} \|f - s_{n,p}^*(f)\|_{L_p(d\mu)}^{1/n} = 1.$$

Together with (41) or (42), this implies (40).

Now from (39) and (40), it follows that there is a subsequence $\Lambda(f) \subset \mathbb{N}$ such that the monic polynomials $s_{n,p}^*(f)/a_{n,p}^*$ satisfy

$$\limsup_{\substack{n \to \infty \\ n \in \Lambda(f)}} \left\| \frac{s_{n,p}^*(f)}{a_{n,p}^*} \right\|_{\partial\Delta}^{1/n} \leqslant 1. \tag{43}$$

But by Lemma 3.1 in [1], (43) implies that, for any closed set $A \subset \mathbb{C} \setminus \Delta$,

$$\lim_{\substack{n \to \infty \\ n \in \Lambda(f)}} \nu(s_{n,p}^*(f))(A) = 0.$$

As in the proof of Theorem 4.2, (43) also gives that

$$\lim_{\substack{n \to \infty \\ n \in \Lambda(f)}} \mathscr{U}(\nu(s^*_{n,p}(f)), z) = \mathscr{U}(\mu^*, z), \qquad z \in \mathbf{C} \backslash \Delta,$$

and so, as before, we conclude that

$$\lim_{\substack{n \to \infty \\ n \in \Lambda(f)}} \mathscr{U}(\hat{\nu}(s^*_{n,p}(f)), z) = \mathscr{U}(\mu^*, z), \qquad z \in \mathbf{C} \backslash \Delta$$

and that any weak-star limit measure of $\{\hat{\nu}(s^*_{n,p}(f))\}_{n \in \Lambda(f)}$ must equal μ^*. This proves the first part of our theorem.

In order to prove the second part, by Theorem 2.1 in [1], it remains to show that, for any closed set $A \subset \Delta^c$,

$$\lim_{\substack{n \to \infty \\ n \in \Lambda(f)}} \nu(s^*_{n,p}(f))(A) = 0. \tag{44}$$

For this purpose, we need the following lemma.

Lemma 4.5. *Let $w(\theta) \geqslant 0$ be Lebesgue integrable on $[0, 2\pi]$ and $\log w \in L_1([0, 2\pi])$. Assume $p > 0$ and $F \in H^\infty$. Then*

$$|F(z)| \leqslant K_{|z|,p} \left(\frac{1}{2\pi} \int_0^{2\pi} |F(e^{i\theta})|^p \, w(\theta) \, d\theta \right)^{1/p}, \qquad z \in \Delta^\circ,$$

where $K_{|z|,p} > 0$ is independent of F.

Proof. The Szegö function (cf. [17, Chap. 10])

$$D(z) := \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log \sqrt{w(\theta)} \, \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\theta \right)$$

is in H^2, has no zeros in Δ°, and satisfies

$$\lim_{r \to 1^-} |D(re^{i\theta})| = |w(\theta)|^{1/2}, \qquad \text{a.e.} \quad \theta \in (0, 2\pi).$$

First, let us assume $F \neq 0$ in Δ°. Then we can define an analytic branch of $[F(z)\, D(z)^{2/p}]^p$ in Δ^c, and so, by Cauchy integral formula, for $|z| < r < 1$,

$$[F(z)\, D(z)^{2/p}]^p = \frac{1}{2\pi i} \int_0^{2\pi} \frac{[F(re^{i\theta})\, D(re^{i\theta})^{2/p}]^p}{re^{i\theta} - z} \, ire^{i\theta} \, d\theta.$$

Thus, by letting $r \to 1^-$, we get (cf. [2, p. 21])

$$|F(z)|^p |D(z)|^2 \leqslant \frac{1}{2\pi} \frac{1}{1-|z|} \int_0^{2\pi} |F(e^{i\theta})|^p w(\theta)\, d\theta,$$

i.e.,

$$|F(z)|^p \leqslant \frac{1}{2\pi} \frac{1}{|D(z)|^2(1-|z|)} \int_0^{2\pi} |F(e^{i\theta})|^p w(\theta)\, d\theta.$$

Thus, with $K_{|z|, p} := |D(z)|^{-2/p}(1-|z|)^{-1/p}$, the lemma is proved when $F \neq 0$. The general case can be proved by factoring out the zeros of F, i.e., by writing $F(z) = B(z)\, g(z)$, where g is in H^∞ and has no zeros in Δ^c and $B(z)$ is a Blaschke product, and applying the first part of the proof to g (cf. [18, Sect. 5.5]). ∎

We now return to the proof of Theorem 4.4. Applying Lemma 4.5 to the functions $f - s_{n,p}^*(f)$, we see that $s_{n,p}^*(f)$ converges locally uniformly to f in Δ^c. Since f has only finitely many zeros on each compact subset of Δ^c, Hurwitz's theorem implies that (44) holds for any closed set $A \subset \Delta^c$. Thus, $\nu(s_{n,p}^*(f))$ converges in the weak-star topology to μ^* as $n \to \infty$, $n \in \Lambda(f)$, by Theorem 2.1 in [1]. ∎

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