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Behavior of the Lagrange Interpolants in the Roots of Unity

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Dedicated to R.S. Varga on the occasion of his sixtieth birthday.

Abstract. Let A_0 be the class of functions f analytic in the open unit disk $|z| < 1$, continuous on $|z| \leq 1$, but not analytic on $|z| \leq 1$. We investigate the behavior of the Lagrange polynomial interpolants $L_{n-1}(f, z)$ to f in the n-th roots of unity. In contrast with the properties of the partial sums of the Maclaurin expansion, we show that for any w, with $|w| > 1$, there exists a $g \in A_0$ such that $L_{n-1}(g, w) = 0$ for all n. We also analyze the size of the coefficients of $L_{n-1}(f, z)$ and the asymptotic behavior of the zeros of the $L_{n-1}(f, z)$.

1. Convergence

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be continuous on $D_1 := \{z \in \mathbf{C} : |z| \leq 1\}$. Then the Lagrange interpolant to f at the n-th roots of unity $e(\frac{k}{n}), k = 0, 1, \ldots, n-1, e(x) := e^{2\pi i x}$, can be written as

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$$L_{n-1}(f,z) = \frac{z^n - 1}{n} \sum_{k=0}^{n-1} \frac{e(\frac{k}{n}) f(e(\frac{k}{n}))}{z - e(\frac{k}{n})} =: \sum_{j=0}^{n-1} c(j,n) z^j,$$

where

$$c(j,n) := \frac{1}{n} \sum_{k=0}^{n-1} e\left(\frac{(n-j)k}{n}\right) f\left(e(\frac{k}{n})\right), \; j = 0, 1, \ldots, n-1.$$

When f is analytic on D_1 (that is, f is analytic on $|z| \le 1+\epsilon$ for some $\epsilon > 0$), several results concerning Walsh's theory of equiconvergence describe the very close behavior of the sequence of Lagrange interpolants $\{L_{n-1}(f,z)\}$ and the sequence of partial sums $\{s_{n-1}(f,z)\}$ of its Taylor series, $s_{n-1}(f,z) := \sum_{k=0}^{n-1} a_k z^k$. For example, for such f's and for any $z \in \mathbf{C}$, the sequences $\{L_{n-1}(f,z)\}_1^\infty$ and $\{s_{n-1}(f,z)\}_1^\infty$ either both converge or both diverge (hence the term *equiconvergence*).

But when f belongs to A_0 — the set of all functions continuous on D_1, analytic on the interior $\overset{\circ}{D}_1$, but not analytic on D_1 —, the behavior of the two sequences may be different. Of course, both $\{L_{n-1}(f,z)\}_1^\infty$ and $\{s_{n-1}(f,z)\}_1^\infty$ converge (to $f(z)$) when $|z| < 1$. When $|z| = 1$ there are several examples of $f \in A_0$ such that $\{L_{n-1}(f,z)\}_1^\infty$ converges but $\{s_{n-1}(f,z)\}_1^\infty$ diverges (the first goes back to du Bois-Reymond who constructed a function $f \in A_0$ with a divergent Maclaurin series at $z = 1$, but $L_{n-1}(f,1) = f(1)$). Conversely, $\{L_{n-1}(f,z)\}_1^\infty$ may diverge at a point on $|z| = 1$ where $\{s_{n-1}(f,z)\}_1^\infty$ converges (if f is continuous and of bounded variation on $|z| = 1$, then $s_{n-1}(f)$ converges *uniformly* to f, but $L_{n-1}(f,z)$ can diverge for appropriate f and z, e.g. $f(z) = (1 - \log(1+z))^{-1/2}$ at $z = -1$). When $|z| > 1$, then $\{s_{n-1}(f,z)\}_1^\infty$ necessarily diverges (the terms $a_n z^n$ do not tend to zero). Surprisingly, it is still possible for $\{L_{n-1}(f,z)\}_1^\infty$ to converge for some z with $|z| > 1$, as the corollary of the following theorem shows.

Theorem 1. *Let Λ be any subset of $\mathbf{N}$ and let $m \in \mathbf{N}$. The following are equivalent:*

(a) *There exists an $f \in A_0$ such that the first m coefficients $c(j,n), j = n-1, \ldots, n-m$ of $L_{n-1}(f,z)$ are zero for every $n \in \Lambda$.*

(b) *There exist distinct points $w_j, |w_j| > 1, j = 1, 2, \ldots, m$, and $g \in A_0$ such that $L_{n-1}(g, w_j) = 0$ for every $j = 1, 2, \ldots, m$ and every $n \in \Lambda$.*

Proof. **(a) $\Rightarrow$ (b).** From (1.2) we have

$$\sum_{k=0}^{n-1} e(\frac{ks}{n}) f(e(\frac{k}{n})) = 0, \quad s = 1, 2, \ldots, m, \quad n \in \Lambda.$$

Let $w_j, j = 1, \ldots, m$, be any m different points in $|z| > 1$ and let $g(z) := f(z)p(z)$, where $p(z) := \prod_{j=1}^m (z - w_j)$. Setting

$$W_{s,j} := (-1)^s \sum_{\substack{\Sigma\alpha_\nu = s \\ \alpha_j = 0 \\ \alpha_\nu = 0 \text{ or } 1}} w_1^{\alpha_1} \ldots w_m^{\alpha_m}$$

and using (1.1) and (1.3) we get that for any $n \in \Lambda$ and $j = 1, 2, \ldots, m$

$$
\begin{aligned}
L_{n-1}(g,w_j) &= \frac{w_j^n-1}{n}\sum_{k=0}^{n-1} f(e(\tfrac{k}{n}))e(\tfrac{k}{n})\frac{p(e(\frac{k}{n}))}{w_j-e(\frac{k}{n})} \\
&\quad -\frac{w_j^n-1}{n}\sum_{k=0}^{n-1} f(e(\tfrac{k}{n}))e(\tfrac{k}{n})\prod_{\substack{\nu=1\\ \nu\neq j}}^{m}(e(\tfrac{k}{n})-w_\nu) \\
&= \frac{w_j^n-1}{n}\sum_{k=0}^{n-1} f(e(\tfrac{k}{n}))e(\tfrac{k}{n})\sum_{s=1}^{m} e\left(\frac{k(s-1)}{n}\right) W_{m-s,j} \\
&= \frac{w_j^n-1}{n}\sum_{s=1}^{m} W_{m-s,j}\sum_{k=0}^{n-1} f(e(\tfrac{k}{n}))e(\tfrac{ks}{n}) = 0.
\end{aligned}
\tag{1.4}
$$

(b) $\Rightarrow$ (a). Keeping the notation from the first part of the proof we again set $f(z) := g(z)/p(z)$. Then $f \in A_0$ because the w_j's are outside D_1. From (1.4) we have

$$\sum_{s=1}^{m} W_{m-s,j}c(n-s,n) = 0, \quad j = 1,2,\dots,m,$$

for any $n \in A$. But $\mathrm{Det}(W_{m-s,j})_{j=1,s=1}^{m\ \ m} = \Pi_{k<l}(w_k - w_l) \neq 0$. Hence $c(n-s,n) = 0$, $s = 1,\dots,m$, which completes the proof. ∎

Corollary 2. *For any $w, |w| > 1$, there is a $g \in A_0$ such that $L_{n-1}(g,w) = 0$ for every $n \in \mathbb{N}$.*

Proof. According to Theorem 1 it is enough to find $f \in A_0$ for which all leading coefficients of the Lagrange interpolants are zero. For the function

$$F(z) := \sum_{k=1}^{\infty} \frac{\mu(k)}{k} z^k,$$

where μ is the Möbius function (of number theory), we know (see [1],[5]) that

$$\sum_{k=0}^{n-1} F(e(\tfrac{k}{n})) = 0, \quad n \in \mathbb{N}.$$

Hence for $f(z) := F(z)/z$ we have $\Sigma_{k=0}^{n-1} f(e(\frac{k}{n}))e(\frac{k}{n}) = 0$ for any n, which proves the corollary. In this case, $g(z) = \Sigma_{k=1}^{\infty}\left(\frac{\mu(k)}{k} - w\frac{\mu(k+1)}{k+1}\right)z^k - w$. ∎

Remark 1. We do not know whether there exists a $g \in A_0$ such that $L_{n-1}(g,w_j) = 0$, $j = 1,2$, for every $n \in \mathbb{N}$, where $|w_j| > 1, w_1 \neq w_2$.

Remark 2. Any g satisfying Theorem 1 or Corollary 2 will not be smooth. For example, no function with absolutely convergent Maclaurin series on $|z| = 1$ can satisfy Corollary 2.

2. Coefficients and the Distribution of Zeros

According to a theorem of Jentzsch [6], for any function $f \in A_0$, every point on the boundary of D_1 is a limit point of the zeros of $\{s_{n-1}(f,z)\}_1^\infty$. One can say even more — the zeros of a special subsequence $\{s_{n_j-1}(f,z)\}$ tend weakly to the uniform distribution on the unit circle $\{z : |z| = 1\}$ (see Szegö [7]). The same behavior can be observed for the zeros of the best polynomial approximants to $f \in A_0$ (see [2,3]). It is natural to ask whether the sequence of Lagrange interpolants $\{L_{n-1}(f,z)\}_1^\infty$ also possesses this property.

A crucial step in establishing the above mentioned facts is the proof that the leading coefficients of the full sequence of polynomials are not geometrically small. For example, for the partial sums of Taylor series, this means

$$\limsup_{n\to\infty} |a_n|^{1/n} = 1,$$

which is equivalent to $f \in A_0$, provided $f \in C(D_1)$.

One cannot expect the same behavior for the leading coefficients of $L_{n-1}(f)$. Indeed, as the example function $f \in A_0$ from the proof of Corollary 2 shows, we may have $c(n-1,n) = 0$ for every n. But results similar to Jentzsch's and Szegö's theorems still can be established by utilizing the following statement, which is a special case of Theorem 1 in Grothmann [4].

Let p_m be an algebraic polynomial of exact degree $\kappa(m)$. Define the *zero-measure* ν_m associated with p_m as

$$\nu_m(A) := \frac{\#\text{ of zeros of } p_m \text{ in } A}{\kappa(m)}$$

for any Borel set $A \subset \mathbf{C}$, where the zeros are counted with their multiplicity.

Theorem A. ([4]) *Let Λ be a sequence of positive integers and assume that the following three conditions hold for the sequence $\{p_m\}_{m\in\Lambda}$ of algebraic polynomials:*

(i) $\limsup\limits_{\substack{m\to\infty\\ m\in\Lambda}} \left(\sup\limits_{z\in D_1} \frac{1}{\kappa(m)} \log|p_m(z)| \right) \le 0;$

(ii) *for every compact set* $M \subset \overset{\circ}{D}_1$,

$$\lim_{\substack{m\to\infty\\ m\in\Lambda}} \nu_m(M) = 0;$$

(iii) *there is a compact set* $K \subseteq \overline{\mathbf{C}}\backslash D_1$ *with*

$$\liminf_{\substack{m\to\infty\\ m\in\Lambda}} \left[\sup_{z\in K} \left(\frac{1}{\kappa(m)} \log|p_m(z)| - \log|z| \right) \right] \ge 0$$

Then, in the weak-star topology, ν_m tends to the uniform distribution $\frac{1}{2\pi}d\theta$ on the unit circle as $m \to \infty, m \in \Lambda$.

This leads us to investigating

$$\sigma(f,\theta) := \limsup_{n\to\infty} \max_{(1-\theta)n\le j<n} |c(j,n)|^{1/n}$$

for $\theta \in (0,1]$, where $c(j,n)$ is defined in (1.2). Obviously $\sigma(f,1) = 1$ and σ is an increasing function of θ for any $f \in A_0$. We shall prove

Theorem 3. *For any $f \in A_0$, we have $\sigma(f,1/3) = 1$.*

Proof. For any $r \in \mathbb{N}$ using (1.2) we get $(0 \le j < n)$

$$\begin{aligned} \textstyle\sum_{l=0}^{r-1} c(j+ln,rn) &= \frac{1}{rn}\sum_{k=0}^{rn-1} f(e(\frac{k}{rn}))\sum_{l=0}^{r-1} e(\frac{-k(j+ln)}{rn}) \\ & \frac{1}{rn}\sum_{k=0}^{rn-1} f(e(\frac{k}{rn}))e(\frac{-kj}{rn})\sum_{l=0}^{r-1} e(-\frac{kl}{r}) \\ & \frac{1}{rn}\sum_{m=0}^{n-1} f(e(\frac{m}{n}))e(\frac{-jm}{n})r = c(j,n). \end{aligned}$$

Let us assume that $\sigma(f,1/3) < 1$. Fix q, such that $\sigma(f,1/3) < q < 1$. Then for any $n > n_0$ and any j, $\frac{2}{3}n \le j < n$, we have

$$|c(j,n)| < q^n.$$

Fix $l > n_0$ such that $\frac{5}{1-q}q^s < \frac{1}{2}q^l, s = [\frac{3l}{2}]$, and $|a_l| > q^l$ ($f \in A_0$ implies that $\limsup_{n\to\infty}|a_n|^{1/n} = 1$). From the continuity of f and (1.2) we get

$$\lim_{n\to\infty} c(l,n) = \frac{1}{2\pi i}\int_{|z|=1} f(z)z^{-l-1}dz = a_l.$$

Therefore $|c(l,n)| > \frac{1}{2}q^l$ for any $n \ge n_1$. Let us fix $m \in \mathbb{N}$ such that $2\cdot 3^m \cdot 5 \ge n_1$. Then

$$|c(l, 2\cdot 3^m\cdot 5)| > \frac{1}{2}q^l.$$

Now our aim is, using (2.1) and (2.2), to obtain an estimate contradicting (2.3). From (2.1) we get

$$c(l,2s) = c(l,s) - c(l+s,2s),$$

$$c(l,6n) + c(l+2n,6n) + c(l+4n,6n) = c(l,2n),$$

$$c(l+2n,6n) + c(l+5n,6n) = c(l+2n,3n).$$

From (2.5) and (2.6) we get

$$c(l,6n)=c(l,2n)+\{c(l+5n,6n)-c(l+4n,6n)-c(l+2n,3n)\}.$$

From (2.7) with $n=3^{k-1}s, k=1,2,\ldots,m$, and (2.4) we obtain

$$c(l,2\cdot 3^m s)=c(l,s)-c(l+s,2s)+\alpha$$

where

$$\alpha=\sum_{k=1}^{m}\{c(l+5\cdot 3^{k-1}s,2\cdot 3^k s)-c(l+4\cdot 3^{k-1}s,2\cdot 3^k s)-c(l+2\cdot 3^{k-1}s,3^k s)\}.$$

It is easy to see that all terms on the right-hand side of (2.8) are of the type $c(j,n)$ with $\frac{2}{3}n\le j<n$, $n\ge s>l>n_0$. By applying (2.2) in (2.8) we get

$$\begin{aligned}|c(l,2\cdot 3^m s)| &< q^s+q^{2s}+\textstyle\sum_{k=1}^{m}\{2q^{2(3^k s)}+q^{3^k s}\}\\ &< q^s+q^{2s}+3q^{3s}/(1-q)<5q^s/(1-q)<\frac{1}{2}q^l\end{aligned}$$

This estimate contradicts (2.3) and proves the theorem. ∎

If one assumes that $f\in A_0$ has an absolutely convergent Maclaurin series on $|z|=1$, then $\limsup_{n\to\infty}|c(n-1,n)|^{1/n}=1$ (cf. [5]). This implies that $\sigma(f,\theta)=1, \theta\in(0,1]$, for such f's.

Theorem 3 and the above observation give some evidence to the following.

Conjecture. *For any $f\in A_0$ and any $0<\theta<1$, we have $\sigma(f,\theta)=1$.*

Now we can establish

Theorem 4. *If the above conjecture is true, then for any $f\in A_0$ there is a subsequence $\{n_j\}$ such that the zero measures ν_{n_j} (corresponding to $L_{n_j-1}(f)$) tend (in the weak-star topology) to the uniform distribution on the unit circle as $j\to\infty$.*

Proof. We are going to apply Theorem A with $\{p_m\}$ an appropriate subsequence $L_{n_j-1}(f)$ of the Lagrange interpolants. The subsequence $L_{n_j-1}(f)$ is chosen so that condition (iii) is satisfied.

Let $r>1$ be fixed. Assume that there exists an ϵ, $0<\epsilon<1/2$, such that

$$\limsup_{n\to\infty}\left[\sup_{|z|=r}\left(\frac{1}{\kappa(n)}\log|L_{n-1}(f,z)|-\log r\right)\right]<-3\epsilon\log r,$$

where $\kappa(n)$ is the precise degree of $L_{n-1}(f,z)$. Then for $n>N_0$ and for every $z, |z|=r$, we have

$$\frac{1}{\kappa(n)}\log|L_{n-1}(f,z)|-\log r<-2\epsilon\log r,$$

that is,

$$|L_{n-1}(f,z)|<r^{(1-2\epsilon)\kappa(n)}\le r^{(1-2\epsilon)n}.$$

Hence for any $j, (1-\epsilon)n\le j<n$, we have

$$|c(j,n)| = \left|\frac{1}{2\pi i}\int_{|z|=r} L_{n-1}(f,z)z^{-j-1}dz\right| \le r^{(1-2\epsilon)n-j} \le r^{-\epsilon n}$$

This inequality implies that $\sigma(f,\epsilon) \le r^{-\epsilon}$, which contradicts the conjecture. Therefore

$$\limsup_{n\to\infty}\left[\sup_{|z|=r}\left(\frac{1}{\kappa(n)}\log|L_{n-1}(f,z)| - \log r\right)\right] \ge 0.$$

Hence there exists a subsequence $\{n_j\}$ such that

$$\liminf_{j\to\infty}\left[\sup_{|z|=r}\left(\frac{1}{\kappa(n_j)}\log|L_{n_j-1}(f,z)| - \log r\right)\right] \ge 0.$$

Consequently, condition (iii) of Theorem A is fulfilled for the sequence $\{L_{n_j-1}(f)\}$. The other two conditions of Theorem A are easily seen to be satisfied (even for the whole sequence of Lagrange interpolants). Indeed, condition (i) follows from the trivial estimate

$$\| L_{n-1}(f) \|_{D_1} \le n \| f \|_{D_1}$$

(see (1.1)) and $\kappa(n) \ge \frac{2}{3}n$ (see Theorem 3 — we do not need the Conjecture here). Condition (ii) follows from the facts that $\{L_{n-1}(f)\}$ approximates f uniformly on any compact set $M \subset \mathring{D}_1$ and that f can have only finitely many zeros on M. Hence Theorem 4 follows from Theorem A. ∎

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