THE REPRESENTATION OF FUNCTIONS IN TERMS OF THEIR DIVIDED DIFFERENCES AT CHEBYSHEV NODES AND ROOTS OF UNITY

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Abstract

For the infinite triangular arrays of points whose rows consist of (i) the nth roots of unity, (ii) the extrema of Chebyshev polynomials $T_n(x)$ on $[-1,1]$, and (iii) the zeros of $T_n(x)$, we consider the corresponding sequences of divided difference functionals $\{I_n\}_1^\infty$ in the successive rows of these arrays. We investigate the totality of such functionals as well as the convergence of the *generalized Taylor series* $\sum_1^\infty (I_n f) P_{n-1}(z)$ for a function f, where the P_k are basic polynomials satisfying $I_{j+1} P_k = \delta_{jk}$. Explicit formulae are given for the basic polynomials involving the Möbius function (of number theory), and examples of non-trivial functions f for which $I_n f = 0$, $n = 1, 2, \ldots$, are constructed.

Introduction

Let f be a function defined on the distinct complex points $z_1, \ldots, z_k$. Recall that if $m_1, \ldots, m_k$ are positive integers with $\sum_{i=1}^k m_i = n$, there exists a unique $p \in \mathscr{P}_{n-1}$ ($\mathscr{P}_k$ denotes the set of polynomials of degree at most k), $p(z) = a_0 + a_1 z + \ldots + a_{n-1} z^{n-1}$, satisfying

$$p^{(i)}(z_j) = f^{(i)}(z_j), \qquad i = 0, 1, \ldots, m_j - 1; j = 1, \ldots, k. \tag{0.1}$$

(An assumption that f has the required derivatives at z_j when $m_j > 1$ is implicit in (0.1).) The leading coefficient of p, that is, a_{n-1}, is called the *divided difference of f with respect to* $z_1, \ldots, z_n$ (where each z_j appears m_j times in this sequence). In a more familiar notation we have $a_{n-1} = f(z_1, \ldots, z_n)$, and it is clear that a_{n-1} is a symmetric function of $z_1, \ldots, z_n$. We shall also use the notation

$$I_n f := a_{n-1}.$$

It is obvious that I_n is a linear functional which satisfies $I_n q = 0$ if $q \in \mathscr{P}_m$, $m < n-1$; and $I_n z^{n-1} = 1$. Note that if $z_1 = \ldots = z_n = 0$, then $I_n f = f^{(n-1)}(0)/(n-1)!$

Let $\boldsymbol{\beta}$ denote an infinite triangular array of complex numbers whose jth row, $j = 0, 1, 2, \ldots$, is $\beta^{(j)} = (\beta_1^{(j)}, \ldots, \beta_{j+1}^{(j)})$ and suppose that f is a function defined on all the entries in $\boldsymbol{\beta}$. It is easy to see that, in view of the elementary properties of the divided difference functionals $I_n f = f(\beta_1^{(n-1)}, \ldots, \beta_n^{(n-1)})$, which we have just mentioned, there exist unique *basic polynomials*, $P_k \in \mathscr{P}_k$, $k = 0, 1, 2, \ldots$, that are monic and satisfy

$$I_{j+1} P_k = \delta_{jk}, \qquad j, k = 0, 1, 2, \ldots. \tag{0.2}$$

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Thus $\{P_k(z), I_{k+1}\}_{k=0}^{\infty}$ is a normalized biorthogonal system, and, given $\boldsymbol{\beta}$, each f defined on it has the biorthogonal expansion

$$\sum_{j=1}^{\infty} (I_j f) P_{j-1}(z) \tag{0.3}$$

associated to it. In particular, if all entries in $\boldsymbol{\beta}$ are zero, (0.3) becomes the Taylor series of f, and so we call (0.3) the *generalized Taylor series* of f with respect to $\boldsymbol{\beta}$. A study of generalized Taylor series with respect to certain arrays is one of our themes in this work.

Another prominent theme in what follows is the question of the *totality* of the sequence of divided differences $I_{j,\boldsymbol{\beta}}$, $j = 1, 2, \ldots$ (where the notation indicates the underlying triangular array of points) for some specified set of functions. That is, if for each $f \in X$, $I_j f = 0, j = 1, 2, \ldots$, implies that $f = 0$, then $\{I_j\}_{j=1}^{\infty}$ is called *total* for X. The totality of $\{I_{j,\boldsymbol{\beta}}\}$ for functions having convergent biorthogonal expansions, such as generalized Taylor series, or series of orthogonal polynomials will be examined. Background material about divided differences may be found in [3, 7].

Before sketching the contents of the five subsequent sections of this paper we present some of the notation that will be used. We write $T_n(x)$ and $U_n(x)$ for the Chebyshev polynomials of degree n, of the first and second kinds, respectively. We set

$$D_\rho := \{z \in \mathbb{C} : |z| \leqslant \rho\}, \quad C_\rho := \{z \in \mathbb{C} : |z| = \rho\}, \quad I := [-1, 1], \quad e(x) := e^{2\pi i x};$$

$\boldsymbol{\xi}$ is the infinite triangular array of points of I with $\xi_k^{(j-1)} = \cos((2k-1)(\pi/2j))$, $k = 1, \ldots, j$, $j \geqslant 1$ (zeros of $T_j(x)$); $\boldsymbol{\eta}$ is the infinite triangular array of points of I with $\eta_k^{(j)} = \cos((k-1)\pi/j)$, $k = 1, \ldots, j+1$, $j \geqslant 1$ (extrema of $T_j(x)$), $\eta_1^{(0)} = 0$; $\boldsymbol{\omega}$ is the infinite triangular array of points of C_1 with $\omega_k^{(j)} = e((k-1)/(j+1))$, $k = 1, \ldots, j+1$, $j \geqslant 0$, (roots of unity); $\mathbb{N}$ denotes the positive integers. The *Möbius function* $\mu(n)$ is defined for $n \in \mathbb{N}$ by

$$\mu(n) := \begin{cases} 1, & n = 1, \\ (-1)^k, & n \text{ is the product of } k \text{ distinct primes}, \\ 0, & \text{all other } n \in \mathbb{N}. \end{cases}$$

The function $\mu(n)$ is multiplicative, that is, $\mu(nm) = \mu(n)\mu(m)$ if 1 is the greatest common divisor of m and n (written $(m, n) = 1$). The *Möbius inversion formula* says that if $n \in \mathbb{N}$

$$\sum_{d \mid n} \mu(d) = \sum_{d \mid n} \mu\left(\frac{n}{d}\right) = \begin{cases} 1, & n = 1, \\ 0, & n \neq 1, \end{cases} \tag{0.4}$$

where $d \mid n$ means that d is a divisor of n. Finally, $d(n) := \sum_{d \mid n} 1$; that is, $d(n)$ is the number of positive integers that are divisors of n. If $\varepsilon > 0$, then $d(n) = O(n^\varepsilon)$, as $n \to \infty$, with a constant that depends on ε. Proofs of these number-theoretic results may be found in [8].

In Section 1 we exhibit the connection between the divided differences of a function with respect to points on C_1 and those of a related function with respect to the projections of those points on I. Expressions for the divided differences, with respect to $\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\omega}$, in terms of the coefficients of the expansion of the function in Taylor series or Chebyshev series are also given.

Section 2 contains results about the totality of the $\{I_{j,\boldsymbol{\beta}}\}_1^{\infty}$ $(\boldsymbol{\beta} = \boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\omega})$ for various classes of functions, while connections between divided differences and analyticity of the function are examined in Section 3. Section 4 is devoted to the explicit construction of the basic polynomials for ξ, η and ω, bounds for these polynomials and convergence of the corresponding biorthogonal expansions. Section 5 presents some counterexamples which delimit the sharpness of the totality results of Section 2.

1. *Relationship between divided differences on I and on C_1*

If f is a complex-valued function defined on a subset of $\mathbb{C}$, we define operators L_n and M_n as follows:

$$L_n(f,z):=z^{n-1}(z^2-1)f(z), \qquad M_n(f,z):=z^{n-1}f(z). \tag{1.1}$$

For points $x_j\in[-1,1]$ we write

$$x_j=\cos\phi_j, \qquad 0\leqslant\phi_j\leqslant\pi, \quad z_j:=e^{i\phi_j}. \tag{1.2}$$

THEOREM 1.1. *Suppose that*

$$p(x)=\sum_{k=0}^{m}A_k T_k(x) \quad \text{and} \quad q(z)=\sum_{k=0}^{m}A_k z^k.$$

If $\{x_j\}_0^n\subset[-1,1]$, $n\geqslant 0$, *then*

$$p(x_0,\dots,x_n)=2^{n-1}L_n q(z_0,\overline{z_0},\dots,z_n,\overline{z_n}), \tag{1.3}$$

where x_j *and* z_j *are related as in* (1.2).

Proof. Because of the linearity of the divided difference operator, it suffices to prove (1.3) for $p(x)=T_k(x)$. Thus $q(z)=z^k$ and $L_n(q,z)=z^{k+n-1}(z^2-1)$. If $k=n=0$, then both sides of (1.3) are equal to 1.

Suppose, then, that $k+n\geqslant 1$. If $\rho>1$, we denote by $\tilde{C}_\rho$ the image of the circle C_ρ under the Joukowski mapping

$$w=\phi(z):=z+(z^2-1)^{\frac{1}{2}}, \qquad z=\phi^{-1}(w)=\tfrac{1}{2}(w+w^{-1}). \tag{1.4}$$

Put

$$h(z):=\prod_{j=0}^{n}(z-x_j), \qquad H(w):=\prod_{j=0}^{n}(w-z_j)(w-\overline{z_j}).$$

Then

$$h(\tfrac{1}{2}(w+w^{-1}))=(2w)^{-n-1}H(w),$$

and, by the Hermite formula for divided differences (cf. [7]), we have

$$\begin{aligned}
p(x_0,\dots,x_n)&=\frac{1}{2\pi i}\int_{\tilde{C}_\rho}\frac{T_k(z)}{h(z)}dz=\frac{1}{2\pi i}\int_{C_\rho}\frac{T_k(\frac{1}{2}(w+w^{-1}))}{h(\frac{1}{2}(w+w^{-1}))}d(\tfrac{1}{2}(w+w^{-1}))\\
&=\frac{1}{2\pi i}\int_{C_\rho}\frac{\frac{1}{2}(w^k+w^{-k})}{(2w)^{-n-1}H(w)}\cdot\tfrac{1}{2}(1-w^{-2})\,dw\\
&=2^{n-1}\frac{1}{2\pi i}\int_{C_\rho}\frac{w^k w^{n-1}(w^2-1)}{H(w)}dw+0\\
&=2^{n-1}L_n q(z_0,\overline{z_0},\dots,z_n,\overline{z_n}).
\end{aligned}$$

Theorem 1.1 allows us to relate the divided differences of two functions f and g, defined on I and C_1, respectively, whenever they can be simultaneously approximated by corresponding polynomials p and q. For example, we have the following.

COROLLARY 1.2. *Suppose that* $\{x_j\}_0^n \subset [-1, 1]$ *are* $n+1$ *distinct points. If* $\sum_{k=0}^{\infty} |A_k| < \infty$, *then*

$$f(x) := \sum_{k=0}^{\infty} A_k T_k(x) \in C(I), \qquad g(z) := \sum_{k=0}^{\infty} A_k z^k \in C(D_1)$$

and

$$f(x_0, \ldots, x_n) = 2^{n-1} L_n g(z_0, \overline{z_0}, \ldots, z_n, \overline{z_n}), \tag{1.5}$$

where x_j *and* z_j *are related as in* (1.2).

Proof. The result follows by applying Theorem 1.1 to the polynomials

$$p_m(x) := \sum_{k=0}^{m} A_k T_k(x), \qquad q_m(z) := \sum_{k=0}^{m} A_k z^k,$$

and then taking the limit as $m \to \infty$. That the limiting process yields (1.5) is obvious for the case of divided differences in distinct points. But, when $x_0 = 1$ (or $x_n = -1$), we have $z_0 = \overline{z_0} = 1$ (or $z_n = \overline{z_n} = -1$), and repeated nodes occur in the divided difference on the right-hand side of (1.5). It is easy to check, however, that these repeated nodes cause no difficulties because $L_n g$ is differentiable at 1 and -1 whenever g is continuous.

COROLLARY 1.3. *Suppose that* $\{x_j\}_0^n \subset [-1, 1]$ *are* $n+1$ *distinct points and that*

$$g(z) := \sum_{k=0}^{\infty} a_k z^k \in C(D_1),$$

where the a_k *are real. Then* (1.5) *holds with*

$$f(x) := \operatorname{Re} g(e^{i \arccos x}).$$

Proof. For each $\varepsilon > 0$ there exists $q \in \mathscr{P}_m$ with real coefficients such that $\|g - q\|_{D_1} < \varepsilon$. Writing $q(z) = \sum_{k=0}^{m} A_k z^k$, we put

$$p(x) := \sum_{k=0}^{m} A_k T_k(x).$$

Then $\|f - p\|_I < \varepsilon$, and the corollary follows from Theorem 1.1 just as in the preceding proof.

For the case when both 1 and -1 are nodes in the divided difference we have the following alternative relation which involves the operator M_n of (1.1).

THEOREM 1.4. *If* $p(x) = \sum_{k=0}^{m} A_k T_k(x)$ *and* $q(z) = \sum_{k=0}^{m} A_k z^k$, *then*

$$p(1, x_1, \ldots, x_{n-1}, -1) = 2^{n-1} M_n q(1, z_1, \overline{z_1}, \ldots, z_{n-1}, \overline{z_{n-1}}, -1), \tag{1.6}$$

where x_j *and* z_j *are related as in* (1.2).

Proof. Let $h(z) := z - a$, $g(z)$ be any polynomial, and $y_0, y_1, \ldots, y_l$ be arbitrary points of $\mathbb{C}$. Then the Leibniz formula for divided differences (cf. [3]) yields

$$(gh)(y_0, \ldots, y_l) = g(y_0, \ldots, y_l) h(y_l) + g(y_0, \ldots, y_{l-1}). \tag{1.7}$$

Putting $g(z) = (z+1) M_n(q; z)$, $h(z) = z-1$, and $y_l = 1$ in (1.7) we obtain

$$L_n q(y_0, \ldots, y_{l-1}, 1) = ((z+1) M_n(q, z))(y_0, \ldots, y_{l-1}). \tag{1.8}$$

Substituting $g(z) = M_n(q, z)$, $h(z) = z+1$, and $y_l = -1$ in (1.7) gives

$$((z+1) M_n(q, z))(y_0, \ldots, y_{l-1}, -1) = M_n q(y_0, \ldots, y_{l-1}). \tag{1.9}$$

Keeping in mind that a divided difference is a symmetric function of its nodes, we see that (1.3), (1.8) and (1.9) imply (1.6).

We show next that for the special choice of nodes $\boldsymbol{\beta} = \xi, \boldsymbol{\eta}$, the relation (1.3) can be simplified further. Suppose that $\phi_j = j\pi/n$, $j = 0, \ldots, n$, so that $x_j = \eta_{j+1}^{(n)}$, $j = 0, \ldots, n$. Then we have (cf. (1.2))

$$\{1, z_1, \overline{z_1}, \ldots, z_{n-1}, \overline{z_{n-1}}, -1\} = \{e(0), e(1/2n), \ldots, e((2n-1)/(2n))\},$$

which is the set of zeros of $\psi(z) := z^{2n} - 1$. Hence

$$\begin{aligned} M_n q(1, z_1, \overline{z_1}, \ldots, z_{n-1}, \overline{z_{n-1}}, -1) &= \sum_{k=0}^{2n-1} \frac{M_n(q, e(k/2n))}{\psi'(e(k/2n))} \\ &= \frac{1}{2n} \sum_{k=0}^{2n-1} \frac{q(e(k/2n))\,(e(k/2n))^{n-1}}{(e(k/2n))^{2n-1}} \\ &= \frac{1}{2n} \sum_{k=0}^{2n-1} (-1)^k q(e(k/2n)). \end{aligned}$$

This identity, Theorem 1.4, and the representation of the divided difference

$$F(e(0), e(1/N), \ldots, e((N-1)/N)) = \frac{1}{N} \sum_{k=0}^{N-1} F(e(k/N))\, e(k/N) \tag{1.10}$$

now yield the following.

THEOREM 1.5. *Let* $p(x) = \sum_{k=0}^{m} A_k T_k(x)$, $q(z) = \sum_{k=0}^{m} A_k z^k$, *and*

$$\tilde{q}(z) := \frac{q(z) - q(0)}{z} = \sum_{k=0}^{m-1} A_{k+1} z^k. \tag{1.11}$$

Then for the points $\eta_j^{(n)} = \cos((j-1)\pi/n)$, $n \geqslant 1$,

$$\begin{aligned} p(\eta_1^{(n)}, \ldots, \eta_{n+1}^{(n)}) &= \frac{2^{n-2}}{n} \sum_{k=0}^{2n-1} (-1)^k q(e(k/2n)) \\ &= 2^{n-1} \left[\tilde{q}\left(e(0), e\left(\frac{1}{n}\right), \ldots, e\left(\frac{n-1}{n}\right)\right) - \tilde{q}\left(e(0), e\left(\frac{1}{2n}\right), \ldots, e\left(\frac{2n-1}{2n}\right)\right)\right]. \end{aligned} \tag{1.12}$$

When $x_{j-1} = \xi_j^{(n-1)} = \cos((2j-1)\pi/2n)$, $j = 1, \ldots, n$, the zeros of $T_n(x)$, we obtain from Theorem 1.1 in an analogous way the following.

THEOREM 1.6. *Let* $p(x) = \sum_{k=0}^{m} A_k T_k(x)$, $q(z) = \sum_{k=0}^{m} A_k z^k$. *Then, for* $n \geqslant 1$,

$$\begin{aligned} p(\xi_1^{(n-1)}, \ldots, \xi_n^{(n-1)}) &= \frac{2^{n-3} i}{n} \sum_{k=1}^{2n} (-1)^k q\left(e\left(\frac{2k-1}{4n}\right)\right)\left[e\left(\frac{2k-1}{4n}\right) - e\left(-\frac{2k-1}{4n}\right)\right] \\ &= \frac{2^{n-2}}{n} \sum_{k=1}^{2n} (-1)^{k-1} q\left(e\left(\frac{2k-1}{4n}\right)\right) \sin\left(\frac{2k-1}{2n}\pi\right). \end{aligned} \tag{1.13}$$

These results can be extended to functions defined on C_1 and on I in the same way as Theorem 1.1 yielded Corollaries 1.2 and 1.3. For example, Theorem 1.5 has the following consequence.

COROLLARY 1.7. *Suppose that* $g(z) = \sum_{k=0}^{\infty} a_k z^k \in C(D_1)$, *where the* a_k *are real, and*

$$f(x) := \operatorname{Re} g(e^{i \arccos x}).$$

Then

$$\begin{aligned} f(\eta_1^{(n)}, \ldots, \eta_{n+1}^{(n)}) &= \frac{2^{n-2}}{n} \sum_{k=0}^{2n-1} (-1)^k g\left(e\left(\frac{k}{2n}\right)\right) \\ &= 2^{n-1}\left[\frac{1}{n}\sum_{k=0}^{n-1} g\left(e\left(\frac{k}{n}\right)\right) - \frac{1}{2n}\sum_{k=0}^{2n-1} g\left(e\left(\frac{k}{n}\right)\right)\right]. \end{aligned} \tag{1.14}$$

For future reference, we state the following result which is a straightforward consequence of the properties of divided differences.

PROPOSITION 1.8. *Suppose that* $g(z) = \sum_{k=0}^{\infty} a_k z^k \in C(D_1)$, *where the* a_k *are real, and set*

$$f(x) := \operatorname{Im} g(e^{i \arccos x}), \qquad \tilde{f}(x) := f(x)/(1-x^2)^{\frac{1}{2}}.$$

Then, for $n \geqslant 1$,

$$\begin{aligned} \tilde{f}(\xi_1^{(n-1)}, \ldots, \xi_n^{(n-1)}) &= \frac{2^{n-1}}{n} \sum_{k=1}^{n} (-1)^{k-1} f(\xi_k^{(n-1)}) \\ &= \frac{2^{n-2} i}{n} \sum_{k=1}^{2n} (-1)^k g\left(e\left(\frac{2k-1}{4n}\right)\right). \end{aligned} \tag{1.15}$$

We further remind the reader of two elementary formulae (cf. [11]) that hold for any complex-valued function f defined on η or ξ:

$$f(\eta_1^{(n)}, \ldots, \eta_{n+1}^{(n)}) = \frac{2^{n-1}}{n}\left[\frac{1}{2} f(1) + \sum_{k=1}^{n-1} (-1)^k f(\eta_{k+1}^{(n)}) + \frac{(-1)^n}{2} f(-1)\right], \tag{1.16}$$

$$f(\xi_1^{(n-1)}, \ldots, \xi_n^{(n-1)}) = \frac{2^{n-1}}{n} \sum_{k=1}^{n} (-1)^{k-1} (1-(\xi_k^{(n-1)})^2)^{\frac{1}{2}} f(\xi_k^{(n-1)}). \tag{1.17}$$

Now suppose that $f(x) = \sum_{k=0}^{\infty} A_k T_k(x)$, where the Chebyshev expansion is uniformly convergent on I. Applying the respective divided difference functionals term-by-term to the Chebyshev expansion yields

$$f(\eta_1^{(n)}, \ldots, \eta_{n+1}^{(n)}) = 2^{n-1} \sum_{j=1}^{\infty} A_{(2j-1)n}, \qquad n \geqslant 1 \tag{1.18}$$

and

$$f(\xi_1^{(n-1)}, \ldots, \xi_n^{(n-1)}) = 2^{n-2} \sum_{j=1}^{\infty} (-1)^j (A_{(2j-1)n+1} - A_{(2j-1)n-1}), \qquad n \geqslant 2 \tag{1.19}$$

(cf. [11]). We also mention that the corresponding formula for the divided difference in the nth roots of unity, derived from (1.10), is

$$f(e(0), e(1/n), \ldots, e((n-1)/n)) = \sum_{j=1}^{\infty} a_{jn-1}, \tag{1.20}$$

where $f(z) = \sum_{j=0}^{\infty} a_j z^j$ and the power series is uniformly convergent on C_1.

Theorem 1.6 and (1.19) can be somewhat simplified by considering representations in terms of the $U_k(x)$, the Chebyshev polynomials of the second kind, rather than the $T_k(x)$. Thus we have the following.

THEOREM 1.9. *If* $p(x) = \sum_{k=0}^{m} B_k U_k(x)$ *and* $r(z) = z\sum_{k=0}^{m} B_k z^k$, *then*

$$p(\xi_1^{(n-1)}, \ldots, \xi_n^{(n-1)}) = \frac{2^{n-1}}{2n} i \sum_{k=1}^{2n} (-1)^k r\left(e\left(\frac{2k-1}{4n}\right)\right).$$

Proof. It is easy to verify that

$$T_i(x) = \tfrac{1}{2}(U_i(x) - U_{i-2}(x)), \qquad i = 0, 1, \ldots \quad (U_{-2} = -1, U_{-1} = 0) \tag{1.21}$$

and, therefore, if

$$p(x) = \frac{A_0}{2} + \sum_{k=1}^{m} A_k T_k(x),$$

we obtain

$$2B_j = A_j - A_{j+2}, \qquad j = 0, \ldots, m \quad (A_{m+1} = A_{m+2} = 0). \tag{1.22}$$

Put

$$q(z) := \frac{A_0}{2} + \sum_{k=1}^{m} A_k z^k.$$

Then in view of the identities

$$\sum_{k=1}^{2n} (-1)^k e\left(-\frac{2k-1}{2n}\right) = -\sum_{k=1}^{2n} (-1)^k e\left(\frac{2k-1}{2n}\right),$$

and $\sum_{k=1}^{2n} (-1)^k = 0$, we deduce from Theorem 1.6 and (1.22) that

$$\begin{aligned} p(\xi_1^{(n-1)}, \ldots, \xi_n^{(n-1)}) &= \frac{2^{n-2}}{2n} i \sum_{k=1}^{2n} (-1)^k \left(\tfrac{1}{2}A_0 + \sum_{j=1}^{m} A_j e\left(\frac{(2k-1)j}{4n}\right)\right) \\ &\quad \times \left(e\left(\frac{2k-1}{4n}\right) - e\left(-\frac{2k-1}{4n}\right)\right) \\ &= \frac{2^{n-2}}{2n} i \left[\tfrac{1}{2}A_0 \sum_{k=1}^{2n} (-1)^k \left(e\left(\frac{2k-1}{4n}\right) - e\left(-\frac{2k-1}{4n}\right)\right)\right. \\ &\quad - A_1 \sum_{k=1}^{2n} (-1)^k - A_2 \sum_{k=1}^{2n} (-1)^k e\left(\frac{2k-1}{4n}\right) \\ &\quad \left. + \sum_{j=1}^{m} (A_j - A_{j+2}) \sum_{k=1}^{2n} (-1)^k e\left(\frac{2k-1}{4n}(j+1)\right)\right] \\ &= \frac{2^{n-2}}{2n} i \sum_{k=1}^{2n} (-1)^k \sum_{j=0}^{m} (A_j - A_{j+2}) e\left(\frac{2k-1}{4n}(j+1)\right) \\ &= \frac{2^{n-1}}{2n} i \sum_{k=1}^{2n} (-1)^k r\left(e\left(\frac{2k-1}{4n}\right)\right). \end{aligned}$$

Notice that if

$$\sum_{k=0}^{\infty} |B_k| < \infty, \tag{1.23}$$

then $\sum_{k=0}^{\infty} B_k U_k(x)$ is absolutely convergent on any compact subset of $(-1, 1)$ since

$$U_k(\cos\phi) = (\sin(k+1)\phi)/\sin\phi, \qquad 0 \leqslant \phi \leqslant \pi.$$

If (1.23) holds and we set

$$f(x) = \sum_{k=0}^{\infty} B_k U_k(x), \tag{1.24}$$

then Theorem 1.9 and the identity

$$\sum_{k=1}^{2n} (-1)^k e\left(\frac{2k-1}{4n}(s+1)\right) = \begin{cases} i(-1)^j 2n, & s = (2j-1)n-1,\ j = 1,2,\dots, \\ 0, & s \neq (2j-1)n-1 \end{cases}$$

yield

$$f(\xi_1^{(n-1)}, \dots, \xi_n^{(n-1)}) = 2^{n-1} \sum_{j=1}^{\infty} (-1)^{j-1} B_{(2j-1)n-1}, \qquad n \geqslant 1, \tag{1.25}$$

since the series in (1.24) is uniformly convergent on $[\xi_n^{(n-1)}, \xi_1^{(n-1)}]$.

The simpler representation (1.25) cannot be obtained directly from Theorem 1.6 via (1.19) and (1.22) since the condition $f(x) = \sum_{k=0}^{\infty} A_k T_k(x)$, $\sum_{k=0}^{\infty} |A_k| < \infty$, is much stronger than conditions (1.24) and (1.23). The former condition implies that $f \in C(I)$ while the latter allows f to tend to infinity at 1 or -1.

2. *Totality of divided differences*

A sequence of linear functionals $\mathscr{L}_j$, $j = 1, 2, \dots$, acting on a linear space F is called *total* if $\mathscr{L}_j f = 0$, $j = 1, 2, \dots$, implies that $f = 0$ for any $f \in F$. In this section we show that the divided differences $\{I_{j,\beta}\}$, $j = 1, 2, \dots$ (where the notation indicates the underlying triangular array of points), for $\beta = \omega, \eta, \xi$ are total, each for an appropriately chosen function space.

Suppose that $\beta_j^{(n)} = e((j-1)/(n+1))$, $j = 1, \dots, n+1$; $n = 0, 1, \dots$, so that $\beta = \omega$. The following result is due to Katai [9], but we present an equally brief proof which suggests the approach to the cases $\beta = \eta, \xi$ that will follow.

THEOREM 2.1. *If* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $\sum_{k=0}^{\infty} |a_k| < \infty$, *and* $I_{n,\omega} f = 0$, $n = 1, 2, \dots$, *then* $f = 0$.

Proof. According to (1.20) we have

$$I_{n,\omega} f = \sum_{j=1}^{\infty} a_{jn-1} = 0, \qquad n = 1, 2, \dots. \tag{2.1}$$

Let $p_1, p_2, \dots$ denote the prime numbers in increasing order and let $N_\nu := p_1 p_2 \cdot \dots \cdot p_\nu$. Then

$$\begin{aligned} \sum_{d \mid N_\nu} \mu(d) I_{nd,\omega} f &= \sum_{d \mid N_\nu} \mu(d) \sum_{j=1}^{\infty} a_{jdn-1} \\ &= \sum_{k=1}^{\infty} a_{kn-1} \sum_{\substack{d \mid k \\ d \mid N_\nu}} \mu(d) = \sum_{k=1}^{\infty} a_{kn-1} \sum_{d \mid (k, N_\nu)} \mu(d) = \sum_{\substack{k=1 \\ (k, N_\nu)=1}}^{\infty} a_{kn-1}, \end{aligned} \tag{2.2}$$

where we use (0.4) to obtain the last equality. In view of the hypothesis that $I_{n,\omega} f = 0$, $n = 1, 2, \dots$, (2.2) yields

$$a_{n-1} = -\sum_{\substack{k > p_\nu \\ (k, N_\nu)=1}} a_{kn-1}, \tag{2.3}$$

since while $(1, N_\nu) = 1$, $(k, N_\nu) > 1$ for $1 < k \leqslant p_\nu$.

Suppose that the theorem is false and that $a_{n-1} \neq 0$ for some n. Choose m so large that $\sum_{k=m}^{\infty} |a_k| < |a_{n-1}|$. Then, if we put $\nu = m$, (2.3) gives

$$|a_{n-1}| = \Big| \sum_{\substack{k > p_m \\ (k, N_m) = 1}} a_{kn-1} \Big| \leqslant \sum_{k \geqslant m} |a_k| < |a_{n-1}|;$$

a contradiction. Thus $a_n = 0$, $n = 0, 1, \ldots$, and so $f = 0$.

When $\boldsymbol{\beta} = \boldsymbol{\eta}$ a result analogous to Theorem 2.1 is the following.

THEOREM 2.2. *If* $f(x) = \sum_{k=0}^{\infty} A_k T_k(x)$, $\sum_{k=0}^{\infty} |A_k| < \infty$ *and* $I_{n,\eta} f = 0$, $n = 1, 2, \ldots,$ *then* $f = 0$.

Theorem 2.2 was proved by Éterman [6]. A proof similar to that of Theorem 2.1 also establishes Theorem 2.2 by using (1.18) instead of (1.20) and choosing N_ν to be the product of the first ν *odd* primes.

We turn next to the case $\boldsymbol{\beta} = \boldsymbol{\xi}$ for which the corresponding result is somewhat more elaborate. We require the following result.

LEMMA 2.3. *Suppose that* $m = 1, 2, \ldots$ *and* $s = 2k - 1$, $k = 1, 2, \ldots$. *Then*

$$\sigma(s, m) := \sum_{d \mid (s, m)} \mu(d) (-1)^{\frac{1}{2}(d + (s/d))} = \begin{cases} (-1)^{\frac{1}{2}(s+1)}, & (s, m) = 1, \\ 0, & (s, m) > 1. \end{cases}$$

Proof. If $(s, m) = 1$ the result is obvious. Suppose that $(s, m) > 1$ and $q = p_1 \cdot \ldots \cdot p_\nu$ is the product of the distinct primes dividing (s, m). Then

$$\sigma = \sum_{d \mid q} \mu(d) (-1)^{\frac{1}{2}(d + (s/d))},$$

since $\mu(d) = 0$ for all the other divisors of (s, m). Fix a prime divisor p of q, and put $r = s/p$. Then if $d \mid q/p$ we obtain $\mu(dp) = -\mu(d)$ and

$$(-1)^{\frac{1}{2}(r/d)(p-1)} = (-1)^{\frac{1}{2}(p-1)} = (-1)^{\frac{1}{2}d(p-1)},$$

since the numbers p, d and r/d are odd. Hence

$$\begin{aligned} \sigma &= \sum_{d \mid (q/p)} \{\mu(d) (-1)^{\frac{1}{2}(d + (rp/d))} + \mu(dp) (-1)^{\frac{1}{2}(dp + (rp/dp))}\} \\ &= \sum_{d \mid (q/p)} \{\mu(d) (-1)^{\frac{1}{2}(d + (r/d))} (-1)^{\frac{1}{2}(r/d)(p-1)} - \mu(d) (-1)^{\frac{1}{2}(d + (r/d))} (-1)^{\frac{1}{2}d(p-1)}\} = 0, \end{aligned}$$

since every summand is zero.

Now we are ready to prove the following.

THEOREM 2.4. *If* $f(x) = \sum_{k=0}^{\infty} B_k U_k(x)$, $\sum_{k=0}^{\infty} |B_k| < \infty$, *and* $I_{n,\xi} f = 0$, $n = 1, 2, \ldots,$ *then* $f = 0$.

Proof. Equation (1.25) yields

$$I_{n,\xi} f = 2^{n-1} \sum_{j=1}^{\infty} (-1)^{j-1} B_{(2j-1)n-1} = 0, \qquad n \geqslant 1. \tag{2.4}$$

Let $p_1, p_2, \ldots$ be the odd primes in increasing order and put $N_\nu = p_1 \cdot \ldots \cdot p_\nu$. Then

$$\begin{aligned}\sum_{d\mid N_\nu} (-1)^{\frac{1}{2}(d-1)} \mu(d)\, 2^{1-nd} I_{nd,\xi} f &= \sum_{d\mid N_\nu} (-1)^{\frac{1}{2}(d-1)} \mu(d) \sum_{\substack{m=1\\ m \text{ odd}}}^{\infty} (-1)^{\frac{1}{2}(m-1)} B_{mdn-1}\\ &= \sum_{\substack{s=1\\ s \text{ odd}}}^{\infty} B_{sn-1} \sum_{\substack{d\mid s\\ d\mid N_\nu}} \mu(d)(-1)^{\frac{1}{2}(d-1)}(-1)^{\frac{1}{2}((s/d)-1)}\\ &= \sum_{\substack{s=1\\ s \text{ odd}\\ (s,N_\nu)=1}}^{\infty} (-1)^{\frac{1}{2}(s-1)} B_{sn-1},\end{aligned} \tag{2.5}$$

where Lemma 2.3 was used in the last equality. From (2.4) and (2.5) we obtain

$$B_{n-1} = -\sum_{\substack{k=1\\ (2k+1,N_\nu)=1}}^{\infty} (-1)^k B_{(2k+1)n-1},$$

which, in view of the convergence of $\sum_{k=0}^{\infty} |B_k|$, implies that $B_n = 0$, $n = 0, 1, 2, \ldots$, as in the proof of Theorem 2.1.

3. *Analyticity and the asymptotic behaviour of divided differences*

If $\beta = \omega$ and $f(z)$ has an absolutely convergent power series on D_1, we show first that the sequence $I_{n,\omega} f$, $n = 1, 2, \ldots$, defines the radius of convergence of f in precisely the same way as the sequence of Taylor coefficients of f does. This result is then used to provide information about the zeros of the sequence of interpolating polynomials to f on ω.

THEOREM 3.1. *If $f(z) = \sum_{k=0}^{\infty} a_k z^k$ with $\sum_{k=0}^{\infty} |a_k| < \infty$, then*

$$\limsup_{k\to\infty} |a_k|^{1/k} = \limsup_{n\to\infty} |I_{n,\omega} f|^{1/n}.$$

Proof. (i) Suppose that

$$\limsup_{k\to\infty} |a_k|^{1/k} = \frac{1}{\rho}, \qquad \rho > 1.$$

Then given $\varepsilon > 0$ we have $a_n = O((\rho-\varepsilon)^{-n})$ as $n \to \infty$, and (1.20) implies that

$$|I_{n,\omega} f - a_{n-1}| = O((\rho-\varepsilon)^{-2n}) \quad \text{as } n \to \infty.$$

Therefore

$$\limsup_{n\to\infty} |I_{n,\omega} f|^{1/n} = \frac{1}{\rho}.$$

(ii) Suppose that

$$\limsup_{k\to\infty} |a_k|^{1/k} = 1,$$

but

$$\limsup_{n\to\infty} |I_{n,\omega} f|^{1/n} = \frac{1}{\rho}, \qquad \rho > 1$$

(ρ cannot be less than 1 since $\{I_{n,\omega} f\}_1^{\infty}$ is a bounded sequence). Choose q such that $1/\rho < q < 1$. Then $|I_{n,\omega} f| < q^n$ for $n > n_0$. Next choose $j > n_0$ so that $q^j < 1 - \sqrt{q}$ and $|a_j| > q^j$ and choose m such that

$$\sum_{k=m}^{\infty} |a_k| < (1-\sqrt{q})\, q^j.$$

Then from (2.2) with $v = m$ and $n-1 = j$ we obtain

$$a_j = \sum_{d|N_m} \mu(d)\, I_{(j+1)d,\omega}f - \sum_{\substack{k>p_m \\ (k,N_m)=1}} a_{k(j+1)-1},$$

and so

$$|a_j| \leqslant \sum_{d=1}^{\infty} q^{(j+1)d} + \sum_{k=m}^{\infty} |a_k| \leqslant \frac{q^{j+1}}{1-q^{j+1}} + (1-\sqrt{q})\,q^j < q^j;$$

a contradiction which establishes the theorem.

Theorem 3.1 has the obvious implication that the radius of convergence of the power-series expansion of f is given by

$$1/\limsup_{n\to\infty} |I_{n,\omega}f|^{1/n}$$

(f being entire if the denominator is zero). Moreover, because of the definition of $I_{n,\omega}f$ as the leading coefficient of $\mathscr{L}_{n-1}(f,z)$, the interpolating polynomial of degree $n-1$ to f at the nth roots of unity, Theorem 3.1 can also be used in examining the behaviour of the zeros of $\mathscr{L}_{n-1}(f,z)$. For example, we have the following.

THEOREM 3.2. *If* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $\sum_{k=0}^{\infty} |a_k| < \infty$, *and*

$$\limsup_{k\to\infty} |a_k|^{1/k} = 1,$$

then there is a subsequence $\Lambda \subset \mathbb{N}$ *such that the zeros of* $\mathscr{L}_{n-1}(f)$, $n \in \Lambda$, *converge weak-star to the uniform distribution on the unit circle.*

Proof. Theorem 3.1 implies the existence of $\Lambda \subset \mathbb{N}$ such that

$$|I_{n,\omega}f|^{1/n} \to 1 \quad \text{as } n \to \infty,\ n \in \Lambda.$$

Then for the sequence of monic polynomials

$$p_{n-1}(z) := \frac{1}{I_{n,\omega}f}\mathscr{L}_{n-1}(f,z),$$

$n \in \Lambda$ (and n large enough so that $I_{n,\omega}f \neq 0$), we have

$$\limsup_{n\to\infty} \|p_{n-1}\|_{D_1}^{1/n} \leqslant 1,$$

in view of $\|\mathscr{L}_{n-1}(f)\|_{D_1} \leqslant n\|f\|_{D_1}$. The theorem now follows from [**2**, Theorem 2.1] of Blatt, Saff and Simkani.

We remark that the preceding two theorems improve results of Simkani [**12**]. The analogues of Theorem 3.1 for Chebyshev nodes are the following.

THEOREM 3.3. *If* $f(x) = \sum_{k=0}^{\infty} A_k T_k(x)$, $\sum_{k=0}^{\infty} |A_k| < \infty$, *then*

$$\limsup_{k\to\infty} |A_k|^{1/k} = \limsup_{n\to\infty} |I_{n,\eta}f|^{1/n}.$$

THEOREM 3.4. *If* $f(x) = \sum_{k=0}^{\infty} B_k U_k(x)$, $\sum_{k=0}^{\infty} |B_k| < \infty$, *then*

$$\limsup_{k\to\infty} |B_k|^{1/k} = \limsup_{n\to\infty} |I_{n,\xi}f|^{1/n}.$$

The proofs of Theorems 3.3 and 3.4 follow the proof of Theorem 3.1.

If we know that

$$\limsup_{n\to\infty} |I_{n,\xi} f|^{1/n} = \frac{1}{\rho} \quad \text{or} \quad \limsup_{n\to\infty} |I_{n,\eta} f|^{1/n} = \frac{1}{\rho}, \qquad \rho > 1,$$

then f is analytic in an ellipse with foci at ± 1 and semiminor axis equal to $\frac{1}{2}(\rho-(1/\rho))$. When f is not analytic on I we have

$$\limsup_{n\to\infty} |A_n|^{1/n} = \limsup_{n\to\infty} |B_n|^{1/n} = 1$$

and, just as before, an application of [**2**, Theorem 2.1] yields the following.

THEOREM 3.5. *If* $f(x) = \sum_{k=0}^{\infty} A_k T_k(x)$, $\sum_{k=0}^{\infty} |A_k| < \infty$ *and*

$$\limsup_{k\to\infty} |A_k|^{1/k} = 1,$$

then there is a subsequence $\Lambda \subset \mathbb{N}$ *such that the zeros of the polynomials interpolating* f *at the extrema of* T_n, $n \in \Lambda$, *converge weak-star to the arcsine distribution on* I.

THEOREM 3.6. *If* $f(x) = \sum_{k=0}^{\infty} B_k U_k(x)$, $\sum_{k=0}^{\infty} |B_k| < \infty$ *and*

$$\limsup_{k\to\infty} |B_k|^{1/k} = 1,$$

then there is a subsequence $\Lambda \subset \mathbb{N}$ *such that the zeros of the polynomials interpolating* f *at the zeros of* $T_n(x)$, $n \in \Lambda$, *converge weak-star to the arcsine distribution on* I.

4. *Basic polynomials for a sequence of divided differences*

The basic polynomials $P_{k,\beta} \in \mathscr{P}_k$ (where the second subscript is the underlying infinite triangular array) were described in the introduction. In particular we recall (0.2):

$$I_{j+1} P_k = \delta_{jk}, \qquad j,k = 0,1,2,\dots. \tag{4.1}$$

As we now show, for $\beta = \omega$, η and ξ, the basic polynomials can be obtained explicitly from the formulae (1.20), (1.18), and (1.25), respectively; or their finite inverses, namely (2.2),

$$\sum_{\substack{d|N \\ d \text{ odd}}} \mu(d)\, 2^{1-nd} I_{nd+1,\eta} f = \sum_{\substack{k=1 \\ (2k-1,N)=1}}^{\infty} A_{(2k-1)n}, \tag{4.2}$$

and (2.5), respectively.

We turn first to the case of the roots of unity, $\beta = \omega$. Let

$$P_m(z) = a_0 + \dots + a_{m-1} z^{m-1} + z^m.$$

Then (2.2) with $\nu = m$ and $f = P_m$ reads

$$a_{n-1} = \sum_{d|N_m} \mu(d)\, I_{nd,\omega} P_m.$$

If we require that equation (4.1) holds for $\beta = \omega$, then a_n can be non-zero only if $(n+1)d = m+1$ in which case $a_n = \mu(d)$. Thus (cf. [**4**]) we have obtained the following.

THEOREM 4.1. *If*

$$P_{m,\omega}(z) = \sum_{d|(m+1)} \mu(d)\, z^{((m+1)/d)-1},$$

then

$$I_{n+1,\omega} P_{m,\omega} = \delta_{n,m}, \qquad n,m = 0,1,2,\dots.$$

Secondly, let us consider the case of the extrema of the Chebyshev polynomials, $\beta = \eta$. Let $P_{m,\eta}(x) = \sum_{j=0}^{m} A_j T_j(x)$ satisfy (4.1). Note that $P_{0,\eta} = 1$ trivially. Suppose that $m \geqslant 1$. If $1 \leqslant j \leqslant m$ then according to (4.2), with $n = j, f = P_{m,\eta}$, and N taken to be the product of the first m odd primes,

$$A_j = \sum_{d|N} \mu(d)\, 2^{1-jd} I_{jd+1,\eta} P_{m,\eta},$$

and $A_j \neq 0$ only if $jd = m$ for some odd d, in which case $A_j = \mu(d)\, 2^{1-m}$. Hence

$$P_{m,\eta}(x) = A_0 + 2^{1-m} \sum_{\substack{d|m \\ d \text{ odd}}} \mu(d)\, T_{m/d}(x).$$

To determine A_0 we observe that, for $m \geqslant 1$, $0 = I_1 P_{m,\eta} = P_{m,\eta}(0)$, and obtain the following.

THEOREM 4.2. *$P_{0,\eta}(x) \equiv 1$, and for $m = 1, 2, \ldots,$*

$$P_{m,\eta}(x) = 2^{1-m} \sum_{\substack{d|m \\ d \text{ odd}}} \mu(d)\,(T_{m/d}(x) - T_{m/d}(0))$$

are the basic polynomials with respect to η.

Finally, we turn to the case of the zeros of the Chebyshev polynomial, $\beta = \xi$. Let $P_{m,\xi} = \sum_{k=0}^{m} B_k U_k(x)$ satisfy (4.1). Note that $P_{0,\xi} = 1$. Suppose that $m \geqslant 1$. If $2 \leqslant k \leqslant m+1$, then according to (2.5), with $\nu = m$, $n = k$, and $f = P_{m,\xi}$,

$$B_{k-1} = \sum_{d|N_m} (-1)^{\frac{1}{2}(d-1)} \mu(d)\, 2^{1-kd} I_{kd,\xi} P_{m,\xi},$$

and $B_{k-1} \neq 0$ only if $kd = m+1$ for some odd integer d, in which case $B_{k-1} = (-1)^{\frac{1}{2}(d-1)} \mu(d)\, 2^{-m}$. Hence

$$P_{m,\xi}(x) = B_0 + 2^{-m} \sum_{\substack{d|m+1 \\ d<m+1 \\ d \text{ odd}}} (-1)^{\frac{1}{2}(d-1)} \mu(d)\, U_{((m+1)/d)-1}(x).$$

But for $m \geqslant 1$, $0 = I_{1,\xi} P_{m,\xi} = P_{m,\xi}(0)$ determines B_0 and we obtain the following.

THEOREM 4.3. *$P_{0,\xi}(x) = 1$ and, for $m = 1, 2, \ldots,$*

$$P_{m,\xi}(x) = 2^{-m} \sum_{\substack{d|m+1 \\ d \text{ odd}}} (-1)^{\frac{1}{2}(d-1)} \mu(d)\,(U_{((m+1)/d)-1}(x) - U_{((m+1)/d)-1}(0))$$

are the basic polynomials with respect to ξ.

Theorems 4.1, 4.2 and 4.3 easily give the following estimates for the basic polynomials involved. If we recall that $d(k)$ denotes the number of positive divisors of k, $\|T_k\|_I = 1$ and $\|(1-x^2)^{\frac{1}{2}} U_k(x)\|_I = 1$, we obtain

$$\|P_{m-1,\omega}\|_{D_1} \leqslant \sum_{d|m} |\mu(d)| \leqslant d(m), \tag{4.3}$$

$$2^{m-1} \|P_{m,\eta}\|_I \leqslant 2d(m), \tag{4.4}$$

and

$$2^{m-1} \|(1-x^2)^{\frac{1}{2}} P_{m-1,\xi}(x)\|_I \leqslant 2d(m). \tag{4.5}$$

We are now in a position to present some sufficient conditions for the absolute convergence of biorthogonal expansions when $\beta = \omega$, η and ξ.

THEOREM 4.4. *Let $f(z)=\sum_{k=0}^{\infty} a_k z^k$, $\sum_{k=0}^{\infty}|a_k|<\infty$ and suppose that*

$$\sum_{m=1}^{\infty} |I_{m,\omega}f|\, d(m) < \infty. \tag{4.6}$$

Then

$$f(z) = \sum_{m=1}^{\infty} (I_{m,\omega}f)\, P_{m-1,\omega}(z), \qquad z\in D_1,$$

the convergence being uniform and absolute in D_1.

Proof. In view of (4.3), inequality (4.6) implies that

$$g(z) := \sum_{m=1}^{\infty} (I_{m,\omega}f)\, P_{m-1,\omega}(z)$$

is analytic in $|z|<1$ and continuous in D_1. Write

$$g(z) = \sum_{k=0}^{\infty} c_k z^k.$$

In order to prove the theorem it suffices to show that $c_{k-1}=a_{k-1}$, $k\in\mathbb{N}$. Fix k and $\varepsilon>0$. Let ν be chosen so that $\sum_{m=k\nu}^{\infty}|a_m|<\varepsilon$ and $\sum_{m=k\nu}^{\infty}|I_{m,\omega}f|\, d(m)<\varepsilon$. Put $N=\prod_{p\leqslant\nu} p$, where p is prime, and

$$\begin{aligned} g_\nu(z) &:= \sum_{m=1}^{k\nu} (I_{m,\omega}f)\, P_{m-1,\omega}(z) + \sum_{\substack{m=k\nu+1\\ m\mid kN}}^{kN} (I_{m,\omega}f)\, P_{m-1,\omega}(z)\\ &= \sum_{j=0}^{kN-1} \tilde{c}_j z^j. \end{aligned}$$

Then from (4.3) we get

$$\|g-g_\nu\|_{D_1} \leqslant \left\| g - \sum_{m=1}^{k\nu} (I_{m,\omega}f)\, P_{m-1,\omega} \right\|_{D_1} + \left\| \sum_{\substack{m=k\nu+1\\ m\mid kN}} (I_{m,\omega}f)\, P_{m-1,\omega} \right\|_{D_1} < 2\varepsilon$$

and so

$$|c_{k-1}-\tilde{c}_{k-1}| \leqslant \frac{1}{2\pi}\int_{|z|=1} |(g(z)-g_\nu(z))\, z^{-k-1}|\, |dz| < 2\varepsilon. \tag{4.7}$$

Since, according to Theorem 4.1,

$$P_{m-1,\omega}(z) = \sum_{d\mid m} \mu(d)\, z^{(m/d)-1},$$

we obtain

$$\tilde{c}_{k-1} = \sum_{\substack{d\leqslant\nu\\ d\mid N}} \mu(d)\, I_{kd,\omega}f + \sum_{\substack{d>\nu\\ d\mid N}} \mu(d)\, I_{kd,\omega}f = \sum_{d\mid N} \mu(d)\, I_{kd,\omega}f = \sum_{\substack{j=1\\ (j,N)=1}}^{\infty} a_{jk-1},$$

in view of (2.2). Hence

$$|\tilde{c}_{k-1}-a_{k-1}| = \left| \sum_{\substack{j=\nu+1\\ (j,N)=1}}^{\infty} a_{jk-1} \right| < \varepsilon. \tag{4.8}$$

Now (4.7) and (4.8) imply that $|a_{k-1}-c_{k-1}|<3\varepsilon$ and hence, since $\varepsilon>0$ is arbitrary, $a_{k-1}=c_{k-1}$, $k=1,2,\ldots$.

COROLLARY 4.5. *If* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, *and* $a_k = O(k^{-1-\delta})$, $\delta > 0$, *as* $k \to \infty$, *then*

$$f(z) = \sum_{m=1}^{\infty} (I_{m,\omega}f)\, P_{m-1,\omega}(z), \qquad z \in D_1,$$

the convergence being absolute in D_1.

Proof. Equation (1.20) yields $I_{m,\omega}f = O(m^{-1-\delta})$, $m \to \infty$ and therefore (4.6) holds in view of the bound on $d(m)$ mentioned in the introduction.

THEOREM 4.6. *Let* $f(x) = \sum_{k=0}^{\infty} A_k T_k(x)$, $\sum_{k=0}^{\infty} |A_k| < \infty$, *and*

$$\sum_{m=0}^{\infty} 2^{1-m} |I_{m+1,\eta}f| d(m) < \infty. \tag{4.9}$$

Then

$$f(x) = \sum_{m=0}^{\infty} (I_{m+1,\eta}f)\, P_{m,\eta}(x), \qquad x \in I,$$

the convergence being uniform and absolute in I.

Proof. Set

$$g(x) := \sum_{m=0}^{\infty} (I_{m+1,\eta}f)\, P_m(x).$$

Because of (4.9) and (4.4) the series for g is absolutely and uniformly convergent in I, and $g \in C(I)$. Let

$$C_k := \frac{2}{\pi} \int_{-1}^{1} g(x)\, T_k(x) \frac{dx}{(1-x^2)^{\frac{1}{2}}}, \qquad k = 1, 2, \ldots.$$

We claim that $C_k = A_k$. Fix k and $\varepsilon > 0$. Choose ν so that $\sum_{m=k\nu}^{\infty} |A_m| < \varepsilon$ and $\sum_{m=k\nu}^{\infty} 2^{1-m} |I_{m+1,\eta}f| d(m) < \varepsilon$. Put $N := \prod_{3 \leqslant p \leqslant \nu} p$, where p is prime, and

$$g_\nu(x) := \sum_{m=0}^{k\nu} (I_{m+1,\eta}f)\, P_m(x) + \sum_{\substack{m=k\nu+1 \\ m \mid kN}} (I_{m+1,\eta}f)\, P_m(x) = \sum_{j=0}^{kN} \tilde{C}_j T_j(x).$$

Then $\|g - g_\nu\|_I < 2\varepsilon$, which implies that

$$|C_k - \tilde{C}_k| \leqslant \frac{2}{\pi} \int_{-1}^{1} |g(x) - g_\nu(x)| \frac{dx}{(1-x^2)^{\frac{1}{2}}} < 4\varepsilon. \tag{4.10}$$

But, according to Theorem 4.2, for $m \geqslant 1$

$$P_{m,\eta}(x) = 2^{1-m} \sum_{\substack{d \mid m \\ d \text{ odd}}} \mu(d) (T_{m/d}(x) - T_{m/d}(0)),$$

and so

$$\tilde{C}_k = \sum_{d \mid N} \mu(d)\, 2^{1-dk} (I_{dk+1,\eta}f) = \sum_{\substack{j=1 \\ (2j-1,N)=1}}^{\infty} A_{(2j-1)k},$$

because of (4.2). Hence

$$|\tilde{C}_k - A_k| = \left| \sum_{\substack{j=1 \\ (2j-1,N)=1 \\ 2j-1>\nu}}^{\infty} A_{(2j-1)k} \right| \leqslant \sum_{m=k\nu}^{\infty} |A_m| < \varepsilon, \tag{4.11}$$

since the only odd number less than v and relatively prime to N is 1. Note that (4.10) and (4.11) imply that $C_k = A_k$, $k \in \mathbb{N}$. When $m = 0$, $g(0) = I_{1,\eta} f = f(0)$ and so $A_0 - C_0 = f(0) - g(0) = 0$ and $f = g$.

COROLLARY 4.7. *If $f(x) = \sum_{k=0}^{\infty} A_k T_k(x)$, $A_k = O(k^{-1-\varepsilon})$, $\varepsilon > 0$, as $k \to \infty$, then*

$$f(x) = \sum_{k=0}^{\infty} (I_{m+1,\eta} f) P_m(x), \qquad x \in I,$$

the convergence being absolute and uniform in I.

Proof. Equation (1.18) yields $2^{1-m} I_{m+1,\eta} f = O(\sum_{j=1}^{\infty} (2j-1)^{-1-\varepsilon} m^{-1-\varepsilon}) = O(m^{-1-\varepsilon})$ as $m \to \infty$, which together with $d(m) = O(m^{\varepsilon/2})$ as $m \to \infty$ shows that (4.9) is satisfied. The corollary now follows from Theorem 4.6.

THEOREM 4.8. *Let $f(x) = \sum_{k=0}^{\infty} B_k U_k(x)$, $\sum_{k=0}^{\infty} |B_k| < \infty$, and*

$$\sum_{m=1}^{\infty} 2^{1-m} |I_{m,\xi} f| d(m) < \infty. \tag{4.12}$$

Then

$$f(x) = \sum_{m=1}^{\infty} (I_{m,\xi} f) P_{m-1,\xi}(x), \qquad x \in (-1, 1),$$

the convergence being absolute and uniform on any compact subset of $(-1, 1)$.

Proof. Set

$$g(x) := \sum_{m=1}^{\infty} (I_{m,\xi} f) P_{m-1,\xi}(x),$$

$$G(x) := g(x)(1-x^2)^{\frac{1}{2}} = \sum_{m=1}^{\infty} (I_{m,\xi} f)(P_{m-1,\xi}(x)(1-x^2)^{\frac{1}{2}})$$

and

$$F(x) := f(x)(1-x^2)^{\frac{1}{2}} = \sum_{k=0}^{\infty} B_k (U_k(x)(1-x^2)^{\frac{1}{2}}).$$

The series for F and G converge absolutely in I, the latter being the case because of (4.5) and (4.12). Next we note that $G(0) = I_{1,\xi} f = f(0) = F(0)$ and by an argument similar to that given in the proof of Theorem 4.6 we obtain

$$\int_{-1}^{1} (F(x) - G(x)) U_k(x) \, dx = 0, \qquad k \in \mathbb{N}.$$

Thus $F = G$ and $f = g$.

COROLLARY 4.9. *If $f(x) = \sum_{k=0}^{\infty} B_k U_k(x)$, $B_k = O(k^{-1-\varepsilon})$, $\varepsilon > 0$, as $k \to \infty$, then*

$$f(x) = \sum_{m=1}^{\infty} (I_{m,\xi} f) P_{m-1,\xi}(x).$$

Both representations of f converge absolutely and uniformly on any compact subset of $(-1, 1)$.

5. *Counterexamples*

The main results in Sections 2 and 3 hold for functions represented as *absolutely* convergent series. This condition cannot be replaced by the assumption that f is continuous, nor even by the *uniform* convergence of those series, as we shall show.

The first example of a non-zero function f, analytic in $|z| < 1$, with *uniformly* convergent Taylor series in D_1, that satisfies

$$\sum_{k=0}^{n-1} f(e(k/n)) = 0, \qquad n = 1, 2, \ldots, \tag{5.1}$$

seems to be due to Ching (see [1]). The function is

$$F_1(z) := \sum_{m=1}^{\infty} \frac{\mu(m)}{m} z^m.$$

Observe that $I_{k,\omega}(F_1(z)/z) = 0$, $k = 1, 2, \ldots$, because of (1.10) and the fact that $F_1(1) = 0$. Now Corollary 1.7 implies that $I_{k,\eta} f_1 = 0$, $k = 1, 2, \ldots$, where

$$f_1(x) := \sum_{m=1}^{\infty} \frac{\mu(m)}{m} T_m(x).$$

A rediscovery of this consequence of the Ching example is due to Newman and Rivlin [10].

The uniform convergence of the series for F_1 on C_1 (and hence of the series for f_1 on I) follows by Abel summation from the following remarkable estimate of Davenport [5, Theorem 1]:

$$\sum_{n=1}^{m} \mu(n) e(n\theta) = O(m (\log m)^{-\sigma}), \qquad m \to \infty, \tag{5.2}$$

uniformly for $\theta \in [0, 1]$, where σ is any fixed positive number. As for the proof of (5.1) for $f = F_1$ we have

$$\sum_{k=0}^{n-1} F_1\left(e\left(\frac{k}{n}\right)\right) = \sum_{m=1}^{\infty} \frac{\mu(m)}{m} \sum_{k=0}^{n-1} e\left(\frac{km}{m}\right) = n \sum_{j=1}^{\infty} \frac{\mu(nj)}{nj}$$
$$= \sum_{\substack{j=1 \\ (n,j)=1}}^{\infty} \frac{\mu(nj)}{j} = \mu(n) \sum_{\substack{j=1 \\ (n,j)=1}}^{\infty} \frac{\mu(j)}{j} = 0,$$

since (see [5, Lemma 12])

$$\sum_{\substack{j=1 \\ (n,j)=1}}^{\infty} \frac{\mu(j)}{j} = 0, \qquad n \in \mathbb{N} \tag{5.3}$$

(a generalization of the prime-number theorem).

Next we note that if we put $f_2(x) := f_1(-x)$ and recall that $T_m(-x) = (-1)^m T_m(x)$ and $\eta_k^{(j)} = -\eta_{j+2-k}^{(j)}$, $k = 1, \ldots, j+1$; $j \geqslant 1$ ($\eta_1^{(0)} = 0$), then $I_{k,\eta} f_2 = 0$, $k = 1, 2, \ldots$, where

$$f_2(x) = \sum_{m=1}^{\infty} (-1)^m \frac{\mu(m)}{m} T_m(x)$$

is uniformly convergent in I. Additionally the divided differences of $\frac{1}{2}(f_1 + f_2)$ and $\frac{1}{2}(f_1 - f_2)$ at the Chebyshev extrema are all zero.

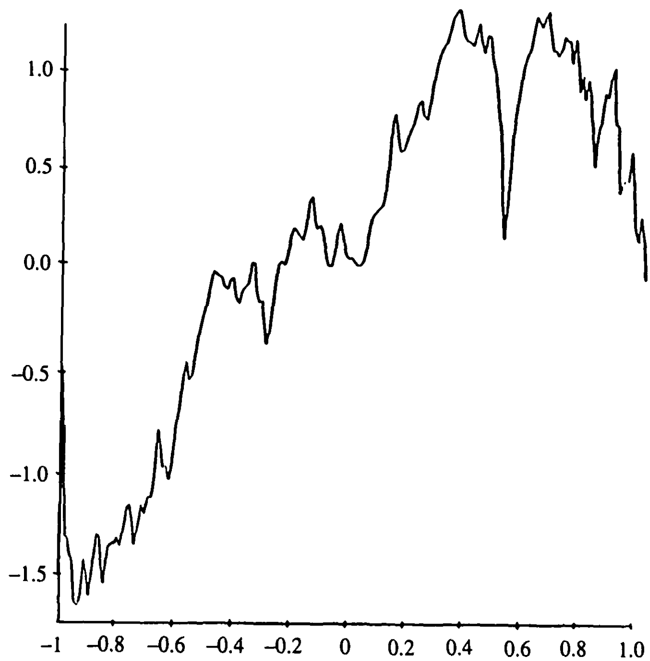

FIG. 1. Graph of the partial sum $\sum_{m=1}^{200} \frac{\mu(m)}{m} T_m(x)$ of $f_1(x)$

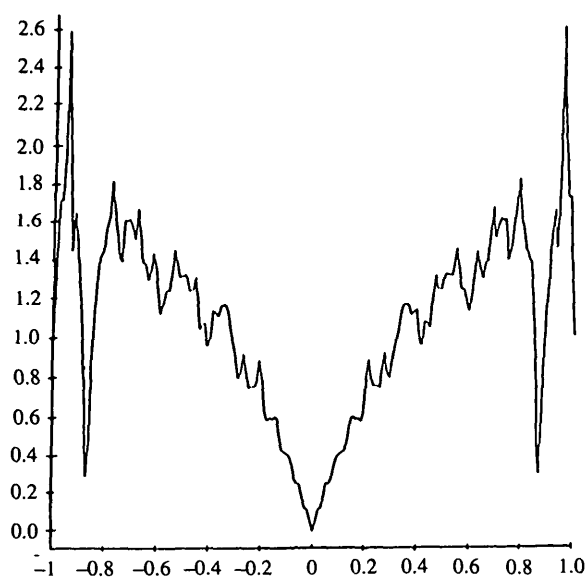

FIG. 2. Graph of the partial sum $\sum_{j=1}^{200} (-1)^{j-1} \frac{\mu(2j-1)}{2j-1} U_{2(j-1)}(x)$ of $h(x)$

Finally, we turn to the array of zeros of the Chebyshev polynomials. Put

$$F_2(z) := \sum_{\substack{m=1\\ m \text{ odd}}}^{\infty} (-1)^{\frac{1}{2}(m-1)} \frac{\mu(m)}{m} z^m.$$

From (5.2) with $\theta = \phi + \frac{1}{4}$ and $\theta = \phi + \frac{3}{4}$ we get

$$\sum_{\substack{n=1\\ n \text{ odd}}}^{m} (-1)^{\frac{1}{2}(n-1)} \mu(n)\, e(n\phi) = \frac{1}{2i}\left(\sum_{n=1}^{m} \mu(n)\, e(n(\phi+\tfrac{1}{4})) - \sum_{n=1}^{m} \mu(n)\, e(n(\phi+\tfrac{3}{4}))\right)$$

$$= O(m(\log m)^{-\sigma}), \qquad m \to \infty,$$

and hence the series for F_2 is uniformly convergent on C_1.

Therefore,

$$\sum_{k=1}^{2n} (-1)^k F_2\left(e\left(\frac{2k-1}{4n}\right)\right) = \sum_{\substack{m=1\\ m \text{ odd}}}^{\infty} (-1)^{\frac{1}{2}(m-1)} \frac{\mu(m)}{m} \sum_{k=1}^{2n} (-1)^k e\left(\frac{(2k-1)\,m}{4n}\right)$$

$$= \sum_{\substack{m=1\\ m \text{ odd}}}^{\infty} (-1)^{\frac{1}{2}(m-1)} \frac{\mu(m)}{m} e\left(-\frac{m}{4n}\right) \sum_{k=1}^{2n} e\left(\frac{k(m+n)}{2n}\right)$$

$$= \sum_{s=1}^{\infty} (-1)^{s-1} \frac{\mu(n(2s-1))}{n(2s-1)} i(-1)^s\, 2n$$

$$= -2i \sum_{s=1}^{\infty} \frac{\mu(n(2s-1))}{2s-1} = -2i\mu(n) \sum_{\substack{j=1\\ (j,2n)=1}}^{\infty} \frac{\mu(j)}{j} = 0,$$

in view of (5.3). Now Proposition 1.8 implies that

$$h(x) := \frac{1}{(1-x^2)^{\frac{1}{2}}} \operatorname{Im} F_2(e^{i \arccos x}) = \sum_{\substack{m=1\\ m \text{ odd}}}^{\infty} (-1)^{\frac{1}{2}(m-1)} \frac{\mu(m)}{m} U_{m-1}(x) \tag{5.4}$$

satisfies $I_{n,\xi} h = 0$, $n \in \mathbb{N}$.

The function h is continuous in $(-1, 1)$. We do not know whether $h \in C(I)$ or even whether h is bounded. But, of course, the series for h converges uniformly on every compact subset of $(-1, 1)$. The accompanying Figures 1 and 2 are computer generated graphs of the first 200 terms in series representations of $f_1(x)$ and $h(x)$.

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