Szegõ Asymptotics for Non-Szegõ Weights on [-1,1]

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Abstract. We discuss strong asymptotics of L_p extremal polynomials for weights on $[-1,1]$ not satisfying Szegö's condition. Typical weights treated are $\exp(-(1-x^2)^{-\alpha})$ or $\exp(-exp_k(1-x^2)^{-\alpha})$, where $\alpha > 0$, and $\exp_k$ denotes the k^{th} iterated exponential.

§1. Outline of Results

Let w and $w^2 \in L_1[-1,1]$, with supp(w) having positive measure. Let

$$p_n(w^2,x) = \gamma_n(w^2)x^n + \cdots, \gamma_n(w^2) > 0, \tag{1}$$

$n = 0,1,2,\ldots$, denote the orthonormal polynomials for w^2, satisfying

$$\int_{-1}^{1} p_n(w^2,x)p_m(w^2,x)w^2(x)dx = \delta_{mn}, \quad m,n = 0,1,2,.. \tag{2}$$

Define

$$\mu(\theta) := w^2(\cos\theta)|\sin\theta|, \theta \in [-\pi,\pi]. \tag{3}$$

If w satisfies *Szegõ's condition*

$$\int_{-1}^{1} \log w(x)dx/\sqrt{1-x^2} > -\infty, \tag{4}$$

define the *Szegõ function*

$$D(\mu;z) := \exp\left(\frac{1}{4\pi}\int_{-\pi}^{\pi} \log\mu(t)\frac{1+ze^{-it}}{1-ze^{-it}}dt\right), |z| < 1. \tag{5}$$

If $\phi(z) := z + (z^2-1)^{1/2}$ denotes the usual conformal map of $\mathbb{C}\backslash[-1,1]$ onto $\{z : |z| > 1\}$, Szegõ [9] proved under (4) the strong (or power, or Szegõ) asymptotic

$$\lim_{n\to\infty} \gamma_n(w^2)2^{-n} = (2\pi)^{-1/2}D^{-1}(\mu;0), \tag{6}$$

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and uniformly in closed subsets of $\mathbb{C}\backslash[-1,1]$,

$$\lim_{n\to\infty} p_n(w^2,z)/\phi(z)^n = (2\pi)^{-1/2}D^{-1}\left(\mu;\frac{1}{\phi(z)}\right). \tag{7}$$

Strong asymptotics have been established for only a few weights violating (4). One example [8] is the Pollaczek-like weight $\exp(-(1-x^2)^{-1/2})$. Comparative asymptotics, introduced by Nevai in [8] and subsequently investigated by many authors, extend the applicability of a given strong asymptotic. Some related general conjectures appear in [3]. Recent results on spacing of zeros for non-Szegő weights appear in [1].

In this paper we briefly discuss some new results for non-Szegő weights, for orthogonal polynomials and their L_p cousins. Define for $1\le p\le\infty$, the L_p extremal error,

$$E_{np}(w) := \inf_{\deg(P)<n} \|\{x^n - P(x)\}w(x)\|_{L_p[-1,1]}, \tag{8}$$

and let $T_{np}(w,x)$ denote any monic polynomial of degree n satisfying

$$\|T_{np}(w,x)w(x)\|_{L_p[-1,1]} = E_{np}(w). \tag{9}$$

The *normalized L_p extremal polynomial* is

$$p_{np}(w,x) := T_{np}(w,x)/E_{np}(w).$$

Note that $\gamma_n(w^2) = 1/E_{n2}(w)$ and $p_n(w^2,x) = p_{n2}(w,x)$.

Now let $w(x) := e^{-Q(x)}, x\in(-1,1)$, where Q is even, convex in $(-1,1)$ and $Q(1-) = Q'(1-) = Q''(1-) = \infty$. Associated with Q is the Mhaskar–Rahmanov–Saff number a_n, (cf. [7]) defined as the root of

$$n = \frac{2}{\pi}\int_0^1 a_n t Q'(a_n t)dt/\sqrt{1-t^2}, n>0.$$

Further, for $1\le p\le\infty$ and $n\ge 1$, we define for $\theta\in[-\pi,\pi]$,

$$\mu_{n,p}(\theta) := w^2(a_n\cos\theta)|\sin\theta|^{2/p} \tag{12}$$

For a class of weights including $\exp(-(1-x^2)^{-\alpha})$ or $\exp(-\exp_k(1-x^2)^{-\alpha}), \alpha>0$, we have obtained the following results: For $1<p\le\infty$,

$$\lim_{n\to\infty} p_{np}(w,a_n z)/\left\{\phi(z)^n/D\left(\mu_{n,p};\frac{1}{\phi(z)}\right)\right\} = (2\sigma_p)^{-1}$$

uniformly in closed subsets of $\mathbb{C}\backslash[-1,1]$, and for $1\le p\le\infty$,

$$\lim_{n\to\infty} E_{np}(w)(2/a_n)^n/D(\mu_{n,p};0) = 2\sigma_p, \tag{14}$$

where

$$\sigma_p := \begin{cases} [\Gamma(1/2)\Gamma((p+1)/2)/\Gamma(p/2+1)]^{1/p} & ,1 \le p < \infty \\ 1, & p = \infty. \end{cases} \tag{15}$$

Furthermore, if the zeros of $p_{np}(w, x)$ are denoted by

$$-1 < x_{nn}^{(p)} < x_{n-1,n}^{(p)} < \cdots < x_{2n}^{(p)} < x_{1n}^{(p)} < 1,$$

then for each fixed j, and uniformly for $2 \le p \le \infty$,

$$\lim_{n\to\infty} (1 - x_{jn}^{(p)})/(1 - a_n) = 1.$$

Note that (16) is not true for classical Jacobi weights.

To describe the pointwise asymptotics on $[-1, 1]$ for the orthogonal polynomials, we need the function

$$T(x) := 1 + xQ''(x)/Q'(x), \quad x \in (-1, 1),$$

and for some fixed, but large enough s,

$$c_n := a_n \left(1 + s\left[\frac{\log n}{nT(a_n)}\right]^{2/3}\right), n \ge 1$$

For $n \ge 1$, let

$$\nu_n(\theta) := w^2(c_n \cos\theta)|\sin\theta|, \quad \theta \in [-\pi, \pi],$$

and

$$\Gamma_n(\theta) := \frac{1}{4\pi}\int_{-\pi}^{\pi} \{\log \nu_n(t) - \log \nu_n(\theta)\} \cot\left(\frac{\theta - t}{2}\right) dt, \tag{20}$$

$\theta \in [-\pi, \pi]$. Then there exists $\delta > 0$ such that uniformly for $|x| \le 1 - n^{-\delta}, x = \cos\theta$,

$$p_n(w^2, c_n x)\nu_n(\theta)^{1/2} = \left(\frac{2}{\pi}\right)^{1/2} \cos(n\theta + \Gamma_n(\theta)) + o(1). \tag{21}$$

The proofs provide rates in (13), (14), (16), and (21), and involve one-sided approximations by weighted polynomials of the form $P_n(x)w(c_n x)$. The latter are constructed as in [2,4,5]. We can also treat non-even weights, but these require additional restrictions, including analyticity of w^{-1} in $(-1, 1)$.

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