Szegő Asymptotics for Non-Szego Weights on [-1,1]

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Abstract. We discuss strong asymptotics of \( L_p \) extremal polynomials for weights on \([-1, 1]\) not satisfying Szegő’s condition. Typical weights treated are \( \exp(-(1-x^2)^{-\alpha}) \) or \( \exp(-c p_k (1-x^2)^{-\alpha}) \), where \( \alpha > 0 \), and \( \exp_k \) denotes the \( k \)th iterated exponential.

§1. Outline of Results

Let \( w \) and \( w^2 \in L_1[-1,1] \), with \( \text{supp}(w) \) having positive measure. Let

\[
p_n(w^2, x) = \gamma_n(w^2)x^n + \cdots, \gamma_n(w^2) > 0, \quad n = 0, 1, 2, \ldots
\]

denote the orthonormal polynomials for \( w^2 \), satisfying

\[
\int_{-1}^{1} p_n(w^2, x)p_m(w^2, x)w^2(x)dx = \delta_{mn}, \quad m, n = 0, 1, 2, \ldots
\]

Define

\[
\mu(\theta) := w^2(\cos \theta)|\sin \theta|, \theta \in [-\pi, \pi].
\]

If \( w \) satisfies Szegő’s condition

\[
\int_{-1}^{1} \log w(x)dx/\sqrt{1-x^2} > -\infty,
\]

define the Szegő function

\[
D(\mu; z) := \exp \left( \frac{1}{4\pi} \int_{-\pi}^{\pi} \log \mu(t) \frac{1+ze^{-it}}{1-ze^{-it}} dt \right), |z| < 1.
\]

If \( \phi(z) := z + (z^2 - 1)^{1/2} \) denotes the usual conformal map of \( C \setminus [-1,1] \) onto \( \{ z : |z| > 1 \} \), Szegő [9] proved under (4) the strong (or power, or Szegő) asymptotic

\[
\lim_{n \to \infty} \frac{\gamma_n(w^2)}{2^{-n}} = (2\pi)^{-1/2} D^{-1}(\mu; 0),
\]
and uniformly in closed subsets of \( \mathbb{C} \setminus [-1, 1] \),

\[
\lim_{n \to \infty} p_n(w^2, z)/\phi(z)^n = (2\pi)^{-1/2}D^{-1}\left(\mu; \frac{1}{\phi(z)}\right).
\]

(7)

Strong asymptotics have been established for only a few weights violating (4). One example [8] is the Pollaczek–like weight \( \exp(- (1 - x^2)^{-1/2}) \). Comparative asymptotics, introduced by Nevai in [8] and subsequently investigated by many authors, extend the applicability of a given strong asymptotic. Some related general conjectures appear in [3]. Recent results on spacing of zeros for non-Szego weights appear in [1].

In this paper we briefly discuss some new results for non-Szego weights, for orthogonal polynomials and their \( L_p \) cousins. Define for \( 1 \leq p \leq \infty \), the \( L_p \) extremal error,

\[
E_{np}(w) := \inf_{\text{deg}(P) \leq n} \| \{x^n - P(x)\}w(x)\|_{L_p[-1,1]},
\]

(8)

and let \( T_{np}(w, x) \) denote any monic polynomial of degree \( n \) satisfying

\[
\|T_{np}(w, x)w(x)\|_{L_p[-1,1]} = E_{np}(w).
\]

(9)

The normalized \( L_p \) extremal polynomial is

\[
p_{np}(w, x) := T_{np}(w, x)/E_{np}(w).
\]

Note that \( \gamma_n(w^2) = 1/E_{n2}(w) \) and \( p_n(w^2, x) = p_{n2}(w, x) \).

Now let \( w(x) := \exp(-Q(1-x^2)) \), \( x \in (-1, 1) \), where \( Q \) is even, convex in \((-1, 1)\) and \( Q(1-) = Q'(1-) = Q''(1-) = \infty \). Associated with \( Q \) is the Mhaskar–Rahmanov–Saff number \( a_n \), (cf. [7]) defined as the root of

\[
n = \frac{2}{\pi} \int_0^1 a_n Q'(a_n t) dt / \sqrt{1-t^2}, \quad n > 0.
\]

Further, for \( 1 \leq p \leq \infty \) and \( n \geq 1 \), we define for \( \theta \in [-\pi, \pi] \),

\[
\mu_{n,p}(\theta) := w^2(a_n \cos \theta)|\sin \theta|^{2/p}
\]

(12)

For a class of weights including \( \exp(-(1-x^2)^{-\alpha}) \) or \( \exp(-\exp_\delta(1-x^2)^{-\alpha}) \), \( \alpha > 0 \), we have obtained the following results: For \( 1 < p \leq \infty \),

\[
\lim_{n \to \infty} p_{np}(w, a_n z) / \left\{ \phi(z)^n / D \left( \mu_{n,p}; \frac{1}{\phi(z)} \right) \right\} = (2\sigma_p)^{-1}
\]

uniformly in closed subsets of \( \mathbb{C} \setminus [-1, 1] \), and for \( 1 \leq p \leq \infty \),

\[
\lim_{n \to \infty} E_{np}(w)(2/a_n)^n / D(\mu_{n,p}; 0) = 2\sigma_p,
\]

(14)
Szegő Asymptotics

where

\[ \sigma_p := \begin{cases} \left[ \Gamma(1/2) \Gamma((p + 1)/2)/\Gamma(p/2 + 1) \right]^{1/p}, & 1 \leq p < \infty \\ 1, & p = \infty. \end{cases} \]  \hspace{1cm} (15) \]

Furthermore, if the zeros of \( p_n(w, x) \) are denoted by

\[ -1 < x_{n}^{(p)} < x_{n-1}^{(p)} < \ldots < x_{2n}^{(p)} < x_{1n}^{(p)} < 1, \]

then for each fixed \( j \), and uniformly for \( 2 \leq p \leq \infty \),

\[ \lim_{n \to \infty} \left( 1 - x_j^{(p)} \right)/(1 - a_n) = 1. \]

Note that (16) is not true for classical Jacobi weights.

To describe the pointwise asymptotics on \([-1, 1]\) for the orthogonal polynomials, we need the function

\[ T(x) := 1 + x Q''(x)/Q'(x), \quad x \in (-1, 1), \]

and for some fixed, but large enough \( s \),

\[ c_n := a_n \left( 1 + s \left[ \log n \right]^{2/3} \right), \quad n \geq 1 \]

For \( n \geq 1 \), let

\[ \nu_n(\theta) := w^2(c_n \cos \theta) |\sin \theta|, \quad \theta \in [-\pi, \pi]. \]

and

\[ \Gamma_n(\theta) := \frac{1}{4\pi} \int_{-\pi}^{\pi} \{ \log \nu_n(t) - \log \nu_n(\theta) \} \cot \left( \frac{\theta - t}{2} \right) dt, \]  \hspace{1cm} (20) \]

\[ \theta \in [-\pi, \pi]. \] Then there exists \( \delta > 0 \) such that uniformly for \( |x| \leq 1 - n^{-\delta} \), \( x = \cos \theta \),

\[ p_n(w^2, c_n x) \nu_n(\theta)^{1/2} = \left( \frac{2}{\pi} \right)^{1/2} \cos(n\theta + \Gamma_n(\theta)) + o(1). \]  \hspace{1cm} (21) \]

The proofs provide rates in (13), (14), (16), and (21), and involve one-sided approximations by weighted polynomials of the form \( P_n(x)w(c_n x) \). The latter are constructed as in [2,4,5]. We can also treat non-even weights, but these require additional restrictions, including analyticity of \( w^{-1} \) in \((-1, 1)\).
References

2. He, X. and X. Li, Uniform convergence of polynomials associated with varying Jacobi weights, Rocky Mountain J. Math., to appear.

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