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ZEROS OF CHEBYSHEV POLYNOMIALS ASSOCIATED WITH A COMPACT SET IN THE PLANE*

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Abstract. It is proved that the zeros of the Chebyshev polynomials associated with a compact set in the plane having connected interior and complement stay away from the boundary if and only if the set is bounded by an analytic curve.

Key words. Chebyshev polynomials, zeros, analytic curve, Faber polynomials

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Let K be an infinite compact subset of the complex plane $\mathbb{C}$. The unique nth-degree monic polynomial $T_n^K(z) = T_n(z) = z^n + \cdots$ with minimal supremum norm on K is called the nth *Chebyshev polynomial* associated with K. It is well known that the zeros of T_n^K lie in the convex hull of the set K. For the case when K is the unit disk, $T_n^K(z) = z^n$, $n = 0, 1, \cdots$, so it is possible for all the zeros of T_n^K to lie in the interior of K. The aim of this paper is to characterize those sets K for which the zeros stay away from the boundary of K.

Let G_∞ be the unbounded component of the complement $\mathbb{C}\backslash K$ of K. Obviously the Chebyshev polynomials associated with K are the same as those associated with $\mathbb{C}\backslash G_\infty$; therefore in what follows we will assume that $K = \mathbb{C}\backslash G_\infty$, i.e., the complement of K is connected. Widom [5] has proved that for every closed subset S of G_∞ there is a natural number n_S such that each T_n^K can have at most n_S zeros in S. Thus, most of the zeros are close to K. In the case where K has empty interior we actually know the asymptotic distribution of the zeros of T_n^K; namely, it coincides with the equilibrium measure of the set K (see [1]). This result is no longer true if K has nonempty interior, as the above-mentioned example of the unit disk shows. It seems to be a very difficult problem to determine the distribution of the zeros (if it exists at all) for general K's. In connection with this question our aim is to prove the following theorem.

THEOREM. *Let K be a compact subset of $\mathbb{C}$ with connected interior and complement. Then the zeros of the Chebyshev polynomials T_n^K stay away from the boundary of K if and only if K is bounded by an analytic curve.*

By "staying away from the boundary" we mean that for some neighborhood of the boundary there is no zero of T_n^K in this neighborhood for all large n. The proof shows that the same result holds if by "staying away from the boundary" we mean that for some neighborhood of the boundary there are at most $o(n)$ zeros of T_n^K in this neighborhood for $n \to \infty$.

By an analytic curve we mean a simple closed curve γ that has a parametric representation $\gamma_1(t) + i\gamma_2(t)$, $t \in [0, 2\pi]$, where γ_1 and γ_2 are analytic functions on $[0, 2\pi]$.

It seems likely that our result is valid in a somewhat more general form; namely, if K has disconnected interior, then the zeros stay away from the boundary exactly when K is bounded by a finite number of (in this case not necessarily simple) analytic

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curves. However, in this formulation "staying away" must mean the weaker $o(n)$ version discussed above as can be seen from the example: $K=\{z \mid |z^2+1|\leqq 1\}$. In fact, this K is bounded by an analytic (though not simple) curve, but the symmetry of K with respect to the origin implies that $T^K_{2n+1}(0)=0$ for all n.

Proof. (Sufficiency.) We need the Faber polynomials associated with the set K.

Our assumption is that K is bounded by a simple closed analytic curve γ. Thus the complement G_∞ of K in $\bar{\mathbb{C}}:=\mathbb{C}\cup\{\infty\}$ can be mapped conformally onto the exterior of a circle $C_R=\{w \mid |w|=R\}$ by a function φ normalized by $\varphi(\infty)=\infty$, $\lim_{z\to\infty}\varphi(z)/z=1$ (cf. [2, § 14]). Then R is the logarithmic capacity of K and since, without loss of generality, this may be assumed to be 1, in what follows we take $R=1$, i.e., φ maps G_∞ conformally onto the exterior of the unit disk. If

$$\varphi(z)=z+\alpha_0+\frac{\alpha_{-1}}{z}+$$

is the Laurent expansion of φ at infinity, then the expansion of φ^n is of the form

$$\varphi^n(z)=z^n+\alpha^{(n)}_{n-1}z^{n-1}+\cdots+\alpha^{(n)}_0+\frac{\alpha^{(n)}_{-1}}{z}+\cdots.$$

The polynomials

$$F_n(z):=z^n+\alpha^{(n)}_{n-1}z^{n-1}+\cdots+\alpha^{(n)}_0$$

are called the *Faber polynomials* of K.

Since γ is analytic, there is an $r<1$ such that φ can be extended to a conformal mapping of the unbounded component of the complement of a curve $\gamma_r\subseteq K^0:=\operatorname{int}(K)$ onto the exterior of the circle C_r (cf. [2, p. 45]). Clearly, if for $r\leqq\rho\leqq 1$, K_ρ denotes the compact set bounded by the curve $\gamma_\rho=\varphi^{-1}(C_\rho)$, then the Faber polynomials of $K_1=K$ and K_ρ are identical (in what follows we may assume $r<1$ so large that γ_ρ is a simple closed analytic curve for $r\leqq\rho\leqq 1$). But then there exist constants $A>0$ and $0<a<1$ such that on γ_1 the modulus of the difference between $F_n(z)$ and $\varphi^n(z)$ is at most Aa^n for all n (see [2, p. 108]).

We will show that for $\max\{a^{1/2}, r\}<\rho<1$ all the zeros of $T^K_n=T_n$ lie in K_ρ for large n, and proving this will complete the sufficiency part. Let $a^{1/2}<b<\rho$. First we claim that for $z\in\gamma_1$ we have $|T_n(z)-F_n(z)|\leqq Bb^n$ for some constant B independent of z and n. To prove this claim we expand T_n in its Faber series:

$$T_n(z)=F_n(z)+c_1F_{n-1}(z)+\cdots+c_nF_0(z).$$

It is known (see [3, p. 58]) that the Fourier expansion of $T_n(\varphi^{-1}(e^{i\theta}))$ has the form

$$T_n(\varphi^{-1}(e^{i\theta}))\sim e^{ik\theta}+c_1e^{i(k-1)\theta}+\cdots+c_n+q_1e^{-i\theta}+\cdots$$

and so from the Parseval formula we get

$$1+|c_1|^2+\cdots+|c_n|^2\leqq\frac{1}{2\pi}\int_0^{2\pi}|T_n(\varphi^{-1}(e^{i\theta}))|^2\,d\theta.$$

We have already remarked that $|F_n(z)-\varphi^n(z)|\leqq Aa^n$ for $z\in\gamma_1=\partial K$, which implies that $\|F_n\|_K\leqq 1+Aa^n$, and so $\|T_n\|_K\leqq 1+Aa^n$. Substituting this into the previous estimate we get

$$|c_1|^2+\cdots+|c_n|^2\leqq 2Aa^n+A^2a^{2n},$$

from which the inequality $|T_n(z)-F_n(z)|\leqq D\sqrt{n}\cdot a^{n/2}$ immediately follows for $z\in\gamma_1$ with a constant D (note that the Faber polynomials are uniformly bounded on γ_1), and this proves our claim.

Next we note that the Bernstein–Walsh lemma (cf. [4, p. 77]) yields the following inequality for the supremum norms:

$$\|T_n - F_n\|_{\gamma_s} \leqq s^n \|T_n - F_n\|_{\gamma_1} \text{ for any } s \geqq 1$$

and since on γ_s we have already seen that $|F_n| = s^n(1+o(1))$ uniformly in $s \geqq 1$, the inequality $|T_n(z) - F_n(z)| < |F_n(z)|$ follows for every large n, say $n \geqq n_0$, and any $z \notin K$. Hence T_n has no zeros outside K for large n.

Now let $b < b_1 < \rho$. From what we have discussed above concerning F_n and φ^n it also follows that for $z \notin K_\rho$ we have uniformly $|F_n(z)| \geqq db_1^n$ for some positive constant d, and at the same time $|T_n(z) - F_n(z)| \leqq Bb^n$ inside γ_1. Thus we can conclude again that T_n has no zero in $K \backslash K_\rho$ for large n.

This completes the sufficiency part of the proof.

(Necessity.) Now suppose that the zeros stay away from the boundary. Let

$$\nu_n := \frac{1}{n} \sum_{k=1}^{n} \delta_{z_k^{(n)}}$$

be the normalized counting measure on the zeros $z_k^{(n)}$ of T_n. Since the zeros of the T_n's lie in the convex hull of K, we can select a subsequence $\{\nu_{n_k}\}$ converging in the weak-star topology (on Borel measures with compact support) to some measure ν. According to our assumption and the Widom theorem mentioned in the Introduction, ν is supported in a compact subset of the interior K^0 of K. By assumption, K has connected interior and so there is a compact set $H \subset K^0$ such that H has connected interior containing the support of ν and $T_n(z) \neq 0$ for $z \in K \backslash H$ and n large. Let

$$g(z) := \exp\left(\int \log \frac{1}{z-t} \, d\nu(t) \right),$$

where we take that branch of the logarithm that is positive for positive z. Then g is defined, analytic and single-valued in $\mathbb{C} \backslash H$ (note that ν is a probability measure). In $K \backslash H$ and also in a neighborhood of the boundary of K

$$T_{n_k}(z)|^{1/n_k} \to \exp\left(\int \log |z-t| \, d\nu(t) \right),$$

and this combined with the fact that

$$\lim_{n \to \infty} \|T_n\|_K^{1/n} = \operatorname{cap}(K)$$

yields the result that the function

$$\log |g(z)| = \int \log |z-t| \, d\nu(t)$$

is a harmonic function in $\mathbb{C} \backslash H$, is of the form $\log |z| + o(1)$ around the infinity, and is at most as large as $\log(\operatorname{cap}(K))$ on $K \backslash H$. If $\mathcal{G}(z)$ denotes the Green's function with pole at infinity for the complement of K, then we have again $\mathcal{G}(z) + \log(\operatorname{cap}(K)) = \log |z| + o(1)$ as $z \to \infty$, but $\mathcal{G}(z) + \log(\operatorname{cap}(K)) \geqq \log(\operatorname{cap}(K))$ in $\mathbb{C} \backslash K$. Therefore, from the maximum principle for harmonic functions, we get first that $\mathcal{G}(z) + \log(\operatorname{cap}(K)) \geqq \log |g(z)|$ in $\mathbb{C} \backslash K$ and then that these two functions actually coincide because their difference is zero at infinity. From this we get that $\log |g(z)| > \log(\operatorname{cap}(K))$ outside K. In the interior of $K \backslash H$ we obtain from (1) and (2) and the maximum principle that $\log|g(z)|<\log(\text{cap}(K))$. These facts imply that on the boundary of K we must have $\log|g(z)|=\log(\text{cap}(K))$ and that at no other point of $\mathbb{C}\backslash H$ can we have equality. Thus,

$$\partial K=\{z\in\mathbb{C}\backslash H \,|\, |g(z)|=\text{cap}(K)\}$$

and from this we will deduce that ∂K is in fact an analytic curve.

Without loss of generality we may assume $\text{cap}(K)=1$. First of all we show that ∂K is locally an analytic curve. Let z_0 be an arbitrary point on the boundary of K. If $g'(z_0)\neq 0$, then g has an analytic inverse g^{-1} in a neighborhood U of z_0, and in this neighborhood ∂K coincides with the image of a portion of the unit circle under the mapping g^{-1}. Hence, for some neighborhood $U_1\subset U$ of z_0, the intersection $\partial K\cap U_1$ is the analytic image of an arc on the unit circle, and so it is analytic.

Now suppose that $g'(z_0)=\cdots=g^{(k-1)}(z_0)=0$, $g^{(k)}(z_0)\neq 0$, with $k\geqq 2$. Then g can be represented in a neighborhood U of z_0 as $g(z)=c+(h(z))^k$, where $|c|=|g(z_0)|=1$, h is analytic in U, and $h(z_0)=0$ but $h'(z_0)\neq 0$. For some small $\delta>0$ the set

$$\{w \,|\, |c+w^k|=1, |w|\leqq\delta\}$$

is the union of k analytic arcs intersecting the x axis at zero with angle $(\pi/2+\arg c)/k+j\pi/k$, $0\leqq j<k$. According to what we have said above, this implies that, in some neighborhood $U_1\subset U$ of z_0, the part of the boundary ∂K lying in U_1 is the union of k analytic arcs such that their tangent lines at their common point z_0 divide the plane into $2k$ congruent sectors. Let γ_δ be the inverse image of the circle $|w|=\delta$ under the mapping $w=h(z)$, $z\in U_1$. Then it follows from h being conformal around z_0 that, for small $\delta>0$, γ_δ is a simple closed curve such that ∂K divides it into $2k$ connected pieces: $\gamma_{\delta,0},\cdots,\gamma_{\delta,2k-1}$, where each of these Jordan arcs is considered without its endpoints. Let $P_j\in\gamma_{\delta,j}$, $j=0,\cdots,2k-1$. Then P_0 belongs either to K^0 or to G_∞; for definiteness, suppose that $P_0\in G_\infty$. As we move away from P_0 we stay in G_∞ until we reach ∂K. This implies that $P_1\in K^0$, since in the opposite case we would have $P_1\in G_\infty$, which would mean that the common endpoint S of $\gamma_{\delta,0}$ and $\gamma_{\delta,1}$ had a neighborhood disjoint from K^0, contradicting the maximum principle (recall that outside K^0 we have $|g|\geqq 1$ and that $|g(S)|=1$ because $S\in\partial K$). In a similar fashion we can see that $P_2\in G_\infty$ and $P_3\in K^0$. Now, since G_∞ is connected, the points P_0 and P_2 can be joined by an arc Γ_∞ lying in G_∞. Since it is not possible to join P_0 and P_2 inside γ_δ (the possibility of joining P_1 and P_3 to z_0 in K^0 inside γ_δ prevents this), we can assume that Γ_∞ lies exterior to γ_δ (except for its endpoints). Similarly, since K^0 is connected, the points P_1 and P_3 can be joined by an arc Γ_0 in K^0 that also lies exterior to γ_δ. But clearly such a pair of arcs must intersect, which is absurd because $G_\infty\cap K^0\neq\emptyset$. This contradiction shows that $g'(z_0)=0$ cannot occur.

We have thus shown that ∂K locally is an analytic and simple curve. To complete the proof we have only to mention that ∂K must be connected because K^0 is connected.

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