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APPROXIMATION BY POLYNOMIALS WITH LOCALLY GEOMETRIC RATES

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ABSTRACT. In contrast to the behavior of best uniform polynomial approximants on $[0, 1]$ we show that if $f \in C[0, 1]$ there exists a sequence of polynomials $\{P_n\}$ of respective degree $\leq n$ which converges uniformly to f on $[0, 1]$ and geometrically fast at each point of $[0, 1]$ where f is analytic. Moreover we describe the best possible rates of convergence at all regular points for such a sequence.

1. INTRODUCTION

Our paper is related to the fact that best polynomial approximants are very far from giving good approximation on *subsets* of the original set. In fact, let $\|\cdot\|_{[0,1]}$ denote the sup norm on $[0, 1]$, let f be continuous and real-valued on $[0, 1]$, and $Q_n = Q_n(f)$ be the best uniform approximant to f out of Π_n, the set of polynomials of degree at most n. A celebrated result of Kadec [2] says that the extremal points of $\{|f - Q_n(f)|\}_{n=0}^{\infty}$ are dense on $[0, 1]$, and so on any subinterval $I \subseteq [0, 1]$ the approximation given by $\{Q_n(f)\}_{n=0}^{\infty}$ (considering the whole sequence) is not better than on the whole interval $[0, 1]$, no matter how smooth f is on I.

In [3] it was shown that the situation radically changes if one considers *near best* approximants instead of best ones. For example, when f is piecewise analytic on $[0, 1]$ and otherwise k-times continuously differentiable at the

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non-regular points it was shown that for each $\beta > 1$ there are constants C, $c > 0$ and polynomials $p_n \in \Pi_n$, $n = 1, 2, \dots$, such that

$$|f(x) - p_n(x)| \le \frac{C}{n^{k+1}} \exp(-cn[d(x)]^\beta), \qquad x \in [0,\ 1],$$

where $d(x)$ measures the distance from x to the nearest non-regular point of f. It was also shown that a similar estimate with $\beta = 1$ is, in general, impossible. These polynomials p_n, unlike the polynomials of best approximation, yield geometric convergence on (closed) intervals of analyticity even though $\{E_n(f)\}$, $E_n(f) := \|f - Q_n(f)\|_{[0,1]}$, has order only $\{n^{-k-1}\}$.

The problem whether similar results hold for more general sets of functions (not just for piecewise analytic ones) has however remained open. To be more precise we ask the following: let $f \in C[0,\ 1]$ be analytic on the (relative to $[0,\ 1]$) open subset D of $[0,\ 1]$. Is it possible to find polynomials $P_n \in \Pi_n$, $n = 0,\ 1, \dots$, such that

$$\|f - P_n\|_{[0,1]} \to 0 \qquad \text{as } n \to \infty,$$

and at every point of D we have geometric convergence, i.e.

$$\varlimsup_{n\to\infty} |f(x) - P_n(x)|^{1/n} < 1, \qquad x \in D\ ? \tag{1}$$

We will show that this is always possible and describe the behavior of the left-hand side of (1) which is, in a certain sense, best possible.

We will assume that D is the exact set of analyticity, i.e. D contains every regular point of f. For $x \in [0,\ 1]$ let $d(x)$ be the distance from x to the nearest singularity of f, where f is considered to be extended to the complex plane and we also count the singularities outside $[0,\ 1]$. In other words, $d(x)$ is the largest radius such that the Taylor expansion of f about x converges in $\{z \in \mathbf{C}\colon |z - x| < d(x)\}$. Of course, if x is not a regular point of f, then $d(x) = 0$.

If $d(x) > 0$ for every $x \in [0,\ 1]$, i.e. if f is analytic on $[0,\ 1]$, then the best uniform approximants converge geometrically to f and so in what follows we assume that f has a singularity somewhere on $[0,\ 1]$.

Theorem 1. *Suppose that $\beta > 1$ and $f \in C[0,\ 1]$ has a singularity on $[0,\ 1]$. There are polynomials $P_n \in \Pi_n$, $n = 0, 1, \dots$, such that*

$$\|f - P_n\|_{[0,1]} \to 0 \qquad \textit{as } n \to \infty, \tag{2}$$

and for $x \in [0,\ 1]$

$$|f(x) - P_n(x)| \le C_{f,x} \exp(-cn[d(x)]^\beta), \tag{3}$$

where $c > 0$ is an absolute constant and the constant $C_{f,x}$ is bounded for x in any compact subset of D.

Next we show that Theorem 1 is best possible in the sense that (3) with $\beta = 1$ is, in general, impossible.

Theorem 2. *There are no positive constants C_x, $x \in [-1, 1]$, and $c > 0$ such that C_x, $x \in D$, are bounded for every compact subset D of $[-1, 1]$ not containing the origin and for every n there are polynomials $P_n \in \Pi_n$ with*

$$||x| - P_n(x)| \leq C_x \exp(-cn|x|), \qquad x \in [-1, 1].$$

For convenience here the basic interval $[0, 1]$ has been replaced by $[-1, 1]$. Note that then for $f(x) = |x|$ we have $d(x) = |x|$.

Finally, we also show that Theorem 1 cannot be sharpened by putting a constant C_f into (3)—the constant must depend on x, even allowing c and β to depend on f.

Theorem 3. *There exists $f \in C[0, 1]$ such that for no constants β, C, $c > 0$ can one find polynomials $P_n \in \Pi_n$, $n = 1, 2, \ldots$, with the property*

$$|f(x) - P_n(x)| \leq C \exp(-cn[d(x)]^\beta) \qquad n = 1, 2, \ldots, \qquad x \in [0, 1]$$

2. Proof of Theorem 1

It is enough to prove the theorem for $\beta \in (1, 3/2]$ because $d(x) \leq 1$. Also it is enough to prove the theorem for $n \geq N_0$ because afterward we can increase $C_{f,x}$ so that (3) will be fulfilled for any natural number n. In the beginning n is arbitrary and only when necessary we place restrictions on N_0.

For fixed $n \geq 1$ we set $\tilde{d}(x) := d(x) + n^{-1/\beta}$ and define the points $0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = 1$ by $x_0 := 0$, $x_{k+1} := x_k + (1/40)\tilde{d}(x_k)$ whenever this defines a number x_{k+1} with $x_{k+1} + (1/40)\tilde{d}(x_{k+1}) \leq 1$; in the opposite case, which occurs, say, for $k = m$, we set $x_{k+1} = x_{m+1} := 1$.

From the definition of $d(x)$ we have

$$|\tilde{d}(x) - \tilde{d}(x_k)| = |d(x) - d(x_k)| \leq |x - x_k|,$$

and so for $x \in [x_k, x_k + (1/40)\tilde{d}(x_k)]$, $0 \leq k \leq m$,

$$\frac{39}{40}\tilde{d}(x_k) \leq \tilde{d}(x) \leq \frac{41}{40}\tilde{d}(x_k).$$

This easily implies that for each $k = 0, \ldots, m$,

$$|x_{k+1} - x_k| \leq \frac{1}{19}\tilde{d}(x_k),$$

and thus

$$\frac{18}{19}\tilde{d}(x_k) \leq \tilde{d}(x) \leq \frac{20}{19}\tilde{d}(x_k), \qquad x \in [x_k, x_{k+1}].$$

We claim that

$$[x_{k-2}, x_{k+3}] \subset \left[x - \frac{\tilde{d}(x)}{4}, x + \frac{\tilde{d}(x)}{4}\right] \qquad x \in [x_k, x_{k+1}$$

(for definiteness set $x_{-2} = x_{-1} = 0$ and $x_{m+2} = x_{m+3} = 1$). In fact, we get from (5) and (6) that

$$\begin{aligned} x - \frac{\tilde{d}(x)}{4} &\le x_k + \frac{1}{19}\tilde{d}(x_k) - \frac{1}{4}\cdot\frac{18}{19}\tilde{d}(x_k) \\ &\le [x_{k-1} + \frac{1}{19}\tilde{d}(x_{k-1})] - \frac{7}{38}\cdot\frac{18}{19}\tilde{d}(x_{k-1}) \\ &\le x_{k-2} + \frac{1}{19}\tilde{d}(x_{k-2}) + \frac{1}{19}\cdot\frac{20}{19}\tilde{d}(x_{k-2}) - \frac{7}{-}\left(\frac{18}{19}\right)^2\tilde{d}(x_{k-2}) \\ &\le x_{k-2}, \end{aligned}$$

and similarly

$$\begin{aligned} x + \frac{\tilde{d}(x)}{4} \ge x_k + \frac{1}{4}\cdot\frac{18}{19}\tilde{d}(x_k) &\ge x_k + \frac{1}{19}\left[1 + \frac{20}{19} + \left[\frac{20}{19}\right]^2\right]\tilde{d}(x_k) \\ &\ge x_k + \frac{1}{19}(\tilde{d}(x_k) + \tilde{d}(x_{k+1}) + \tilde{d}(x_{k+2})) \\ &\ge x_{k+3}. \end{aligned}$$

By [1, Theorem 3] there exist two absolute constants C_1, c_1 with $c_1 \le 1$ such that for every n there exists a polynomial $\chi_n^* = \chi$ of degree at most $n/2$ such that on $[-1, 1]$ the polynomial χ is monotone increasing, satisfies $0 \le \chi \le 1$ there and, with $\gamma := (1+\beta)/2 > 1$,

$$\left|\chi(x) - \frac{1 + \operatorname{sign} x}{2}\right| \le C_1 \exp(-c_1 n |x|^\gamma), \qquad x \in [-1, 1],$$

With this $\chi_n^* = \chi$ (n is fixed) we define

$$\begin{aligned} &\chi_0(x) := 1 - \chi(x - x_1), \\ &\chi_j(x) := \chi(x - x_j) - \chi(x - x_{j+1}), \qquad j = 1, \dots, m-1, \end{aligned}$$

$$\chi_m(x) := \chi(x - x_m).$$

Clearly we have

$$\sum_{j=0}^{m} \chi_j(x) = \tag{8}$$

and $0 \le \chi_j(x) \le 1$ on $[0, 1]$. Furthermore, $(1 + 19/18)^{-\gamma} > 1/4$ gives

$$\begin{aligned} \chi_j(x) &\le 2C_1 \exp(-c_1 n \min\{|x - x_j|^\gamma;\ |x - x_{j+1}|^\gamma\}) \\ &\le 2C_1 \exp\left(-\frac{1}{4}c_1 n |x - x_j|^\gamma\right) \end{aligned}$$

provided $x \in [0, 1] \backslash [x_{j-1}, x_{j+2}]$. This "partition of unity" will be used together with local best approximants to produce the required polynomial of degree n.

In fact, let $P_j(f) = P_j$, $j = 0, \dots, m$, be the best uniform polynomial approximant of f on $[x_{j-1}, x_{j+2}]$ of degree

$$n_j := \left[\frac{c_1}{800} n(\tilde{d}(x_j))^\gamma\right],$$

and set

$$P_n^*(x) = P(x) := \sum_{j=0}^{m} \chi_j(x) P_j(x)$$

Then P is a polynomial of degree at most n and below we show that $P_n^* = P$ satisfies the requirements set forth in Theorem 1.

Let $x \in [x_k, x_{k+1}]$ with k arbitrary and let j be different from $k-1$, k and $k+1$. From (9) we get

$$0 \le \chi_j(x) \le 2C_1 \exp\left[-\frac{c_1}{4} n|x - x_j|^\gamma\right]$$

and the estimate (cf. [4, 2.13.27])

$$|Q(y)| \le (2|y|)^{\deg Q} \|Q\|_{[-1,1]}, \qquad y \in \mathbf{R}\setminus[-1, 1],$$

transformed to the interval $[x_{j-1}, x_{j+2}]$ together with (5) and (6) easily implies

$$\begin{aligned}
|f(x) - P_j(x)| &\le \|f\|_{[0,1]} + (100|x - x_j|/\tilde{d}(x_j))^{n_j} \|P_j\|_{[x_{j-1}, x_{j+2}]} \\
&\le 3\|f\|_{[0,1]} (100|x - x_j|/\tilde{d}(x_j))^{n_j} \\
&\le 3\|f\|_{[0,1]} \exp\left[\frac{c_1}{800} n\, \tilde{d}(x_j)^\gamma \log\left(100|x - x_j|/\tilde{d}(x_j)\right)\right] \\
&\le 3\|f\|_{[0,1]} \exp\left(\frac{c_1}{8} n e^{-1}\, \tilde{d}(x_j)^{\gamma-1} |x - x_j|\right) \\
&\le 3\|f\|_{[0,1]} \exp\left(\frac{c_1}{8} n|x - x_j|^\gamma\right)
\end{aligned}$$

because $100|x - x_j|/\tilde{d}(x_j) > 1$ and $\log u \le e^{-1} u$ for $u \ge$ Thus

$$\begin{aligned}
\chi_j(x)|f(x) - P_j(x)| &\le 6C_1 \|f\|_{[0,1]} \exp\left(-\frac{c_1}{8} n|x - x_j|^\gamma\right) \\
&\le 6C_1 \|f\|_{[0,1]} \exp\left(-\frac{c_1}{16} n|x - x_j|^\gamma\right) \exp(-c_2 n^{1-\gamma/\beta}) \\
&\le 6C_1 \|f\|_{[0,1]} \exp(-c_2 n\, d(x)^\beta) \exp(-c_2 n^{1-\gamma/\beta}),
\end{aligned}$$

where $c_2 := (c_1/16)\cdot(1/150) > 0$ and where we used that $|x - x_j| \ge (1/40) n^{-1/\beta}$ (note that $x \notin [x_{j-1}, x_{j+2}]$ and $|x_{\ell+1} - x_\ell| \ge (1/40) n^{-1/\beta}$ for any $0 \le \ell \le m$) and $|x - x_j| \ge (1/50)\, d(x)$. This immediately implies that for $x \in [x_k, x_{k+1}]$,

$$\sum_{\substack{j=0 \\ j \ne k-1, k, k+1}}^{m} \chi_j(x)|f(x) - P_j(x)| \le C_2 \|f\|_{[0,1]} \exp(-c_2 n [d(x)]^\beta \tag{10}$$

for some constant C_2 depending only on β

Since (cf. (8))

$$f(x) - P(x) = \sum_{j=0}^{m} \chi_j(x)(f(x) - P_j(x))$$

we have to estimate $f(x) - P_j(x)$ for $j = k-1$, k, $k+1$, as well.

Again, let $x \in [x_k, x_{k+1}]$. If $d(x) \le n^{-1/\beta}$, then we just write

$$|f(x) - P_j(x)| \le \|f\|_{[0,1]} \le e\|f\|_{[0,1]} \exp(-n[d(x)]^\beta) . \tag{12}$$

If, however, $d(x) > n^{-1/\beta}$, then $\tilde{d}(x) \le 2d(x)$, which, together with (7) yields

$$\begin{aligned} |f(x) - P_j(x)| &\le E_{n_j}(f)_{[x_{k-2}, x_{k+3}]} \\ &\le E_{n_j}(f)_{[x-\tilde{d}(x)/4, x+\tilde{d}(x)/4]} \\ &\le E_{n_j}(f)_{[x-d(x)/2, x+d(x)/2]}, \end{aligned} \tag{13}$$

where $E_m(f)_{[a,b]}$ denotes the error in best uniform approximation to f on $[a, b]$ out of Π_m. To estimate the right-hand member of (13) consider the Taylor expansion of f about x. For the absolute value of the ν-th Taylor coefficient, Cauchy's inequality gives the bound

$$\sup_{|z-x| \le 3d(x)/4} |f(z)| \left(\frac{3}{4} d(x)\right)^{-\nu} =: C(x) \left(\frac{3}{4} d(x)\right)^{-\nu}$$

and so the n_j-th partial sum approximates f on $[x - d(x)/2,\ x + d(x)/2]$ with error at most

$$C(x) \sum_{\nu=n_j+1}^{\infty} \left(\frac{2}{3}\right)^{\nu} = 2C(x) \exp\left(\left(\log\frac{2}{3}\right) n_j\right) \le 2C(x) \exp(-c_3 n[d(x)]^\beta)$$

with $c_3 := (\log \frac{3}{2}) c_1/1000$ and where we used the fact that $n_j > (c_1/1000) n[d(x)]^\beta$ whenever $n_j \ge N_0$ and N_0 satisfies $(c_1/800) N_0^{1-\gamma/\beta} \ge 4$.

Thus

$$E_{n_j}(f)_{[x-d(x)/2, x+d(x)/2]} \le 2C(x) \exp(-c_3 n[d(x)]^\beta). \tag{14}$$

The relations (10)–(14) yield

$$\begin{aligned} |f(x) - P_n^*(x)| &\equiv |f(x) - P(x)| \\ &\le C_2\|f\| \exp(-c_2 n[d(x)]^\beta) + 3\|f\| \le C_3\|f\| \end{aligned} \tag{15}$$

and

$$|f(x) - P_n^*(x)| \equiv |f(x) - P(x)| \le C_4(x) \exp(-c_3 n[d(x)]^\beta),$$

which proves (3). Property (2) also can easily be obtained from (15). In fact, notice that every n_j is at least as large as

$$\frac{c_1}{1000} n(\tilde{d}(x))^\gamma \ge \left[\frac{c_1}{1000} n^{1-\gamma/\beta}\right] =: m_n,$$

and $m_n \to \infty$ as $n \to \infty$. If Q_{m_n} denotes the best polynomial approximation of f on $[0, 1]$ by polynomials of degree at most m_n, then we get from $n_j \geq m_n$

$$P_j(f;\cdot)_{[x_{j-1}, x_{j+2}]} - Q_{m_n}(\cdot) \equiv P_j(f - Q_{m_n};\cdot)_{[x_{j-1}, x_{j+2}]},$$

and so (cf. (8))

$$f(\cdot) - \sum_{j=0}^{m} \chi_j(\cdot) P_j(f;\cdot) \equiv (f - Q_{m_n})(\cdot) \quad \sum_{j=0}^{m} \chi_j(\cdot) P_j(f - Q_{m_n};\cdot)$$

Hence we obtain from (15), with f replaced by $f - Q_{m_n}$ on the right, that

$$|f(x) - P_n^*(x)| \leq C_3 \|f - Q_{m_n}\|_{[0,1]}$$

Since $\|f - Q_{m_n}\|_{[0,1]} \to 0$ as $n \to \infty$, we obtain (2). □

3. Proof of Theorem 2

Suppose to the contrary that for any $a \in (0, 1/2]$ the estimate

$$\text{(16)} \qquad |P_n(x) - |x| \leq C_a \exp(-2cn|x|), \qquad x \in [-1, 1] \backslash [-a, a],$$

is possible with some positive constants C_a and c and polynomials $P_n \in \Pi_n$. Then (16) holds for $(P_n(x) + P_n(-x))/2$, as well, so we may assume each P_n to be even.

Consider the derivative P_n' of P_n, which is odd. Let $x \in [a, 1]$ and $b := \min\{x, 1/2\}$. Since

$$|P_n(u) - u| \leq C_a \exp(-2cnb)$$

if $u \in [b, 1]$, we get from Markoff's inequality, applied to the interval $[b, 1]$, that

$$\begin{aligned} |P_n'(x) - 1| &\leq 4C_a n^2 \exp(-2cnb) \\ &\leq 4C_a n^2 \exp(-cnx), \qquad a \leq x \leq 1. \end{aligned}$$

In a similar way

$$|P_n'(x) + 1| \leq 4C_a n^2 \exp(-cn|x|), \qquad -1 \leq x \leq -a,$$

and so for $a \leq |x| \leq 1$ we have

$$|P_n'(x) - \operatorname{sign} x| \leq 4C_a n^2 \exp(-cn|x|),$$

where C_a is independent of n.

This implies that the polynomials

$$Q_n(x) := 1 - (P_n'(\sqrt{x}))^2$$

of degree at most n satisfy $Q_n(0) = 1$ and, for large n,

$$|Q_n(x)| \leq 8C_a n^2 \exp(-cn\sqrt{x}), \qquad a \leq x \leq 1.$$

However, this is impossible, since in [1, Section 4.I] it was proved that if $Q_n \in \Pi_n$ and $Q_n(0) = 1$, then for $0 < a \leq 1$ the inequality $|Q_n(x)| \leq 1$ on $[a, 1]$ implies

$$\int_a^1 \frac{-\log|Q_n(x)|}{x^{3/2}}\,dx \leq 4n$$

This contradiction proves the theorem. □

4. Proof Theorem 3

Let $a_k := 2^{-k}(1+i)$, where $i = \sqrt{-1}$, and set

$$f(z) := \sum_{k=1}^{\infty} n_k^{-1/8}[2^k(z-a_k)]^{-n_k} =: \sum_{k=1}^{\infty} g_k(z),$$

where $\{n_k\}$ will be completely specified later in the proof. We set $n_1 = 1$, and require $n_{k+1} > 2^k n_k$ for all k so that for every $\ell > 0$ we have

$$n_k/2^{\ell k} \to \infty \tag{17}$$

Since

$$|[2^k(z-a_k)]^{-1}| \leq 1 \qquad \text{if } |z-a_k| \geq 2^{-k},$$

it is easy to see that the series defining f uniformly converges on the real line and also on every compact subset of the complex plane not containing the points 0, $\{a_k\}_{k=1}^{\infty}$; thus these are the singular points of f. In particular,

$$d(x) \geq \frac{1}{\sqrt{2}}x \qquad \text{for } x \in [0, 1]. \tag{18}$$

Now we need some easy estimates:

$$|g_k(2^{-k})| = n_k^{-1/8},$$

$$|g_k(2^{-k} + n_k^{-1/4})| = n_k^{-1/8}(1 + 4^k n_k^{-1/2})^{-n_k/2} < \frac{1}{2}n_k^{-1/8}$$

and so

$$n_k^{1/4}|g_k(2^{-k}) - g_k(2^{-k} + n_k^{-1/4})| > \frac{1}{2}n_k^{1/8}.$$

Using this, by standard gliding-hump arguments we can select n_1, n_2, ... one after the other in such a way that

$$n_k^{1/4}|f(2^{-k}) - f(2^{-k} + n_k^{-1/4})| > \frac{1}{4}n_k^{1/8} \tag{19}$$

is satisfied for all k.

Now if (4) is true for some C, $c > 0$ then

$$|f(x) - P_m(x)| \leq 2^k \exp(-2^{-k} m[d(x)]^{\beta}), \quad m = 1, 2, \ldots, \quad x \in [0, 1], \tag{20}$$

is also true for any large k, $k \geq \max\{\ln C; -\ln c\}/\ln 2$, and below we show that this is impossible.

Fix a k so large that (20) is satisfied and also $(1/16)n_k^{-1/8} < 2^k \exp\{-2^{-\beta/2} n_k^{1/16}\}$, which is always possible because of (17).

Set $m = [2^{k(1+\beta)} n_k^{1/16}] + 1$ into (20). Then for $x \in [2^{-k}, 1]$ we get from (18) and (20)

$$|f(x) - P_m(x)| \le 2^k \exp(-2^{-k} 2^{k(1+\beta)} n_k^{1/16} (2^{-k}/\sqrt{2})^\beta) < \frac{1}{16} n_k^{-1/8}$$

therefore (cf. (19)),

$$n_k^{1/4} |P_m(2^{-k}) - P_m(2^{-k} + n_k^{-1/4})| > \frac{1}{8} n_k^{1/8}. \tag{21}$$

On the other hand, $\{P_m\}$ are uniformly bounded on $[0, 1]$, say $|P_m| \le K$, $m = 1, 2, \dots$, (cf. (4)); hence Bernstein's inequality shows that for $x \in [2^{-k}, 2^{-k} + n_k^{-1/4}]$,

$$|P_m'(x)| \le K m 2^k,$$

which implies

$$\begin{aligned} n_k^{1/4} |P_m(2^{-k}) - P_m(2^{-k} + n_k^{-1/4})| &\le K m 2^k \\ &\le 2K 2^{k(2+\beta)} n_k^{1/16}. \end{aligned} \tag{22}$$

However, (17), (21) and (22) are not compatible which proves our theorem. □

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