APPROXIMATION BY POLYNOMIALS
WITH LOCALLY GEOMETRIC RATES

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ABSTRACT. In contrast to the behavior of best uniform polynomial approximants on \([0, 1]\) we show that if \(f \in C[0, 1]\) there exists a sequence of polynomials \(\{P_n\}\) of respective degree \(\leq n\) which converges uniformly to \(f\) on \([0, 1]\) and geometrically fast at each point of \([0, 1]\) where \(f\) is analytic. Moreover we describe the best possible rates of convergence at all regular points for such a sequence.

1. INTRODUCTION

Our paper is related to the fact that best polynomial approximants are very far from giving good approximation on subsets of the original set. In fact, let \(\|\cdot\|_{[0,1]}\) denote the sup norm on \([0, 1]\), let \(f\) be continuous and real-valued on \([0, 1]\), and \(Q_n = Q_n(f)\) be the best uniform approximant to \(f\) out of \(\Pi_n\), the set of polynomials of degree at most \(n\). A celebrated result of Kadec [2] says that the extremal points of \(\{|f - Q_n(f)|\}_{n=0}^{\infty}\) are dense on \([0, 1]\), and so on any subinterval \(I \subseteq [0, 1]\) the approximation given by \(\{Q_n(f)\}_{n=0}^{\infty}\) (considering the whole sequence) is not better than on the whole interval \([0, 1]\), no matter how smooth \(f\) is on \(I\).

In [3] it was shown that the situation radically changes if one considers near best approximants instead of best ones. For example, when \(f\) is piecewise analytic on \([0, 1]\) and otherwise \(k\)-times continuously differentiable at the
non-regular points it was shown that for each $\beta > 1$ there are constants $C$, $c > 0$ and polynomials $p_n \in \Pi_n$, $n = 1, 2, \ldots$, such that

$$|f(x) - p_n(x)| \leq \frac{C}{n^{\beta+1}} \exp(-cn[d(x)]^\beta), \quad x \in [0, 1],$$

where $d(x)$ measures the distance from $x$ to the nearest non-regular point of $f$. It was also shown that a similar estimate with $\beta = 1$ is, in general, impossible. These polynomials $p_n$, unlike the polynomials of best approximation, yield geometric convergence on (closed) intervals of analyticity even though $\{E_n(f)\}$, $E_n(f) := \|f - Q_n(f)\|_{[0,1]}$, has order only $\{n^{-k-1}\}$.

The problem whether similar results hold for more general sets of functions (not just for piecewise analytic ones) has however remained open. To be more precise we ask the following: let $f \in C[0, 1]$ be analytic on the (relative to $[0, 1]$) open subset $D$ of $[0, 1]$. Is it possible to find polynomials $P_n \in \Pi_n$, $n = 0, 1, \ldots$, such that

$$\|f - P_n\|_{[0,1]} \to 0 \quad \text{as } n \to \infty,$$

and at every point of $D$ we have geometric convergence, i.e.

$$(1) \quad \lim_{n \to \infty} |f(x) - P_n(x)|^{1/n} < 1, \quad x \in D.$$  

We will show that this is always possible and describe the behavior of the left-hand side of (1) which is, in a certain sense, best possible.

We will assume that $D$ is the exact set of analyticity, i.e. $D$ contains every regular point of $f$. For $x \in [0, 1]$ let $d(x)$ be the distance from $x$ to the nearest singularity of $f$, where $f$ is considered to be extended to the complex plane and we also count the singularities outside $[0, 1]$. In other words, $d(x)$ is the largest radius such that the Taylor expansion of $f$ about $x$ converges in $\{z \in \mathbb{C}: |z - x| < d(x)\}$. Of course, if $x$ is not a regular point of $f$, then $d(x) = 0$.

If $d(x) > 0$ for every $x \in [0, 1]$, i.e. if $f$ is analytic on $[0, 1]$, then the best uniform approximants converge geometrically to $f$ and so in what follows we assume that $f$ has a singularity somewhere on $[0, 1]$.

**Theorem 1.** Suppose that $\beta > 1$ and $f \in C[0, 1]$ has a singularity on $[0, 1]$. There are polynomials $P_n \in \Pi_n$, $n = 0, 1, \ldots$, such that

$$(2) \quad \|f - P_n\|_{[0,1]} \to 0 \quad \text{as } n \to \infty,$$

and for $x \in [0, 1]$

$$(3) \quad |f(x) - P_n(x)| \leq C_{f,x} \exp(-cn[d(x)]^\beta),$$

where $c > 0$ is an absolute constant and the constant $C_{f,x}$ is bounded for $x$ in any compact subset of $D$.

Next we show that Theorem 1 is best possible in the sense that (3) with $\beta = 1$ is, in general, impossible.
Theorem 2. There are no positive constants $C_x$, $x \in [-1, 1]$, and $c > 0$ such that $C_x$, $x \in D$, are bounded for every compact subset $D$ of $[-1, 1]$ not containing the origin and for every $n$ there are polynomials $P_n \in \Pi_n$ with

$$|x - P_n(x)| \leq C_x \exp(-cn|x|), \quad x \in [-1, 1].$$

For convenience here the basic interval $[0, 1]$ has been replaced by $[-1, 1]$. Note that then for $f(x) = |x|$ we have $d(x) = |x|$. Finally, we also show that Theorem 1 cannot be sharpened by putting a constant $C_f$ into (3)—the constant must depend on $x$, even allowing $c$ and $\beta$ to depend on $f$.

Theorem 3. There exists $f \in C[0, 1]$ such that for no constants $\beta, C, c > 0$ can one find polynomials $P_n \in \Pi_n$, $n = 1, 2, \ldots$, with the property

$$|f(x) - P_n(x)| \leq C \exp(-cn[d(x)])^\beta \quad n = 1, 2, \ldots, \quad x \in [0, 1].$$

2. Proof of Theorem 1

It is enough to prove the theorem for $\beta \in (1, 3/2]$ because $d(x) \leq 1$. Also it is enough to prove the theorem for $n \geq N_0$ because afterward we can increase $C_{f,x}$ so that (3) will be fulfilled for any natural number $n$. In the beginning $n$ is arbitrary and only when necessary we place restrictions on $N_0$.

For fixed $n \geq 1$ we set $\tilde{d}(x) := d(x) + n^{-1/\beta}$ and define the points $0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = 1$ by $x_0 := 0$, $x_{k+1} := x_k + (1/40)\tilde{d}(x_k)$ whenever this defines a number $x_{k+1}$ with $x_{k+1} + (1/40)\tilde{d}(x_{k+1}) \leq 1$; in the opposite case, which occurs, say, for $k = m$, we set $x_{k+1} = x_{m+1} := 1$.

From the definition of $d(x)$ we have

$$|\tilde{d}(x) - \tilde{d}(x_k)| = |d(x) - d(x_k)| \leq |x - x_k|,$$

and so for $x \in [x_k, x_k + (1/40)\tilde{d}(x_k)]$, $0 \leq k \leq m$,

$$\frac{39}{40} \tilde{d}(x_k) \leq \tilde{d}(x) \leq \frac{41}{40} \tilde{d}(x_k).$$

This easily implies that for each $k = 0, \ldots, m$,

$$|x_{k+1} - x_k| \leq \frac{1}{19} \tilde{d}(x_k),$$

and thus

$$\frac{18}{19} \tilde{d}(x_k) \leq \tilde{d}(x) \leq \frac{20}{19} \tilde{d}(x_k), \quad x \in [x_k, x_{k+1}].$$

We claim that

$$[x_{k-2}, x_{k+3}] \subset \left[ x - \frac{\tilde{d}(x)}{4}, x + \frac{\tilde{d}(x)}{4} \right] \quad x \in [x_k, x_{k+1}].$$
(for definiteness set $x_{-2} = x_{-1} = 0$ and $x_{m+2} = x_{m+3} = 1$). In fact, we get from (5) and (6) that

$$x - \frac{\tilde{d}(x)}{4} \leq x_k + \frac{1}{19} \tilde{d}(x_k) - \frac{1}{4} \cdot \frac{18}{19} \tilde{d}(x_k)$$

$$\leq [x_{k-1} + \frac{1}{19} \tilde{d}(x_{k-1})] - \frac{7}{38} \cdot \frac{18}{19} \tilde{d}(x_{k-1})$$

$$\leq x_{k-1} + \frac{1}{19} \tilde{d}(x_{k-2}) + \frac{1}{19} \cdot \frac{20}{19} \tilde{d}(x_{k-2}) - \frac{7}{18} \left( \frac{18}{19} \right)^2 \tilde{d}(x_{k-2})$$

$$\leq x_{k-2},$$

and similarly

$$x + \frac{\tilde{d}(x)}{4} \geq x_k + \frac{1}{4} \cdot \frac{18}{19} \tilde{d}(x_k) \geq x_k + \frac{1}{19} \left[ 1 + \frac{20}{19} + \left( \frac{20}{19} \right)^2 \right] \tilde{d}(x_k)$$

$$\geq x_k + \frac{1}{19} (\tilde{d}(x_k) + \tilde{d}(x_{k+1}) + \tilde{d}(x_{k+2}))$$

$$\geq x_{k+3}.$$

By [1, Theorem 3] there exist two absolute constants $C_1, c_1$ with $c_1 \leq 1$ such that for every $n$ there exists a polynomial $\chi_n^* = \chi$ of degree at most $n/2$ such that on $[-1, 1]$ the polynomial $\chi$ is monotone increasing, satisfies $0 \leq \chi \leq 1$ there and, with $\gamma := (1 + \beta)/2 > 1$,

$$\left| \chi(x) - \frac{1 + \text{sign } x}{2} \right| \leq C_1 \exp(-c_1 n |x|^\gamma), \quad x \in [-1, 1],$$

With this $\chi_n^* = \chi$ ($n$ is fixed) we define

$$\chi_0(x) := 1 - \chi(x - x_1),$$

$$\chi_j(x) := \chi(x - x_j) - \chi(x - x_{j+1}), \quad j = 1, \ldots, m - 1,$$

$$\chi_m(x) := \chi(x - x_m).$$

Clearly we have

$$\sum_{j=0}^{m} \chi_j(x) =$$

and $0 \leq \chi_j(x) \leq 1$ on $[0, 1]$. Furthermore, $(1 + 19/18)^{-\gamma} > 1/4$ gives

$$\chi_j(x) \leq 2C_1 \exp(-c_1 n \min\{|x - x_j|; |x - x_{j+1}|\})$$

$$\leq 2C_1 \exp\left( -\frac{1}{4} c_1 n |x - x_j|^\gamma \right)$$

provided $x \in [0, 1]\setminus[x_{j-1}, x_{j+2}]$. This "partition of unity" will be used together with local best approximants to produce the required polynomial of degree $n$. 
In fact, let $P_j(f) = P_j$, $j = 0, \ldots, m$, be the best uniform polynomial approximant of $f$ on $[x_{j-1}, x_{j+2}]$ of degree

$$n_j := \left[ \frac{c_1}{800} n(d(x_j))^\gamma \right],$$

and set

$$P_n^*(x) = P(x) := \sum_{j=0}^m \chi_j(x) P_j(x).$$

Then $P$ is a polynomial of degree at most $n$ and below we show that $P_n^* = P$ satisfies the requirements set forth in Theorem 1.

Let $x \in [x_k, x_{k+1}]$ with $k$ arbitrary and let $j$ be different from $k-1$, $k$ and $k+1$. From (9) we get

$$0 \leq \chi_j(x) \leq 2C_1 \exp \left[ -\frac{c_1}{4} n|x-x_j|^\gamma \right]$$

and the estimate (cf. [4, 2.13.27])

$$|Q(y)| \leq (2|y|)^{\deg Q} \|Q\|_{[-1,1]}, \quad y \in \mathbb{R}\setminus[-1,1],$$

transformed to the interval $[x_{j-1}, x_{j+2}]$ together with (5) and (6) easily implies

$$|f(x) - P_j(x)| \leq \|f\|_{[0,1]} + (100|x - x_j|/d(x_j))^{n_j} \|P_j\|_{[x_{j-1}, x_{j+2}]}
\leq 3\|f\|_{[0,1]} (100|x - x_j|/d(x_j))^{n_j}
\leq 3\|f\|_{[0,1]} \exp \left[ \frac{c_1}{800} n d(x_j)^\gamma \log \left( 100|x - x_j|/d(x_j) \right) \right]
\leq 3\|f\|_{[0,1]} \exp \left( \frac{c_1}{8} n e^{-1} d(x_j)^\gamma |x - x_j| \right)
\leq 3\|f\|_{[0,1]} \exp \left( \frac{c_1}{8} n |x - x_j|^\gamma \right)$$

because $100|x - x_j|/d(x_j) > 1$ and $\log u \leq e^{-1} u$ for $u \geq 1$. Thus

$$\chi_j(x)|f(x) - P_j(x)| \leq 6C_1 \|f\|_{[0,1]} \exp \left( -\frac{c_1}{8} n |x - x_j|^\gamma \right)
\leq 6C_1 \|f\|_{[0,1]} \exp \left( -\frac{c_1}{16} n |x - x_j|^\gamma \right) \exp(-c_2 n^{1-\gamma/\beta})
\leq 6C_1 \|f\|_{[0,1]} \exp(-c_2 n d(x)^\beta) \exp(-c_2 n^{1-\gamma/\beta}),$$

where $c_2 := (c_1/16)\cdot(1/150) > 0$ and where we used that $|x - x_j| \geq (1/40)n^{-1/\beta}$ (note that $x \notin [x_{j-1}, x_{j+2}]$ and $|x_{\ell+1} - x_{\ell}| \geq (1/40)n^{-1/\beta}$ for any $0 \leq \ell \leq m$) and $|x - x_j| \geq (1/50)d(x)$. This immediately implies that for $x \in [x_k, x_{k+1}]$,

$$\sum_{j=0}^m \chi_j(x)|f(x) - P_j(x)| \leq C_2 \|f\|_{[0,1]} \exp(-c_2 n [d(x)]^\beta)$$

for some constant $C_2$ depending only on $\beta$. 

Since (cf. (8))
\[ f(x) - P(x) = \sum_{j=0}^{m} \chi_j(x)(f(x) - P_j(x)) \]
we have to estimate \( f(x) - P_j(x) \) for \( j = k - 1, k, k + 1 \), as well.

Again, let \( x \in [x_k, x_{k+1}] \). If \( d(x) \leq n^{-1/\beta} \), then we just write
\[ |f(x) - P_j(x)| \leq \|f\|_{[0,1]} \leq e\|f\|_{[0,1]} \exp(-n[d(x)]^\beta) . \]

If, however, \( d(x) > n^{-1/\beta} \), then \( d(x) \sim 2d(x) \), which, together with (7) yields
\[ |f(x) - P_j(x)| \leq E_n(f)_{[x_k-2, x_{k+3}]} \]
\[ \leq E_n(f)_{[x_d(x)/4, x + d(x)/4]} \]
\[ \leq E_n(f)_{[x-d(x)/2, x+d(x)/2]} , \]
where \( E_m(f)_{[a,b]} \) denotes the error in best uniform approximation to \( f \) on \([a, b]\) out of \( \Pi_m \). To estimate the right-hand member of (13) consider the Taylor expansion of \( f \) about \( x \). For the absolute value of the \( \nu \)-th Taylor coefficient, Cauchy's inequality gives the bound
\[ \sup_{|z-x| \leq 3d(x)/4} |f(z)| \left( \frac{3}{4} d(x) \right)^{-\nu} \leq C(x) \left( \frac{3}{4} d(x) \right)^{-\nu} \]
and so the \( n_j \)-th partial sum approximates \( f \) on \([x - d(x)/2, x + d(x)/2]\) with error at most
\[ C(x) \sum_{\nu=n_j+1}^{\infty} \left( \frac{2}{3} \right)^{\nu} = 2C(x) \exp \left( \left( \log \frac{2}{3} \right) n_j \right) \leq 2C(x) \exp(-c_3 n[d(x)]^\beta) \]
with \( c_3 := (\log \frac{3}{2})c_1/1000 \) and where we used the fact that \( n_j > (c_1/1000)n[d(x)]^\beta \) whenever \( n_j \geq N_0 \) and \( N_0 \) satisfies \( (c_1/800)N_0^{1-\gamma/\beta} \geq 4 \).

Thus
\[ E_n(f)_{[x-d(x)/2, x+d(x)/2]} \leq 2C(x) \exp(-c_3 n[d(x)]^\beta) . \]

The relations (10)-(14) yield
\[ |f(x) - P_n^*(x)| = |f(x) - P(x)| \leq C_2\|f\| \exp(-c_2 n[d(x)]^\beta) + 3\|f\| \leq C_3\|f\| \]
and
\[ |f(x) - P_n^*(x)| = |f(x) - P(x)| \leq C_4(x) \exp(-c_3 n[d(x)]^\beta) , \]
which proves (3). Property (2) also can easily be obtained from (15). In fact, notice that every \( n_j \) is at least as large as
\[ \frac{c_1}{1000}n(d(x))^\gamma \geq \left[ \frac{c_1}{1000}n^{1-\gamma/\beta} \right] =: m_n , \]
and \( m_n \to \infty \) as \( n \to \infty \). If \( Q_{m_n} \) denotes the best polynomial approximation of \( f \) on \([0, 1]\) by polynomials of degree at most \( m_n \), then we get from \( n_j \geq m_n \)

\[
P_j(f; \cdot)[x_{j-1}, x_{j+1}] - Q_{m_n}(\cdot) \equiv P_j(f - Q_{m_n}; \cdot)[x_{j-1}, x_{j+1}],
\]

and so (cf. (8))

\[
f(\cdot) - \sum_{j=0}^{m} x_j(\cdot) P_j(f; \cdot) \equiv (f - Q_{m_n})(\cdot) \sum_{j=0}^{m} x_j(\cdot) P_j(f - Q_{m_n}; \cdot)
\]

Hence we obtain from (15), with \( f \) replaced by \( f - Q_{m_n} \) on the right, that

\[
|f(x) - P^*_n(x)| \leq C_3 \|f - Q_{m_n}\|_{[0,1]}
\]

Since \( \|f - Q_{m_n}\|_{[0,1]} \to 0 \) as \( n \to \infty \), we obtain (2). \( \square \)

3. Proof of Theorem 2

Suppose to the contrary that for any \( a \in (0, 1/2] \) the estimate

\[
|P_n(x) - |x| \leq C_a \exp(-2cn|x|), \quad x \in [-1, 1]\setminus[-a,a],
\]

is possible with some positive constants \( C_a \) and \( c \) and polynomials \( P_n \in \Pi_n \). Then (16) holds for \( (P_n(x) + P_n(-x))/2 \), as well, so we may assume each \( P_n \) to be even.

Consider the derivative \( P'_n \) of \( P_n \), which is odd. Let \( x \in [a, 1] \) and \( b := \min\{x, 1/2\} \). Since

\[
|P_n(u) - u| \leq C_a \exp(-2cnb)
\]

if \( u \in [b, 1] \), we get from Markoff's inequality, applied to the interval \([b, 1]\), that

\[
|P'_n(x) - 1| \leq 4C_a n^2 \exp(-2cnb)
\]

\[
\leq 4C_a n^2 \exp(-cnx), \quad a \leq x \leq 1.
\]

In a similar way

\[
|P'_n(x) + 1| \leq 4C_a n^2 \exp(-cn|x|), \quad -1 \leq x \leq -a,
\]

and so for \( a \leq |x| \leq 1 \) we have

\[
|P'_n(x) - \text{sign} x| \leq 4C_a n^2 \exp(-cn|x|),
\]

where \( C_a \) is independent of \( n \).

This implies that the polynomials

\[
Q_n(x) := 1 - (P'_n(\sqrt{x}))^2
\]

of degree at most \( n \) satisfy \( Q_n(0) = 1 \) and, for large \( n \),

\[
|Q_n(x)| \leq 8C_a n^2 \exp(-cn\sqrt{x}), \quad a \leq x \leq 1.
\]
4. Proof Theorem 3

Let $a_k := 2^{-k}(1 + i)$, where $i = \sqrt{-1}$, and set

$$f(z) := \sum_{k=1}^{\infty} n_k^{-1/8} [2^k(z - a_k)]^{-n_k} =: \sum_{k=1}^{\infty} g_k(z),$$

where $\{n_k\}$ will be completely specified later in the proof. We set $n_1 = 1$, and require $n_{k+1} > 2^k n_k$ for all $k$ so that for every $\ell > 0$ we have

$$n_k/2^{\ell k} \to \infty$$

Since

$$|[2^k(z - a_k)]^{-1}| \leq 1 \quad \text{if } |z - a_k| \geq 2^{-k},$$

it is easy to see that the series defining $f$ uniformly converges on the real line and also on every compact subset of the complex plane not containing the points $0, \{a_k\}_{k=1}^{\infty}$; thus these are the singular points of $f$. In particular,

$$d(x) \geq \frac{1}{\sqrt{2}} x \quad \text{for } x \in [0, 1].$$

Now we need some easy estimates:

$$|g_k(2^{-k})| = n_k^{-1/8},$$

$$|g_k(2^{-k} + n_k^{-1/4})| = n_k^{-1/8} (1 + 4^k n_k^{-1/2} - n_k/2) < \frac{1}{2} n_k^{-1/8}$$

and so

$$n_k^{1/4} |g_k(2^{-k}) - g_k(2^{-k} + n_k^{-1/4})| > \frac{1}{2} n_k^{1/8}.$$

Using this, by standard gliding-hump arguments we can select $n_1, n_2, \ldots$ one after the other in such a way that

$$n_k^{1/4} |f(2^{-k}) - f(2^{-k} + n_k^{-1/4})| > \frac{1}{4} n_k^{1/8}$$

is satisfied for all $k$.

Now if (4) is true for some $C$, $c > 0$ then

$$|f(x) - P_m(x)| \leq 2^k \exp(-2^{-k} m[d(x)]^\theta), \quad m = 1, 2, \ldots, \quad x \in [0, 1],$$

is also true for any large $k$, $k \geq \max\{\ln C; -\ln c\}/\ln 2$, and below we show that this is impossible.
Fix a $k$ so large that (20) is satisfied and also $(1/16)n_k^{-1/8} < 2^k \exp \{ -2^{-\beta/2} n_k^{1/16} \}$, which is always possible because of (17).

Set $m = [2^{k(1+\beta)} n_k^{1/16}] + 1$ into (20). Then for $x \in [2^{-k}, 1]$ we get from (18) and (20)

$$|f(x) - P_m(x)| \leq 2^k \exp(-2^{-k} 2^{k(1+\beta)} n_k^{1/16} (2^{-k}/\sqrt{2})^\beta) < \frac{1}{16} n_k^{-1/8}$$

therefore (cf. (19)),

$$n_k^{1/4} |P_m(2^{-k}) - P_m(2^{-k} + n_k^{-1/4})| > \frac{1}{8} n_k^{1/8}. \tag{21}$$

On the other hand, $\{P_m\}$ are uniformly bounded on $[0, 1]$, say $|P_m| \leq K$, $m = 1, 2, \ldots$, (cf. (4)); hence Bernstein's inequality shows that for $x \in [2^{-k}, 2^{-k} + n_k^{-1/4}]$,

$$|P'_m(x)| \leq Km2^k,$$

which implies

$$n_k^{1/4} |P_m(2^{-k}) - P_m(2^{-k} + n_k^{-1/4})| \leq Km2^k \leq 2K2^{k(2+\beta)} n_k^{1/16}. \tag{22}$$

However, (17), (21) and (22) are not compatible which proves our theorem. \qed

**REFERENCES**


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