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Zeros of expansions in orthogonal polynomials

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Abstract

The theory of bi-orthogonal polynomials is exploited to investigate the location of zeros of truncated expansions in orthogonal polynomials. It turns out that, subject to additional conditions, these zeros can be confined to certain real intervals. Two general techniques are being used: the first depends on a theorem that links strict sign consistency of a generating function to loci of zeros and the second consists of re-expression of transformations from [**3**] in an orthogonal basis.

1. *Introduction*

Expansions in orthogonal polynomials play an important role in many applications of mathematics and it is of some interest to be able to confine their zeros to specific portions of the complex plane. For this, of course, we require additional information on the expansion coefficients.

An early example of a result of this type, due to Turán [**11**], concerns expansions in Hermite polynomials $H_k(x)$. It states that the zeros of $\Sigma_{k=0}^{n} p_k H_k(x)$, where $p_0, \dots, p_n$ are real, $p_n = 1$, lie in the complex strip $\{z \in \mathbb{C} : |\mathrm{Im}\, z| < \tfrac{1}{2}(M^*+1)\}$, where $M^* := \max_{k=0,\dots,n-1} |p_k|$. Further results of this type are listed in Marden [**6**].

A simple, but apparently new, result on zeros of truncated expansions is presented now:

THEOREM 1. *Let the polynomial $\Sigma_0^n q_k x^k$ with real coefficients $q_0, \dots, q_n$ have all its zeros in the open complex unit disc. Then all the zeros of the polynomial $\Sigma_0^n q_k T_k(x)$, where $T_k(x) = \cos(k \cos^{-1} x)$ is the kth degree Chebyshev polynomial, lie in the interval $(-1, 1)$.*

Proof. Since $T_k(\cos\theta) = \cos k\theta$, the theorem follows by applying the argument principle to $q(z) := \Sigma_0^n q_k z^k$ along $|z| = 1$. As q has exactly n zeros in $|z| < 1$ and none on $|z| = 1$, it follows that its argument varies by $2n\pi$. Thus

$$\mathrm{Re}\, q(e^{i\theta}) = \sum_0^n q_k \cos k\theta$$

has exactly n zeros for $0 \leqslant \theta < \pi$. The statement of the theorem follows at once.

A similar result can be readily obtained, by an identical technique, for Chebyshev polynomials U_n of the second kind.

In this paper we present several results that relate loci of zeros of truncated expansions in orthogonal polynomials to certain properties of expansion coefficients. The common denominator of our results is that they are based on a method, introduced by Iserles and Nørsett [3], to locate zeros of some polynomials. We will now review briefly this method.

Let $\phi(x,\mu)$, $x\in(a,b)$, $\mu\in(c,d)$ be a distribution in x for all relevant values of the parameter μ. Further, assume that

$$d\phi(x,\mu) = \omega(x,\mu)\,d\alpha(x),$$

where $\omega(\cdot,\mu)$ is a $C^1[a,b]$, strictly sign-consistent (see Karlin and Studden [5]) function, whereas α is a distribution, independent of μ. Given n parameters $\mu_1,\ldots,\mu_n$ in (c,d), we say that the nth degree polynomial $p_n(x;\mu_1,\ldots,\mu_n)$, which is not identically zero, is *bi-orthogonal* if

$$\int_a^b p_n(x;\mu_1,\ldots,\mu_n)\,d\phi(x,\mu_l) = 0 \quad (l = 1,\ldots,n).$$

Bi-orthogonal polynomials are investigated in detail in Iserles and Nørsett [4]. In particular, it is demonstrated there that, subject to the above conditions, p_n (as a function of x) possesses exactly n zeros in the interval $[a,b]$. Let $\mathscr{T}$ be a transformation that maps the set of nth degree polynomials into itself in the following manner: given a polynomial q_n with the zeros $\mu_1,\ldots,\mu_n\in\mathbb{C}$ we set $\mathscr{T}\{q_n\} = p_n(\cdot;\mu_1,\ldots,\mu_n)$ (given p_n for $\mu_1,\ldots,\mu_n\in(c,d)$, the definition can be typically extended to $\mu_1,\ldots,\mu_n\in\mathbb{C}$, except, possibly, for a finite set of points). It is now clear that the transformation $\mathscr{T}$ maps polynomials with all their zeros in (c,d) into polynomials with all zeros confined to $[a,b]$. Moreover it is frequently possible to prove that the zeros of the transformed polynomial reside, in fact, in the open interval (a,b). Iserles and Nørsett [3] provide a long list of such transformations and a forthcoming paper will address itself to characterizing all transformations of this form, subject to some extra conditions.

This result on *zero-mapping transformations* is used in the present paper in two distinct ways. First, we establish a connection between a specific family of distributions ϕ and generating functions of orthogonal polynomial sequences, a connection that leads to a general theorem on zeros of truncated expansions in the underlying orthogonal polynomials. This is the theme of §2. In §3 we exploit this theorem to derive several results on specific sequences of orthogonal polynomials. For example, we prove the following results.

(i) If the polynomial $\sum_0^n q_k x^k$ has only real zeros, then this is also the case with the polynomial $\sum_0^n q_k H_k(x)$, where H_k is a Hermite polynomial.

(ii) If the polynomial $\sum_0^n q_k x^k$ has all its zeros in $(-1,1)$, then all the zeros of

$$\sum_0^n \frac{k!\,q_k}{(1+\alpha)_k} L_k^{(\alpha)}(x)$$

are non-negative. Here $L_k^{(\alpha)}$ is a Laguerre polynomial, and $\alpha > -1$.

(iii) If the polynomial $\sum_0^n q_k x^k$ has all its zeros in $(-1,1)$, then the expansion in ultraspherical polynomials

$$\sum_0^n \frac{k!\,(k+\alpha+\frac{1}{2})}{(\alpha+1)_k} q_k P_k^{(\alpha,\alpha)}(x)$$

has all its zeros in $[-1,1]$ for all $\alpha \geqslant -\frac{1}{2}$.

(iv) If all the zeros of $\Sigma_0^n q_k x^k$ reside in $(-\alpha, \alpha)$, then $\Sigma_0^n q_k C_k^{(\alpha)}(x)$ has positive zeros for all $\alpha > 0$. Here $C_k^{(\alpha)}$ is a Charlier polynomial (see Chihara [**1**]).

(v) Let $m_k(\cdot\,; \beta, c)$, $\beta, c > 0$, be a Meixner polynomial of the first kind (Chihara [**1**]). If all the zeros of $\Sigma_0^n q_k x^k$ are in $(-c, c)$, then all the zeros of

$$\sum_0^n \frac{c^k q_k}{(\beta)_k} m_k(x; \beta, c)$$

are positive.

(vi) Let $M_k(\cdot\,; \delta, \eta)$, where $\delta \in \mathbb{R}$ and $\eta > 0$, be a Meixner polynomial of the second kind (Chihara [**1**]). If all the zeros of $\Sigma_0^n q_k x^k$ reside in $(-1/|\delta|, 1/|\delta|)$, then all the zeros of

$$\sum_0^n \frac{q_k}{(\delta^2+1)^k (\eta)_k} M_k(x; \delta, \eta)$$

are real.

(vii) Let $\alpha < 0$, $q \in (0,1)$ and $U_k^{(\alpha)}$ be a polynomial of Al-Salam and Carlitz (Chihara [**1**]). Then, given that all the zeros of $\Sigma_0^n q_k x^k$ are in $(0, \min\{1, 1/|\alpha|\})$, all the zeros of

$$\sum_0^n q^{-\frac{1}{2}(k-1)k} (-\alpha)^{n-k} U_k^{(\alpha)}(x)$$

are in $[\alpha, 1]$.

(viii) If all the zeros of the polynomial $\Sigma_0^n q_k x^k$ are in $(0,1)$, then all the zeros of

$$\sum_0^n q_k q^{-(k+1)k} \beta^{n-k} W_k(x; \beta, q)$$

reside in $[0, 1]$. Here $W_k(\cdot\,; \beta, q)$, $\beta, q \in (0,1)$ is a Wall polynomial (Chihara [**1**]).

Another mechanism for generation of transformations of similar type is explored in §4. It is based upon the manipulation of results of Iserles and Nørsett [**4**] and re-expression of transformations thereof in a different basis. This leads to further results, of slightly different flavour.

(ix) If the polynomial $\Sigma_0^n (\beta_k + 1 - x)_k \eta_k / k!$ has only positive zeros, then the same is true for $\Sigma_0^n L_k^{(\beta_k)}(x) \eta_k$, where $L_k^{(\alpha)}$ is a Laguerre polynomial and $\beta_0, \ldots, \beta_n > -1$.

(x) Let $\alpha, \beta_0, \ldots, \beta_n > -1$ and the polynomial $\Sigma_0^n (\beta_k + 1 - x)_k \eta_k / k!$ have only positive zeros. Then all the zeros of

$$\sum_0^n (-1) \left(\frac{1-x}{2}\right)^{n-k} P_k^{(\alpha+n-k, \beta_k)}(x)\, \eta_k$$

reside in $(-1, 1)$. Here $P_k^{(\alpha_1, \alpha_2)}$ denotes a Jacobi polynomial.

(xi) Given that $\alpha_0, \ldots, \alpha_n > 0$ and $\Sigma_0^n \eta_k (x - \alpha_k)^k$ has only positive zeros, then also all the zeros of $\Sigma_0^n \eta_k C_k^{(\alpha_k)}(x)$ are positive. Here $C_k^{(\alpha)}$ is a Charlier polynomial.

(xii) Let $c \in (0,1)$, $\beta > 0$ and $m_k(\cdot\,; \beta, c)$ be a Meixner polynomial of the first kind. If all the zeros of $\Sigma_0^n \eta_k (x - \beta c)^k$ reside in $(0, \beta)$, then all the zeros of

$$\sum_0^n (-1)^k (x+\beta)_k\, m_{n-k}(x; \beta + k, c)\, \eta_{n-k} (\beta c)^{n-k}$$

are positive.

2. *Generating functions and transformations*

Let ψ be a distribution with a supporting set $\mathscr{E} = (a, b)$ (Chihara [1]) and let $\{P_n\}_{n=0}^{\infty}$ be a set of orthogonal polynomials with respect to ψ,

$$\int_{\mathscr{E}} P_n(x) P_m(x)\, d\psi(x) = \begin{cases} 0 & (m \neq n), \\ f_n \neq 0 & (m = n), \end{cases}$$

$$P_n(x) = \kappa_n x^n + \dots, \quad \kappa_n \neq 0.$$

Further, we assume that a generating function

$$G(x, \mu) = \sum_{n=0}^{\infty} \delta_n P_n(x) \mu^n \quad (\delta_0, \delta_1, \ldots \neq 0),$$

exists, is convergent and *strictly sign consistent* (*SSC*) for all $x \in \mathscr{E}$ and $\mu \in \mathscr{D} := (c, d)$, say, where $d > c$. By strict sign consistency we mean that, for every $n \geq 1$ and all monotone sequences $x_1 < x_2 < \ldots < x_n \in \mathscr{E}$ and $\mu_1 < \mu_2 < \ldots < \mu_n \in \mathscr{D}$ the determinant

$$\det \begin{bmatrix} G(x_1, \mu_1) & G(x_1, \mu_2) & & G(x_1, \mu_n) \\ \vdots & \vdots & & \\ G(x_n, \mu_1) & G(x_n, \mu_2) & \ldots & G(x_n, \mu_n) \end{bmatrix}$$

is non-zero and of a sign that depends on n, but not on the choice of the two monotone sequences. Strict sign consistency is surveyed at length in Karlin and Studden [5].

We set

$$d\phi(x, \mu) := G(x, \mu)\, d\psi(x) \quad (x \in \mathscr{E}, \mu \in \mathscr{D})$$

and consider the underlying set of bi-orthogonal polynomials $\{p_n\}$. Let

$$\rho_n(x) := \frac{1}{\kappa_n} P_n(x)$$

and $\rho := \{\rho_n\}_0^{\infty}$. The intermediate goal being to find the explicit form of p_n as a linear combination of the ρ_k's, we next evaluate the generalized moments with respect to ρ:

$$\begin{aligned} L_n(\mu, \rho) :&= \int_{\mathscr{E}} \rho_n(x)\, d\phi(x, \mu) \\ &= \sum_0^{\infty} \frac{\delta_k}{\kappa_n} \int_{\mathscr{E}} P_k(x) P_n(x)\, d\psi(x)\, \mu^k \\ &= \frac{\delta_n f_n}{\kappa_n} \mu^n \quad (n = 0, 1, \ldots). \end{aligned}$$

It has been proved in Iserles and Nørsett [3] and is easy to verify that, given $L_n(\mu, \rho) = \sigma_n \mu^n$, where $\sigma_n \neq 0$ for $n = 0, 1, \ldots$, the explicit form of the nth bi-orthogonal polynomial is, up to a non-zero multiplicative constant,

$$p_n(x; \mu_1, \ldots, \mu_n) = \sum_{k=0}^{n} \frac{q_k}{\sigma_k} \rho_k(x),$$

where

$$\sum_{k=0}^{n} q_k x^k = \prod_{k=1}^{n} (x - \mu_k).$$

Hence, substituting the known values of the σ_k's,

$$p_n(x;\mu_1,\ldots,\mu_n) = \sum_0^n \frac{\kappa_k q_k}{\delta_k f_k}\rho_k(x)$$
$$= \sum_0^n \frac{q_k}{\delta_k f_k} P_k(x).$$

It now follows at once from Iserles and Nørsett [3], as discussed in §1, that

THEOREM 2. *Given that the generating function $G(x,\mu)$ is SSC for all $x \in \mathscr{E}$ and $\mu \in \mathscr{D}$, and that all the zeros of the polynomial $\sum_{k=0}^n q_k x^k$ are in the interval $\mathscr{D}$, all the zeros of the polynomial $\sum_{k=0}^n q_k P_k(x)/(\delta_k f_k)$ reside in the closure of $\mathscr{E}$.*

The last theorem is crucial in deriving the transformations of §3. Given an arbitrary distribution ψ and a generating function G, the task in hand is twofold: (*a*) determine the range of μ, necessarily symmetric with respect to the origin, such that G converges; (*b*) check whether G is SSC for all (or part) of the range of convergence.

Determination of the range of convergence is relatively straightforward, bearing in mind that, if G is analytic (in μ) in the neighbourhood of the origin then it remains analytic (and the series converges) within the open complex disc of radius $r > 0$, r being the distance of the nearest singularity from the origin.

Verification of SSC is much more intricate and has been treated (in §3) on a case-by-case basis. Nonetheless, several criteria for strict sign consistency, or, to be precise, for a slightly stronger concept of *strict total positivity* (*STP*), whereby the determinant (2·1) is positive, are very useful within this framework.

Criterion (*a*). The function $G(x,y) = e^{xy}$ is STP for all $x, y \in \mathbb{R}$ (see Karlin and Studden [5]).

Criterion (*b*). The *composition rule*: given that $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ are non-empty open intervals, that G and H are STP for $(x,y) \in \mathscr{A} \times \mathscr{B}$ and $(x,y) \in \mathscr{B} \times \mathscr{C}$, respectively, and that ζ is a distribution with support in $\mathscr{B}$, the function

$$\int_{\mathscr{B}} G(x,\tau)\, H(\tau,y)\, d\zeta(\tau)$$

is STP for all $(x,y) \in \mathscr{A} \times \mathscr{C}$ (see Pólya and Szegö [7])

Criterion (*c*). Let a be monotone and f positive and non-increasing for all $x > 0$. Then

$$G(x,y) = (a(x)+a(y)+1)\int_0^\infty \tau^{a(x)+a(y)} f(\tau)\, d\tau$$

is STP for $x, y > 0$ (see Pólya and Szegö [7]).

3. *Zeros of expansions obtained from a generating function*

(i) *The Hermite distribution.* Let

$$d\psi(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}\, dx, \quad \mathscr{E} = (-\infty,\infty).$$

Thus $P_n \equiv H_n$, a *Hermite polynomial*, and, setting $\delta_n = 1/n!$ for $n = 0, 1, \ldots$ we obtain the well known generating function

$$G(x,\mu) = e^{2x\mu-\mu^2}$$

(see Rainville [7]). It is clear that the series for G converges for all $x, \mu \in \mathbb{R}$. Moreover, it follows easily from Criterion (*a*) in §2 that it is STP there. Since $f_n = 2^n n!$, we obtain

PROPOSITION 1. *If the polynomial $\Sigma_0^n q_k x^k$ has only real zeros, then so has the polynomial $\Sigma_0^n q_k H_k(x)$.*

Proof. Since $\Sigma_0^n q_k (2x)^k$ has only real zeros, the proposition is a straightforward consequence of Theorem 2.

Note the different result of Turán (already mentioned in §1).

Proposition 1 can be recast in a slightly different manner. Let D: d/dx be the differential operator. It is easy to verify that, formally,

$$e^{-D^2} \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k H_k\left(\frac{x}{2}\right)$$

Thus we have

PROPOSITION 1 A. *If the polynomial q has only real zeros, then so has the polynomial $e^{-D^2} q$.*

COROLLARY 1 B. *If f is a function in the Laguerre–Pólya class, then $e^{-D^2} f$ has only real zeros.*

Proof. Since f is in the Laguerre–Pólya class, it is a real entire function that is the uniform limit on compact subsets of the plane of polynomials $\{Q_n\}$ with only real zeros. The corollary then follows on applying Proposition 1 A to the Q_n's and appealing to Hurwitz's theorem (cf. Hille [2]) on the zeros of limits of analytic functions.

(ii) *The Laguerre distribution.* We choose

$$d\psi(x) = x^{\alpha} e^{-x}\, dx, \quad \alpha > -1, \quad \mathscr{E} = (0, \infty).$$

This leads to the (generalized) *Laguerre polynomials*, with the classical generating function

$$G(x,\mu) = \sum_0^{\infty} L_k^{(\alpha)}(x)\, \mu^k = (1-\mu)^{-1-\alpha} e^{-x\mu/(1-\mu)},$$

which is convergent for all $|\mu| < 1$. Thus $\delta_n \equiv 1$. The above generating function is SSC for all $x > 0$ and $-1 < \mu < 1$: the positive factor in front, independent of x, makes no difference to total positivity and our assertion follows at once from Criterion (*a*) of §2. Since $f_n = \Gamma(\alpha+1)(\alpha+1)_n/n!$, where Γ is the gamma function and $(a)_n$ is a Pochhammer symbol (see Rainville [8]), we obtain

PROPOSITION 2. *If all the zeros of $\Sigma_0^n q_k x^k$ are in $(-1, 1)$, then the polynomial $\Sigma_0^n k!\, q_k L_k^{(\alpha)}(x)/(\alpha+1)_k$ has only non-negative zeros.*

Note that if $\alpha = 0$ (the 'simple' Laguerre polynomial) then the zeros are, in fact, positive: this follows readily from the fact that $(1)_k = k!$ and $L_k^{(0)}(0) = 1$ and because $\Sigma\, q_k x^k$ does not vanish at 1.

(iii) *The ultraspherical (Gegenbauer) distribution*. Given $\alpha > -1$, we choose

$$d\psi(x) = (1 \quad x^2)^\alpha \, dx, \quad \mathscr{E} = \quad -1, 1)$$

This leads to the *ultraspherical polynomials* (i.e. Jacobi polynomials with equal parameters) $P_n^{(\alpha,\alpha)}$ or, with different normalization, to Gegenbauer polynomials (see Rainville [**8**]).

We have

$$G(x,\mu) \quad \sum_{n=0}^{\infty} \frac{(1+2\alpha)_n}{(1+\alpha)_n} P_n^{(\alpha,\alpha)}(x)\,\mu^n \quad \frac{1}{(1-2x\mu+\mu^2)^{\alpha+\frac{1}{2}}},$$

$$\int_{-1}^{1} (1 \quad x^2)^\alpha [P_n^{(\alpha,\alpha)}(x)]^2 \, dx \quad 2^{1+2\alpha} \frac{(\Gamma(1+\alpha+n))^2}{n!(1+2\alpha+2n)\,\Gamma(1+2\alpha+n)},$$

hence

$$\frac{1}{\delta_{k,k}} - \frac{c_\alpha k!}{(1+\alpha)_k}\left(k+\alpha+\tfrac{1}{2}\right), \quad \text{where} \quad c_\alpha \quad \frac{\Gamma(\alpha+\frac{1}{2})}{(\sqrt{\pi})\,\Gamma(\alpha+1)}$$

We will now demonstrate that, given $\alpha > -\frac{1}{2}$, the function G is STP for all $|\mu| < 1$ The generating function is

$$G(x,\mu) \quad G_\alpha(x,\mu) \quad \frac{1}{(1+\mu^2)^{\alpha+\frac{1}{2}}(1-(2\mu x)/(1+\mu^2))^{\alpha+\frac{1}{2}}}$$

Set $y := 2\mu/(1+\mu^2)$, $|y| < 1$, and perform the transformation $x \mapsto (x \quad 1)/(x+1)$, $y \mapsto (y-1)/(y+1)$. Then G_α is STP if

$$\left(\frac{1}{2}\frac{(x+1)(y+1)}{x+y}\right)^\beta, \quad \text{where} \quad \beta = \alpha+\tfrac{1}{2},$$

is STP for $x > 0$, $y > 0$, which, in turn, is equivalent to the condition that

$$\tilde{G}_\beta(x,\mu) \quad \frac{1}{(x+\mu)^\beta}$$

is STP for $x, \mu > 0$. We now set

$$a(x) := x \quad \tfrac{1}{2}, \quad (x > 0)$$

$$f(x) := \begin{cases} (-\ln x)^\beta/\Gamma(\beta+1 & (0 < x < 1) \\ 0 & (1 \leqslant x), \end{cases}$$

and use Criterion (*c*) from §2: a is monotone and f weakly monotonically decreasing within the range for all $\beta > 0$; thus

$$\frac{1}{\Gamma(\beta+1)}(x+\mu)\int_0^1 t^{x+\mu-1} \quad \ln t)^\beta \, dt = \frac{x+\mu}{\Gamma(\beta+1)}\int_0^\infty e^{-(x+\mu)\tau}\tau^\beta \, d\tau$$

$$\frac{1}{\Gamma(\beta+1)(x+\mu)^\beta}\int_0^\infty \tau^\beta e^{-\tau} \, d\tau$$

$$\frac{1}{(x+\mu)^\beta} = \tilde{G}_\beta(x,\mu),$$

and $\tilde{G}$ is, indeed, STP for all $\beta > 0$. Hence we have

PROPOSITION 3. *Given any* $\alpha \geqslant -\frac{1}{2}$, *if the polynomial* $\Sigma_0^n q_k x^k$ *has all its zeros in* $(-1, 1)$, *then*

$$\sum_{k=0}^{n} \frac{k!}{(\alpha+1)_k}(k+\alpha+\tfrac{1}{2})\, q_k P_k^{(\alpha,\alpha)}(x)$$

has all its zeros in $[-1, 1]$.

Proof. The proposition follows from the above discussion whenever $\alpha > -\frac{1}{2}$. The borderline case $\alpha = -\frac{1}{2}$, corresponding to Chebyshev polynomials, has been already dealt with in Theorem 1.

The case when $\alpha < -\frac{1}{2}$ must be considered, for the time being, as open: it is easy to see that G cannot be STP, but there are some indications that it is, nonetheless, SSC.

(iv) *The Charlier distribution*. Let α be a positive number and ψ be a step function with jumps of $e^{-\alpha}\alpha^l/l!$ at $l = 0, 1, \ldots$, and let $\mathscr{E} = (0, \infty)$. The underlying orthogonal polynomials are the *Charlier polynomials* $C_n^{(\alpha)}$ (see Chihara [1]).

PROPOSITION 4. *If all the zeros of the polynomial* $\Sigma_0^n q_k x^k$ *are in the interval* $(-\alpha, \alpha)$, *then the polynomial* $\Sigma_0^n q_k C_k^{(\alpha)}(x)$ *has all zeros positive*.

Proof. We have

$$G(x, \mu) = e^{-\alpha\mu}(1+\mu)^x = \sum_{n=0}^{\infty} \frac{1}{n!} C_n^{(\alpha)}(x)\, \mu^n$$

(from Chihara [1]), which is convergent and STP (by Criterion (a)) for all $|\mu| < 1$. Thus, by Theorem 2 and as $f_k = k!\alpha^k$ for $k = 0, 1, \ldots$, if $\Sigma q_k x^k$ has all its zeros in $\mathscr{D} = \{x \in \mathbb{R} : |x| < \alpha\}$ then $\Sigma q_k(\alpha x)^k$ has all its zeros in $(-1, 1)$, implying that all the zeros of $\Sigma q_k C_k^{(\alpha)}(x)$ are non-negative. Moreover $C_k^{(\alpha)}(0) = (-\alpha)^k$. Thus $\Sigma q_k C_k^{(\alpha)}$ may not vanish at $x = 0$, since this would have been equivalent to a zero of $\Sigma q_k x^k$ being on the boundary of the open interval that, by assumption, contains all its zeros. This completes the proof.

(v) *The Meixner distribution of the first kind*. We at present let $c \in (0, 1)$, $\beta > 0$ and define ψ as a step function with jumps of $c^l(\beta)_l/l!$ at $l = 0, 1, \ldots$, letting $\mathscr{E} = (0, \infty)$. This leads to the *Meixner polynomials of the first kind* $m_n(x; \beta, c)$ (see Chihara [1]).

It is known that $f_k = (1-c)^{-\beta} c^{-k} k!(\beta)_k$ for $k = 0, 1, \ldots$, and that, with $\delta_k = 1/k!$,

$$G(x, \mu) = \left(1 - \frac{\mu}{c}\right)^x (1-\mu)^{-x-\beta},$$

is convergent for all $|\mu| < c$. Strict sign consistency follows readily from Criterion (a) by the change of variable

$$\bar{\mu} = \ln\left(\frac{1-\mu/c}{1-\mu}\right).$$

As zeros of the expansion are prevented from migrating to the origin by an argument identical to that for Charlier polynomials, we have

PROPOSITION 5. *If all the zeros of* $\Sigma_0^n q_k x^k$ *reside in* $(-c, c)$, *then all the zeros of* $\Sigma_0^n c^k q_k m_k(x; \beta, c)/(\beta)_k$ *are positive*.

(vi) *The Meixner distribution of the second kind*. Now δ is real, η is positive, $\mathscr{E} \equiv \mathbb{R}$ and

$$d\psi(x) = \Gamma\left(\frac{\eta}{2}\right)^{-2} \left|\Gamma\left(\frac{\eta+ix}{2}\right)\right|^2 e^{-x\tan^{-1}\delta}\, dx.$$

The underlying orthogonal system comprises *Meixner polynomials of the second kind* $M_n(x;\delta,\eta)$ (see Chihara[1]). It is known that $f_k = (1+\delta^2)^k\, k!(\eta)_k \int_{-\infty}^{\infty} d\psi(x)$ for $k = 0, 1, \ldots$ and that

$$\sum_{n=0}^{\infty} \frac{1}{n!} M_n(x;\delta,\eta)\,\mu^n = ((1+\delta\mu)^2+\mu^2)^{-\frac{1}{2}\eta}\, e^{x\tan^{-1}(\mu/(1+\delta\mu))}.$$

The generating function is convergent and SSC for all $|\delta\mu| < 1$: the second statement follows in a similar manner to that on Meixner polynomials of the first kind. Therefore we have established

PROPOSITION 6. *If all the zeros of* $\Sigma_0^n q_k x^k$ *reside in* $\quad 1/|\delta|, 1/|\delta|)$, *then all the zeros of*

$$\sum_{0}^{n} \frac{1}{(\delta^2+1)^k\,(\eta)_k} q_k M_k(x;\delta,\eta)$$

are real.

(vii) *The Al-Salam–Carlitz distribution.* Let $\alpha < 0$, $q \in (0,1)$ and ψ be a step function with jumps of

$$\frac{(\alpha q;q)_\infty\, q^l}{[q]_l (q/\alpha;q)_l}$$

at q^l for $l \quad 0, 1, \ldots$ and jumps of

$$\frac{\alpha(q/\alpha;q)_\infty\, q^l}{[q]_l (\alpha q;q)_l}$$

at αq^l for $l = 0, 1, \ldots$. Here $(a;q)_m = \Pi_{k=0}^{m-1}(1-q^k a)$ is the *Gauss–Heine* symbol and $[q]_m \equiv (q;q)_m$ (see Slater[9]). This produces the polynomials $U_n^{(\alpha)}$ of *Al-Salam and Carlitz* (see Chihara[1]). It is known that

$$f_k = C(\quad \alpha)^k\, q^{\frac{1}{2}(k-1)k}\, [q]_k \quad (k \quad 0, 1, \ldots),$$

where C is independent of k. Moreover

$$\sum_{n=0}^{\infty} \frac{1}{[q]_n} U_n^{(\alpha)}(x)\,\mu^n = \frac{(\mu;q)_\infty\,(\alpha\mu;q)_\infty}{(x\mu;q)_\infty}.$$

The generating function converges for all $\alpha < x < 1$ and $|\mu| < \min\{1, 1/|\alpha|\}$. Its strict total positivity within this range of x and for $\mu > 0$ can be proved from Criterion (*c*), by an approach that has been presented in Iserles and Nørsett[3]. We again use Theorem 2 in order to derive

PROPOSITION 7. *If* $\Sigma_0^n q_k x^k$ *has all its zeros in* $(0, \min\{1, 1/|\alpha|\})$, *then all the zeros of*

$$\sum_{0}^{n} q^{-\frac{1}{2}(k-1)k}(-\alpha)^{n-k}\, U_k^{(\alpha)}(x)$$

reside in $[\alpha, 1]$.

Note that, in contrast to the previous examples, the range of convergence and the range of strict total positivity of the generating function are different here: our statement holds, of course, for the intersection of these intervals.

(viii) *The Wall distribution.* Let β and q be two numbers in $(0,1)$ and ψ be a step function with jumps of

$$\frac{\beta^l}{(\beta;q)_\infty\, [q]_l}$$

at q^{l+1} for $l = 0, 1, \ldots$. This leads to the *Wall polynomials* $W_n(x;\beta,q)$ (see Chihara [1]), with $f_k = (\beta;q)_k[q]_k\beta^k q^{k(k+1)}$ for $k = 0, 1, \ldots$. Moreover it is known that setting $\delta_n = 1/((\beta;q)_n[q]_n)$ for $n = 0, 1, \ldots$ yields

$$G(x,\mu) = (\mu;q)_\infty\, {}_0\Phi_1\left[\begin{matrix}-; \\ \beta;\end{matrix}\quad q, x\mu\right],$$

where ${}_0\Phi_1$ is a *q-hypergeometric function* (see Slater [9]). The series converges for $|\mu| < 1$. To prove strict total positivity (for $\mu > 0$) we refer to lemma 25 in Iserles and Nørsett [3], where we replace in the proof the function ϕ by a step function with jumps of δ_l at q^l for $l = 0, 1, \ldots$. This leads at once to

PROPOSITION 8. *If all the zeros of* $\sum_0^n q_k x^k$ *reside in* $(0, 1)$, *then all the zeros of*

$$\sum_0^n q_k q^{-(k+1)k} \beta^{n-k} W_k(x;\beta,q)$$

are in $[0, 1]$

4. *Zeros of expansions and Iserles–Nørsett transformations*

The paper of Iserles and Nørsett, introducing the technique of bi-orthogonality to analyse zeros of transformed polynomials, presents a long list of transformations. Some of these transformations can be re-expressed in a different form, yielding statements on zeros of truncated expansions in orthogonal polynomials

(ix) *The Laguerre transformation.* Theorem 9 in Iserles and Nørsett [3] states that the transformation

$$\mathscr{T}\left\{\sum_{k=0}^{n}(x)_k r_k\right\} = \sum_{k=0}^{n} r_k x^k$$

maps polynomials with positive zeros into polynomials with positive zeros. We now re-express $\sum_0^n (x)_k r_k$ in a different basis:

LEMMA 3. *Given* $\beta_0, \beta_1, \quad, \beta_n > -1$, *if*

$$\sum_{k=0}^{n}(x)_k r_k = \sum_{k=0}^{n}\frac{(\beta_k+1-x)_k}{k!}\eta_k \quad (x\in\mathbb{R}),$$

where $\eta_0, \ldots, \eta_n$ *are given, then it is true that*

$$r_l = (-1)^l \sum_{k=l}^{n}\binom{k}{l}\frac{(\beta_k+l+1)_{k-l}}{k!}\eta_k \quad (l = 0, 1, \ldots, n)$$

Proof. We substitute the postulated values of $r_0, \ldots, r_n$ and apply the Vandermonde theorem (see for example Slater [9]) to sum ${}_2F_1$ hypergeometric series with unit argument:

$$= \sum_{k=0}^{n}\frac{(\beta_k+1)_k}{k!}\,{}_2F_1\left[\begin{matrix}-k, x; \\ \beta_k+1;\end{matrix}\ 1\right]\eta_k$$

$$= \sum_{k=0}^{n}\frac{(\beta_k+1-x)_k}{k!}\eta_k,$$

and the lemma is true.

Proposition 9. *If $\sum_0^n (\beta_k+1-x)_k \eta_k/k!$ has only positive zeros, then so does $\sum_0^n L_k^{(\beta_k)}(x)\,\eta_k$.*

Proof. We substitute the values for $r_0, \ldots, r_n$ from Lemma 3

$$\sum_{l=0}^{n} r_l x^l = \sum_{k=0}^{n} \frac{(\beta_k+1)_k}{k!}\,{}_1F_1\left[\begin{matrix}-k; \\ \beta_k+1;\end{matrix}\; x\right]\eta_k = \sum_{k=0}^{n} L_k^{(\beta_k)}(x)\,\eta_k,$$

where we have used the standard representation of a Laguerre polynomial as a confluent hypergeometric function (see Rainville [**8**]).

Corollary. *If all the zeros of $\sum_0^n (-x)_k \eta_k/k!$ reside in $(-1-\beta, \infty)$ for some $\beta > -1$, then all the zeros of $\sum_0^n L_k^{(\beta)}(x)\,\eta_k$ are positive.*

Proof. This follows on setting $\beta_k \equiv \beta$ in the last proposition.

Proposition 9 is sharp, in the following sense. Let

$$q(x) := \sum_{k=0}^{n} \frac{(\beta_k+1-x)_k}{k!}\eta_k \quad \text{and} \quad p(x) := \sum_{k=0}^{n} L_k^{(\beta_k)}(x)\,\eta_k.$$

Then

$$p(0) = \sum_0^n L_k^{(\beta_k)}(0)\,\eta_k = \sum_0^n \frac{(\beta_k+1)_k}{k!}\eta_k = q(0)$$

and a real zero of p can migrate outside $(0, \infty)$ if and only if this is also the case with a zero of q.

(x) *The Jacobi transformation.* Let $\alpha > -1$. Then from Iserles and Nørsett [3] the transformation

$$\mathscr{T}\left\{\sum_{k=0}^{n} (x)_k r_k\right\} = \sum_{k=0}^{n} (-1)^k (-n-\alpha)_k x^k (1-x)^{n-k} r_k \tag{4·1}$$

maps polynomials with positive zeros into polynomials with zeros in $(0, 1)$.

Proposition 10. *Given $\alpha, \beta_0, \beta_1, \ldots, \beta_n > -1$, if the polynomial*

$$q(x) = \sum_0^n \frac{(\beta_k+1-x)_k}{k!}\eta_k$$

has only positive zeros, then all the zeros of

$$p(x) := \sum_{k=0}^{n} (-1)^k \left(\frac{1-x}{2}\right)^{n-k} P_k^{(\alpha+n-k,\, \beta_k)}(x)\,\eta_k$$

are in $(-1, 1)$.

Proof. By Lemma 3

$$\sum_0^n (x)_k r_k = \sum_0^n \frac{(\beta_k+1-x)_k}{k!}\eta_k$$

and hence $\quad r_l = (-1)^l \sum_{k=l}^{n} \binom{k}{l} \frac{(\beta_k+l+1)_{k-l}}{k!}\eta_k \quad (l = 0, 1, \ldots, n).$

Thus, by the Vandermonde theorem,

$$\sum_{l=0}^{n} (-1)^l (-\alpha-n)_l x^l (1-x)^{n-l} r_l$$

$$= (1-x)^n \sum_{k=0}^{n} \frac{(\beta_k+1)_k}{k!} {}_2F_1\left[\begin{matrix} -k, -\alpha-n; \\ \beta_k+1; \end{matrix} \quad \frac{x}{x-1}\right] \eta_k$$

$$= \sum_{k=0}^{n} (-1)^k (1-x)^{n-k} P_k^{(\alpha+n-k,\beta_k)}(2x-1)\, \eta_k,$$

where we have used one of the standard forms of a Jacobi polynomial as a terminating ${}_2F_1$ hypergeometric series (see Rainville [**8**]). The desired result follows at once from (4·1) on using the transformation $(2x-1) \mapsto x$.

Note that, again, the result is sharp, in the sense that

$$p(-1) = \sum_{0}^{n} (-1)^k P_k^{(\alpha+n-k,\beta_k)}(-1)\, \eta_k = \sum_{0}^{n} \frac{(1+\beta_k)_k}{k!} \eta_k = q(0),$$

$$p(1) = (-1)^n P_n^{(\alpha,\beta_n)}(1)\, \eta_n \neq 0.$$

A special case of the last proposition is of interest, since it involves a hitherto unknown summation formula for Jacobi polynomials. We set $\beta_k \equiv \beta$, $\eta_k \equiv 1$. It is easy to verify by induction on n that, for every $c \in \mathbb{C}$,

$$\sum_{k=0}^{n} \frac{(c)_k}{k!} = \frac{(c+1)_n}{n!} \tag{4·2}$$

$$q(x) = \sum_{0}^{n} \frac{(\beta+1-x)_k}{k!} = \frac{(\beta+2-x)_n}{n!},$$

with positive zeros. Let $\zeta := \frac{1}{2}(1-x)$. Then from Rainville [**8**],

$$p(x) = \sum_{0}^{n} (-1)^k \zeta^{n-k} P_k^{(\alpha+n-k,\beta)}(1-2\zeta)$$

$$= \sum_{0}^{n} (-1)^k \zeta^{n-k} \frac{(1+\alpha+n-k)_k}{k!} {}_2F_1\left[\begin{matrix} -k, 1+\alpha+\beta+n; \\ 1+\alpha+n-k; \end{matrix} \quad \zeta\right]$$

$$= (1+\alpha)_n \sum_{k=0}^{n} \frac{(-1)^k}{k!} \sum_{l=0}^{k} (-1)^l \binom{k}{l} \frac{(1+\alpha+\beta+n)_l}{(1+\alpha)_{n-k+l}} \zeta^{n-k+l}$$

$$= (-1)^n (1+\alpha)_n \sum_{l=0}^{n} \frac{(-1)^l}{(1+\alpha)_l (n-l)!} \left(\sum_{k=0}^{l} \frac{(1+\alpha+\beta+n)_k}{k!} \right) \zeta^l.$$

We again use (4·2), to produce

$$p(x) = (-1)^n \frac{(1+\alpha)_n}{n!} \sum_{0}^{n} (-1)^l \binom{n}{l} \frac{(2+\alpha+\beta+n)_l}{(1+\alpha)_l} \zeta^l = (-1)^n P_n^{(\alpha,\beta+1)}(x)$$

Obviously all the zeros of $P_n^{(\alpha,\beta+1)}$ reside in $(-1,1)$: one does not need Proposition 10 to prove this! However it follows from our proof, by re-substituting ζ for $\frac{1}{2}(1-x)$, that

$$\sum_{k=0}^{n} \left(\frac{x-1}{2}\right)^{n-k} P_k^{(\alpha+n-k,\beta)}(x) = P_n^{(\alpha,\beta+1)}(x), \tag{4·3}$$

a formula that might be of some interest.

(xi) *The Charlier transformation.* Another transformation from Iserles and Nørsett [3] is

$$\mathscr{T}\left\{\sum_{k=0}^{n} q_k x^k\right\} = \sum_{k=0}^{n} \; -1)^k q_k(-x)_k;$$

this transforms polynomials with positive zeros to polynomials with positive zeros.

PROPOSITION 11. *Let $\alpha_0, \ldots, \alpha_n > 0$ be given and $q(x) = \Sigma_0^n \eta_k (x-\alpha_k)^k$ have only positive zeros. Then so has $p(x) = \Sigma_0^n \eta_k C_k^{(\alpha_k)}(x)$, where $C_k^{(\alpha)}$ is a Charlier polynomial.*

Proof. It is obvious that $q(x) = \Sigma_0^n q_l x^l$, where

$$q_l = \sum_{k=l}^{n} (-1)^{k-l} \binom{k}{l} \eta_k \alpha_k^{k-l} \quad (l = 0, 1, \ldots, n).$$

$$\mathscr{T} q(x) = p(x) = \sum_0^n (-1)^l (-x)_l q_l = \sum_{k=0}^{n} (-1)^k \left(\sum_{l=0}^{k} \binom{k}{l} (-x)_l \alpha_k^{k-l}\right) \eta_k = \sum_0^n \eta_k C_k^{(\alpha_k)}(x)$$

Again, the result is sharp

$$p(0) = \sum_0^n \eta_k C_k^{(\alpha_k)}(0) = \sum_0^n (-\alpha_k)^k \eta_k = q(0).$$

Let $\alpha_k \equiv \alpha$ and $q(x) = (x-\alpha-\beta)^n$, where $\alpha+\beta > 0$. In other words

$$\eta_k = (-1)^{n-k} \binom{n}{k} \beta^{n-k} \quad (k = 0, \ldots, n)$$

$$C_n^{(\alpha)}(x) = \sum_{k=0}^{n} \; -1)^k \binom{n}{k} (-x)_k (-\alpha)^{n-k}$$

from Chihara [1], we have

$$\begin{aligned} p(x) &= \sum_{l=0}^{n} \binom{n}{l} C_l^{(\alpha)}(x) (-\beta)^{n-l} \\ & \sum_{k=0}^{n} (-1)^k \frac{n!}{k!} (-x)_k \sum_{l=k}^{n} \frac{(-1)^{n-l}}{(l-k)!\,(n-l)!} \beta^{n-l} \; -\alpha)^{l-k} \\ &= (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-x)_k \beta^{n-k} \sum_{l=0}^{n-k} \binom{n-k}{l} \left(\frac{\alpha}{\beta}\right)^l \\ &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (-x)_k (-\alpha-\beta)^{n-k} = C_n^{(\alpha+\beta)}(x) \end{aligned}$$

This leads to a summation formula for Charlier polynomials that is apparently new:

$$C_n^{(\alpha+\beta)}(x) = \sum_{k=0}^{n} \binom{n}{k} C_k^{(\alpha)}(x) (-\beta)^{n-k} \quad (\alpha > 0, \alpha+\beta > 0) \tag{4·4}$$

Furthermore it is well known that

$$C_n^{(\alpha)}(x) = n!\, L_n^{(x-n)}(\alpha)$$

(note that the formula in Szegö [10] is different, due to different normalization). Thus (4·4) implies that

$$L_n^{(x-n)}(\alpha+\beta) = \sum_{k=0}^{n} \frac{1}{k!} L_{n-k}^{(x-n+k)}(\alpha)(-\beta)^k$$

We now replace $x-n$ by a, α by x and β by y, producing

$$L_n^{(a)}(x+y) = \sum_{k=0}^{n} \frac{1}{k!} L_{n-k}^{(a+k)}(x)(-y)^k.$$

Compare with the well known summation formula for Laguerre polynomials

$$L_n^{(a+b+1)}(x+y) = \sum_{k=0}^{n} L_k^{(a)}(x) L_{n-k}^{(b)}(y)$$

(see Rainville [8])

(xii) *The Meixner transformation.* Let $\lambda > 0$. Theorem 10A in Iserles and Nørsett [3] states that, subject to all zeros of $\Sigma_0^n q_k x^k$ being in the interval $(0, \lambda)$, the transformation

$$\mathscr{T}\left\{\sum_{k=0}^{n} q_k x^k\right\} = \sum_{k=0}^{n} (-1)^k (-x)_k (\lambda+x)_{n-k} \lambda^k q_k \tag{4·5}$$

produces polynomials with all zeros positive.

The Meixner polynomial of the first kind (see Chihara [1]) can be explicitly expressed as

$$m_n(x;\lambda,c) = \sum_{k=0}^{n} \binom{n}{k} (-x)_k (x+\lambda)_{n-k} c^{-k}$$

where $c \in (0,1)$ and $\lambda > 0$. We will now express the transformation (4·5) in a basis comprising of polynomials m_n:

PROPOSITION 12. *If all the zeros of $q(x) = \Sigma_0^n \eta_k (x-\lambda c)^k$ are in $(0,\lambda)$ (i.e. if the polynomial $\Sigma_0^n \eta_k x^k$ has all its zeros in $(-\lambda c, (1-c)\lambda)$, then all the zeros of*

$$p(x) = \sum_{k=0}^{n} (-1)^k (x+\lambda)_{n-k}\, m_k(x;\lambda+n-k,c)\, \eta_k (\lambda c)^k$$

are positive.

Proof. We have

$$q_l = \sum_{k=l}^{n} -1)^{k-l} \binom{k}{l} \eta_k (\lambda c)^{k-l} \quad (l = 0,1,\ldots,n)$$

Thus, by (4·5),

$$\begin{aligned}\mathscr{T}q(x) = p(x) &= \sum_{l=0}^{n} (-1)^l \; -x)_l (x+\lambda)_{n-l} \lambda^l \sum_{k=l}^{n} (-1)^{k-l} \binom{k}{l} (\lambda c)^{k-l} \eta_k \\ &= \sum_{k=0}^{n} \; -1)^k (x+\lambda)_{n-k} (\lambda c)^k \eta_k \left(\sum_{l=0}^{k} \binom{k}{l} (-x)_l (\lambda+n-k+x)_{n-l} c^{-l} \right) \\ &= \sum_{k=0}^{n} \; -1)^k (x+\lambda)_{n-k} (\lambda c)^k \eta_k\, m_k(x;\lambda+n-k,c),\end{aligned}$$

and the proposition follows

COROLLARY. *If all the zeros of* $\bar{q}(x) = \sum_0^n \eta_k x^k$ *reside in* $(1-1/c, 1)$, *then all the zeros of*

$$\sum_{k=0}^{n} (x+\lambda)_k \, m_{n-k}(x; k+\lambda, c) \, \eta_k$$

are positive.

Proof. Set $\tilde{q}(x) := \bar{q}(1 - x/(\lambda c))$. Thus, by the assumption on the zeros of $\bar{q}$, all the zeros of $\tilde{q}$ are in $(0, \lambda)$. Moreover

$$\tilde{q}(x) = \sum_0^n \frac{\eta_k}{(-\lambda c)^k} (x - \lambda c)^k.$$

The statement about location of the zeros now follows from the last proposition.

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Added in proof. As pointed out by M. Ismail, the identities (4·3), (4·4) can also be derived from general expansion theorems for hypergeometric functions; see J. L. Fields and J. Wimp, 'Expansions of hypergeometric functions in hypergeometric functions', *Math. Comp.* **15** (1961), 390–395.

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