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Interpolation and Functions of Class $H(k, \alpha, 2)$[1]

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In the present note we consider the problem of finding a function $f(z)$, analytic in the open unit disk, which takes specified values in certain given points and also satisfies an integrated continuity condition over the unit circumference γ. Specifically, we shall deal with functions of the type described in the following

DEFINITION. If k is a nonnegative integer, a function $f(z)$, analytic in $|z| < 1$, is said to be *of class* $H(k,\alpha,p)$, $0 < \alpha < 1, p \geqslant 1$, *on* γ, if $f^{(k)}(z)$ is of Hardy class H_p on γ, i.e., the integral

$$\int_0^{2\pi} |f^{(k)}(r\, e^{i\theta})|^p \, d\theta$$

is uniformly bounded for $0 < r < 1$, and if there exists a constant A, independent of ϕ and θ, such that

$$\int_0^{2\pi} |f^{(k)}(e^{i\theta}) - f^{(k)}(e^{i(\theta+\phi)})|^p \, d\theta \leqslant A|\phi|^{\alpha p}.$$

Hardy and Littlewood were the first to point out that degree of approximation in the mean by trigonometric polynomials is closely related to the integrated Lipschitz conditions which are satisfied by the approximated function. The proofs of the theorems stated by Hardy and Littlewood were first given by Quade [*3*]. Walsh and Russell [*1*] used the results on mean approximation by trigonometric polynomials to prove analogues for approximation in the mean on γ by polynomials in z. They established

THEOREM 1. *Let* $g(z)$ *be defined almost everywhere on* γ. *A necessary and sufficient condition that* $g(z)$ *be equivalent* (*i.e. equal* a.e.) *on* γ *to a function* $f(z)$ *of*

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class $H(k,\alpha,p)$, $0<\alpha<1$, $p\geqslant 1$, *on* γ, *is that there exist polynomials* $p_n(z)$ *of respective degrees n, such that*

$$\left(\int_\gamma |g(z)-p_n(z)|^p\,|dz|\right)^{1/p}\leqslant A_1/n^{k+\alpha}. \tag{1}$$

For $p=2$, this theorem yields the following result on interpolation:

COROLLARY 1. *If values* ω_i, $i=0, 1, \ldots$, *are given, a necessary and sufficient condition that there exist a function* $f(z)$ *of class* $H(k,\alpha,2)$, $0<\alpha<1$, *on* γ, *satisfying* $f^{(i)}(0)=\omega_i$ *for all i, is that*

$$\left(\sum_{i=n}^{\infty}|\omega_i/i!|^2\right)^{1/2}=O(1/n^{k+\alpha}).$$

The proof follows from the fact that the series $\sum_0^\infty(\omega_i z^i/i!)$ is both a Fourier series on γ and a series of interpolation in the origin.

We shall show that Corollary 1 is a special case of a theorem which applies to more general points of interpolation. Before proceeding with this result, we prove an extension of Theorem 1 which applies to approximation by certain types of rational functions. By a rational function *of type* (m,n) we mean a function of the form

$$\frac{a_0 z^m + a_1 z^{m-1}+\ldots+a_m}{b_0 z^n + b_1 z^{n-1}+\ldots+b_n}, \qquad \sum_0^n |b_k|\neq 0.$$

THEOREM 2. *Let* $g(z)$ *be defined almost everywhere on* γ *and let* β_j *be a sequence of points such that* $|\beta_j|\leqslant\rho<1$ *for* $j=1, 2, \ldots$. *A necessary and sufficient condition that* $g(z)$ *be equivalent on* γ *to a function* $f(z)$ *of class* $H(k,\alpha,p)$, $0<\alpha<1$, $p\geqslant 1$, *on* γ, *is that there exist a sequence of rational functions* $r_n(z)$ *of respective types* $(n-1,n)$ *with formal poles in the points* $1/\bar\beta_1, 1/\bar\beta_2, \ldots, 1/\bar\beta_n$, *i.e.,* $r_n(z)=q_n(z)/(1-\bar\beta_1 z)\ldots(1-\bar\beta_n z)$ *for some polynomial* $q_n(z)$ *of degree* $n-1$, *such that*

$$\left(\int_\gamma |g(z)-r_n(z)|^p\,|dz|\right)^{1/p}\leqslant A_2/n^{k+\alpha}. \tag{2}$$

We first establish necessity. Let $p_n(z)$ be a sequence of polynomials of respective degrees n which satisfies (1), and let $R_{n,m}(z)$ be the rational function of type $(m-1,m)$ with formal poles in the points $1/\bar\beta_1, 1/\bar\beta_2, \ldots, 1/\bar\beta_m$ that interpolates to $p_n(z)$ in the points $\beta_1, \beta_2, \ldots, \beta_m$. By the extension of the Hermite formula ([*2*], p. 186), we have

$$p_n(z)-R_{n,m}(z)=\frac{1}{2\pi i}\int_{|t|=\sigma}\frac{B_m(z)\,p_n(t)}{B_m(t)\,(t-z)}\,dt, \qquad |z|<\sigma, \tag{3}$$

where

$$1<\sigma<1/\rho \quad\text{and}\quad B_m(z)\equiv\prod_{i=1}^{m}(z-\beta_i)/(\bar\beta_i z-1).$$

The convergence of the $p_n(z)$ in the mean of order p on γ, implies the existence of a constant L_1 such that

$$\int_\gamma |p_n(t)|^p \, |dt| \leqslant L_1^p, \qquad n = 1, 2, \ldots,$$

and hence ([2], §5.2)

$$|p_n(t)| \leqslant LL_1 \sigma^n, \qquad |t| = \sigma, \tag{4}$$

where L is a constant independent of n. Since $|B_m(z)| = 1$ whenever $|z| = 1$, and ([2], p. 229)

$$|(\bar{\beta}_i t - 1)/(t - \beta_i)| \leqslant (1 + \rho\sigma)/(\rho + \sigma), \qquad |t| = \sigma,$$

we obtain from (3) and (4),

$$|p_n(z) - R_{n,m}(z)| \leqslant M\sigma^n[(1 + \rho\sigma)/(\rho + \sigma)]^m, \qquad z \text{ on } \gamma.$$

Now select a positive integer λ so large that $\mu \equiv \sigma[(1 + \rho\sigma)/(\rho + \sigma)]^\lambda < 1$. Then we have

$$\left(\int_\gamma |p_n(z) - R_{n,\lambda n}(z)|^p \, |dz|\right)^{1/p} \leqslant M_1 \mu^n,$$

and so from (1) there follows

$$\left(\int_\gamma |g(z) - R_{n,\lambda n}(z)|^p \, |dz|\right)^{1/p} \leqslant A_1/n^{k+\alpha} + M_1 \mu^n \leqslant A_3/n^{k+\alpha}.$$

Finally, if we set

$$r_n(z) \equiv \begin{cases} 0, & n = 1, 2, \ldots, \lambda - 1, \\ R_{s,\lambda s}(z), & n = \lambda s, \lambda s + 1, \ldots, \lambda s + \lambda - 1, \end{cases}$$

then the $r_n(z)$ are rational functions of the desired types which satisfy (2) for a suitable choice of the constant A_2. Indeed, it suffices to choose A_2 larger than the quantity

$$A_3(2\lambda)^{k+\alpha} + (\lambda - 1)^{k+\alpha} \left(\int_\gamma |g(z)|^p \, |dz|\right)^{1/p}.$$

To prove sufficiency, assume that rational functions $r_n(z)$ of respective types $(n - 1, n)$ with formal poles in the points $1/\bar{\beta}_1, 1/\bar{\beta}_2, \ldots, 1/\bar{\beta}_n$ satisfy (2). Since the $r_n(z)$ converge in the mean of order p on γ, we have

$$\int_\gamma |r_n(z)|^p \, |dz| \leqslant L_2^p, \qquad n = 1, 2, \ldots,$$

and hence ([2], p. 255)

$$|r_n(z)| \leqslant L_3 L_2[(\sigma - \rho)/(1 - \rho\sigma)]^n, \qquad |z| = \sigma,$$

where L_3 is a constant independent of n, and $1 < \sigma < 1/\rho$. The extension of Theorem 1 to approximation by bounded analytic functions ([1], p. 368) now yields the desired conclusion.

If a function $f_0(z)$ is defined in the points β_j, then $f_0(z)$ may be expanded in a formal series found by interpolation in the β_j. Such a series is

$$f_0(z) \sim a_1/(1-\bar{\beta}_1 z) + \sum_{n=2}^{\infty} a_n(z-\beta_1)\dots(z-\beta_{n-1})/(1-\bar{\beta}_1 z)\dots(1-\bar{\beta}_n z), \quad (5)$$

where $a_1 = f_0(\beta_1)(1-\bar{\beta}_1\beta_1)$, and where the coefficient a_n is determined by the condition that (if precisely k of the points $\beta_1, \dots, \beta_{n-1}$ are equal to β_n) the kth derivative of the sum of the first n terms of the series in (5) will coincide with $f_0^{(k)}(z)$ in the point β_n. It then follows inductively that the sum of the first n terms interpolates to $f_0(z)$ in the points $\beta_1, \dots, \beta_n$, and hence the formal expansion (5) converges to $f_0(z)$ in each β_k.

If $f_0(z)$ is of class H_2 on γ, then it is known ([2], §9.1) that the above expansion is also the generalized Fourier series expansion of $f_0(z)$ in terms of the orthogonal functions

$$\phi_n(z) \equiv \begin{cases} 1/(1-\bar{\beta}_1 z), & n=1, \\ (z-\beta_1)\dots(z-\beta_{n-1})/(1-\bar{\beta}_1 z)\dots(1-\bar{\beta}_n z), & n>1, \end{cases}$$

on γ. The equivalence of these series together with Theorem 2 imply the following generalization of Corollary 1:

THEOREM 3. *Let the points β_j, $|\beta_j| \leqslant \rho < 1, j = 1, 2, \dots,$ and functional values $f_0(\beta_j)$ be given. A necessary and sufficient condition that there exist a function $f(z)$ of class $H(k,\alpha,2)$, $0<\alpha<1$, on γ, satisfying $f(\beta_j) = f_0(\beta_j)$ for all j, is that*

$$\left(\sum_{i=n}^{\infty} |a_i|^2\right)^{1/2} = O(1/n^{k+\alpha}), \quad (6)$$

where the a_i are the coefficients in the formal expansion (5), *found by interpolation in the points β_j, using the functional values $f_0(\beta_j)$.*

Assume that a function $f(z) \in H(k,\alpha,2)$ on γ exists having the desired interpolation properties, and let $s_n(z)$ denote the sum of the first n terms of the series in (5). Since $f(z)$ is of class H_2 on γ ([1], p. 359), $s_n(z)$ is the Fourier expansion of $f(z)$ in terms of the orthogonal functions $\phi_1(z), \dots, \phi_n(z)$. Clearly, any rational function of type $(n-1,n)$, with formal poles in the points $1/\bar{\beta}_1, \dots, 1/\bar{\beta}_n$, is a linear combination of the functions $\phi_1(z), \dots, \phi_n(z)$, and hence $s_n(z)$ is the rational function of that type which is of least squares approximation to $f(z)$ on γ. By Theorem 2, we thus have

$$\left(\int_\gamma |f(z)-s_n(z)|^2 \, |dz|\right)^{1/2} \leqslant A_2/n^{k+\alpha}.$$

An easy calculation yields

$$\int_\gamma |f(z)-s_n(z)|^2 \, |dz| = \int_\gamma |f(z)|^2 \, |dz| - \sum_{i=1}^{n} |b_i|^2,$$

where $|b_i|^2 = 2\pi|a_i|^2/(1-|\beta_i|^2)$. Thus

$$\sum_{i=n+1}^{\infty} |b_i|^2 = \int_\gamma |f(z)-s_n(z)|^2\,|dz| \leqslant (A_2/n^{k+\alpha})^2,$$

which implies the necessity of (6).

Now assume that (6) holds. By the Riesz–Fischer Theorem, the series $\sum_1^\infty a_i\phi_i(z)$ converges in the mean on γ to a function $g(z)$ of class L_2 on γ. In fact,

$$\left(\int_\gamma \left|g(z)-\sum_1^n a_i\,\phi_i(z)\right|^2 |dz|\right)^{1/2} = \left(\sum_{i=n+1}^{\infty} |b_i|^2\right)^{1/2} \leqslant A_3/n^{k+\alpha},$$

and so by Theorem 2, the function $g(z)$ is equivalent on γ to a function $f(z)$ of class $H(k,\alpha,2)$ on γ. Since ([2], p. 107) the series in (5) converges to $f(z)$ interior to γ, $f(z)$ has the desired interpolation properties, and the proof is complete.

We remark that since the β_j have no limit point on γ, the solution (if it exists) of the interpolation problem of Theorem 3 is unique and is given by the series in (5).

References

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